On the generalized numerical radii of operators

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Abstract

It is shown that if $\mathcal{A},\mathcal{B}$,$\mathcal{X}$, and $\mathcal{Y}$ are operators acting on a finite dimensional Hilbert space, then

$$\begin{aligned}&\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*\pm\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)\\&\qquad\le 2\left\|\mathcal{A}\right\|\left\|\mathcal{B}\right\|\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right),\end{aligned}$$

where $\omega_u\left(\mathcal{T}\right)$, $\left\|\mathcal{T}\right\|$, are, respectively, the U-numerical radius, the spectral norm, of an operator $\mathcal{T}$.

Keywords: Numerical radius; Spectral norm; Hilbert Schmidt norm; Hermitian operator; Positive operator; Inequality

1 Introduction

Let $\mathbb{B}(\mathcal{H}^{(n)})$ be the $\mathcal{C}^*$-algebra of all operators on an n-dimensional Hilbert space $\mathcal{H}^{(n)}$. The singular values $\mathfrak{s}_1(\mathcal{X}),\mathfrak{s}_2(\mathcal{X}),...,\mathfrak{s}_n(\mathcal{X})$ of an operator $\mathcal{X}\in\mathbb{B}(\mathcal{H}^{(n)})$ are the eigenvalues of $(\mathcal{X}^*\mathcal{X})^{1/2}$ presented in decreasing order and repeated based on multiplicity.

The spectral norm $\left\|\cdot\right\|$ is the norm defined on $\mathbb{B}(\mathcal{H}^{(n)})$ by

$$\left\|\mathcal{X}\right\|=\sup\{\left\|\mathcal{X}x\right\|:x\in\mathcal{H}^{(n)},\left\|x\right\|=1\}.$$

For $1\le p<\infty$, the Schatten p-norm $\left\|\cdot\right\|_p$ is the norm defined on $\mathbb{B}(\mathcal{H}^{(n)})$ by

$$\left\|\mathcal{X}\right\|_p=\left(_{j=1}^{n}\mathfrak{s}_1^p(\mathcal{X})\right)^{1/p}.$$

In particular, when $p=2$, the Schatten 2-norm $\left\|\cdot\right\|_2$ is called the Hilbert Schmidt norm. In fact,

$$\left\|\mathcal{X}\right\|_2=tr\left(\left|\mathcal{X}\right|^2\right),$$

where $tr\left(\cdot\right)$ is the trace functional defined on $\mathbb{B}(\mathcal{H}^{(n)})$.

The Ky Fan k-norms $\left\|\cdot\right\|$ is the norm defined on $\mathbb{B}(\mathcal{H}^{(n)})$ by

$$\left\|\mathcal{X}\right\|_{(k)}={}_{j=1}^{k}\,\mathfrak{s}_1(\mathcal{X})\text{ for }k=1,2,...,n.$$

In fact, one can see that $\left\|\mathcal{X}\right\|=\left\|\mathcal{X}\right\|_{(1)}$ and $\left\|\mathcal{X}\right\|_1=\left\|\mathcal{X}\right\|_{(n)}$.

A unitarily invariant norm on $\mathbb{B}(\mathcal{H}^{(n)})$, written as $\|\cdot\|_u$, is a norm satisfying the invariance property

$$\|\mathcal{U}\mathcal{X}\mathcal{V}\|_u = \|\mathcal{X}\|_u$$

for every $\mathcal{U}$,$\mathcal{V}$,$\mathcal{X}$ $\in\mathbb{B}(\mathcal{H}^{(n)})$ such that $\mathcal{U}$ and $\mathcal{V}$ are unitaries. It is known that the spectral norm, the Schatten p-norms and the Ky Fan k-norms are typical examples of unitarily invariant norms.

One of the interesting properties that relate to the Ky Fan k-norms and the unitarily invariant norms is Fan Dominance Theorem (see, e.g., [5, p. 93]) which says the following: Let $\mathcal{A}$,$\mathcal{B} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then

$$\|\mathcal{A}\|_{(k)} \leq \|\mathcal{B}\|_{(k)} \text{ for } k = 1, 2, ..., n$$

iff

$$||\mathcal{A}||_u \leq ||\mathcal{B}||_u$$

for every unitarily invariant norm $||\cdot||_u$. Moreover, it is known that (see, e.g., [5, p. 93]) if $\mathcal{A} \in \mathbb{B}(\mathcal{H}^{(n)})$, then

$$\|\mathcal{A}\| \leq ||\mathcal{A}||_u \tag{1}$$

for every unitarily invariant norm $||\cdot||_u$.

The numerical radius $\omega(\cdot)$ of an operator $\mathcal{X}$ in $\mathbb{B}(\mathcal{H}^{(n)})$ is defined by

$$\omega(\mathcal{X}) = \sup\{|\langle \mathcal{X}a, a\rangle| : a \in \mathcal{H}^{(n)}, \|a\| = 1\}, \tag{2}$$

where $\langle \cdot, \cdot\rangle$ is the inner product defined on $\mathcal{H}^{(n)}$. In fact, $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H}^{(n)})$, which is equivalent to the spectral operator norm $\|\cdot\|$, that is if $\mathcal{X}$ $\in\mathbb{B}(\mathcal{H}^{(n)})$, then

$$\frac{1}{2}\|\mathcal{X}\| \leq \omega(\mathcal{X}) \leq \|\mathcal{X}\|.$$

For more details on the numerical radius, we refer the reader to,[?,?,8,10] and references therein.

Some of the inequalities for the numerical radius of operators that we are interested in are the following:[6] If $\mathcal{A}$,$\mathcal{B}$,$\mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$, then

$$\omega\left(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}\right) \leq 2\sqrt{2}\,\|\mathcal{B}\|\,\omega\left(\mathcal{A}\right) \tag{3}$$

and

$$\omega\left(\mathcal{A}^*\mathcal{X} + \mathcal{X}\mathcal{A}\right) \leq 2\,\|\mathcal{A}\|\,\omega\left(\mathcal{X}\right). \tag{4}$$

Generalizations of inequalities (3) and (4) have been given in.[7] It has been found that if $\mathcal{A}$,$\mathcal{B}$,$\mathcal{X}$,$\mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$, then

$$\omega\left(\mathcal{A}^*\mathcal{X}\mathcal{B} \pm \mathcal{B}^*\mathcal{X}\mathcal{A}\right) \leq 2\,\|\mathcal{A}\|\,\|\mathcal{B}\|\,\omega\left(\mathcal{X}\right). \tag{5}$$

One of the interesting characterizations of the numerical radius has been given in[9] as follows: If $\mathcal{X}$ $\in\mathbb{B}(\mathcal{H}^{(n)})$, then

$$\omega(\mathcal{X}) = \sup_{\theta\in R}\|Re\left(e^{i\theta}\mathcal{X}\right)\| = \sup_{\theta\in R}\|Im\left(e^{i\theta}\mathcal{X}\right)\|,$$

where $Re\left(e^{i\theta}\mathcal{X}\right) = \frac{e^{i\theta}\mathcal{X}+e^{-i\theta}\mathcal{X}^*}{2}$ and $Im\left(e^{i\theta}\mathcal{X}\right) = \frac{e^{i\theta}\mathcal{X}-e^{-i\theta}\mathcal{X}^*}{2i}$. This characterization attracted several mathematician and motivated them to define some considerable generalizations of the numerical radius of operators. One of these generalized numerical radius (see, e.g.,[1]) which is called the generalized N-numerical radius, which asserts the following: Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H}^{(n)})$. Define the generalized N-numerical radius radius on $\mathbb{B}(\mathcal{H}^{(n)})$ by

$$\omega_N(\mathcal{X}) = \sup_{\theta\in R} N\left(Re\left(e^{i\theta}\mathcal{X}\right)\right).$$

When $N(\cdot) = \|\cdot\|_2$, the generalized N-numerical radius is called the generalized 2-numerical radius (or the Hilbert Schmidt numerical radius) which is written as

$$\omega_2(\mathcal{X}) = \sup_{\theta\in R}\left\|Re\left(e^{i\theta}\mathcal{X}\right)\right\|_2.$$

In this paper, we emphasize our selves to the N-generalized numerical radius in the case when the norm $N(\cdot)$ is unitarily invariant and we call it as the generalized U-numerical radius. So, let $\|\cdot\|_u$ be a unitarily invariant norm on $\mathbb{B}(\mathcal{H}^{(n)})$. Define the generalized U-numerical radius $\omega_u(\cdot)$ on $\mathbb{B}(\mathcal{H}^{(n)})$ by

$$\omega_u(\mathcal{X}) = \sup_{\theta\in R} \left\| Re\left(e^{i\theta}\mathcal{X}\right)\right\|_u \text{ for } \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)}). \tag{6}$$

Equivalently,

$$\omega_u(\mathcal{X}) = \sup_{\theta\in R} \left\| Im\left(e^{i\theta}\mathcal{X}\right)\right\|_u \text{ for } \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)}).$$

It is in clear view $\omega_u(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H}^{(n)})$ which is not unitarily invariant, it is weakly unitarily invariant, that is

$$\omega_u(\mathcal{U}\mathcal{X}\mathcal{U}^*) = \omega_u(\mathcal{X})$$

for each $\mathcal{U}$,$\mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ such that $\mathcal{U}$ is unitary. Also, it can be seen that if $\mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ is Hermitian, then

$$\omega_u(\mathcal{X}) = \|\mathcal{X}\|_u$$

The triangle inequality for $\omega_u(\cdot)$ is given by

$$\omega_u(\mathcal{X}+\mathcal{Y}) \leq \omega_u(\mathcal{X}) + \omega_u(\mathcal{Y}) \tag{7}$$

for $\mathcal{X}$,$\mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$.

In this paper, we give inequalities for the generalized U-numerical radii. In Section 2, we introduce a generalization of inequality (5) in terms of the generalized U-numerical radii. In Section 3, we are interested in the generalized 2-numerical radius of operators.

2 Generalized U-numerical radius of the operator $\mathcal{A}\mathcal{X}\mathcal{B}^* + \mathcal{B}\mathcal{Y}\mathcal{A}^*$

In this section, we introduce a generalization of inequality (5) in terms of the generalized U-numerical radius. In this approach, first we show the following lemma in order to obtain our main .

Lemma 2.1. *Let $\mathcal{A}, \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{A}^*\right) \leq \|\mathcal{A}\|^2 \omega_u(\mathcal{X}).$$

Proof. It can be inferred from the fact $\left|\left|Re\left(e^{i\theta}\left(\mathcal{A}\mathcal{X}\mathcal{A}^*\right)\right)\right|\right|_u = \left|\left|\mathcal{A}Re\left(e^{i\theta}\mathcal{X}\right)\mathcal{A}^*\right|\right|_u$ that

$$\left|\left|Re\left(e^{i\theta}\left(\mathcal{A}\mathcal{X}\mathcal{A}^*\right)\right)\right|\right|_u \leq \|\mathcal{A}\|^2 \left|\left|Re\left(e^{i\theta}\mathcal{X}\right)\right|\right|_u.$$

So,

$$\begin{aligned} w_u\left(\mathcal{A}\mathcal{X}\mathcal{A}^*\right) &= \sup_{\theta\in R} \left|\left|Re(e^{i\theta}\left(\mathcal{A}\mathcal{X}\mathcal{A}^*\right))\right|\right|_u \\ &\leq \sup_{\theta\in R} \|\mathcal{A}\|^2 \left|\left|Re\left(e^{i\theta}\mathcal{X}\right)\right|\right|_u \\ &= \|\mathcal{A}\|^2 \omega_u(\mathcal{X}). \end{aligned}$$

□

The main result of this part, which presents a generalization of inequality (5) is as follows: This result

Theorem 2.2. *Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then*

(a)

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^* \pm \mathcal{B}\mathcal{Y}\mathcal{A}^*\right) \leq 2\|\mathcal{A}\|\|\mathcal{B}\|\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

More precisely, if we consider $\mathcal{B} = \mathcal{I}_n$, we have

$$\omega_u\left(\mathcal{A}\mathcal{X} \pm \mathcal{Y}\mathcal{A}^*\right) \leq 2\|\mathcal{A}\|\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

(b)

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*\pm\mathcal{B}\mathcal{X}\mathcal{A}^*\right)\leq 2\left\|\mathcal{A}\right\|\left\|\mathcal{B}\right\|\omega_u\left(\mathcal{X}\oplus\mathcal{X}\right).$$

In particular, letting $\mathcal{B}=\mathcal{I}_n$, we have

$$\omega_u\left(\mathcal{A}\mathcal{X}\pm\mathcal{X}\mathcal{A}^*\right)\leq 2\left\|\mathcal{A}\right\|\omega_u\left(\mathcal{X}\oplus\mathcal{X}\right).$$

Proof. Let $T=\begin{bmatrix}\mathcal{A}&\mathcal{B}\\0&0\end{bmatrix}$ and $E=\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}$. Then $\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*+\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)=\omega_u\left(TET^*\right)$. Using Lemma 2.1, we get

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*+\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)=\omega_u\left(TET^*\right)\leq\left\|T\right\|^2\omega_u\left(E\right). \tag{8}$$

Since

$$\left\|T\right\|^2=\left\|\mathcal{A}\mathcal{A}^*+\mathcal{B}\mathcal{B}^*\right\|\leq\left\|\mathcal{A}\right\|^2+\left\|\mathcal{B}\right\|^2 \tag{9}$$

It can be inferred, from inequalities (8) and (9), that

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*+\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)\leq\left(\left\|\mathcal{A}\right\|^2+\left\|\mathcal{B}\right\|^2\right)\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right). \tag{10}$$

Now, in the inequality (10), a substitute of $\mathcal{A}$ and $\mathcal{B}$ by $t\mathcal{A}$ and $\frac{1}{t}\mathcal{B}$, respectively, in which $t=\sqrt{\frac{\|\mathcal{B}\|}{\|\mathcal{A}\|}}$ implies that

$$\begin{aligned}\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*+\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)&\leq\left(t^2\left\|\mathcal{A}\right\|^2+\frac{1}{t^2}\left\|\mathcal{B}\right\|^2\right)\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right)\\&=2\left\|\mathcal{A}\right\|\left\|\mathcal{B}\right\|\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right).\end{aligned} \tag{11}$$

On the other hand, in inequality (11), replacing $\mathcal{A}$ by $i\mathcal{A}$ implies that

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*-\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)\leq 2\left\|\mathcal{A}\right\|\left\|\mathcal{B}\right\|\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right). \tag{12}$$

Part (a) follows from inequalities (11) and (12).

For part (b), it can be seen that

$$\begin{aligned}\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{X}&0\end{bmatrix}\right)&=\sup_{\theta\in R}\left\|Re\left(e^{i\theta}\begin{bmatrix}0&\mathcal{X}\\\mathcal{X}&0\end{bmatrix}\right)\right\|_u\\&=\sup_{\theta\in R}\left\|\begin{bmatrix}0&Re\left(e^{i\theta}\mathcal{X}\right)\\Re\left(e^{i\theta}\mathcal{X}\right)&0\end{bmatrix}\right\|_u\\&=\sup_{\theta\in R}\left\|\begin{bmatrix}Re\left(e^{i\theta}\mathcal{X}\right)&0\\0&Re\left(e^{i\theta}\mathcal{X}\right)\end{bmatrix}\right\|_u\\&=\sup_{\theta\in R}\left|\left|\left|Re\left(e^{i\theta}\left(\mathcal{X}\oplus\mathcal{X}\right)\right)\right|\right|\right|\\&=\omega_u\left(\mathcal{X}\oplus\mathcal{X}\right).\end{aligned} \tag{13}$$

Now, part (b) follows from part (a) by setting $\mathcal{Y}=\mathcal{X}$ and then using identity (13). □

Remark 2.3. We can easily show that

$$\omega_u\left(\begin{bmatrix}0&\mathcal{X}\\\mathcal{Y}&0\end{bmatrix}\right)=\frac{1}{2}\sup_{\theta\in R}\left|\left|\left(\mathcal{X}+e^{i\theta}\mathcal{Y}^*\right)\oplus\left(\mathcal{X}+e^{i\theta}\mathcal{Y}^*\right)\right|\right|_u. \tag{14}$$

So, our result in Theorem 2.2(a) can be formulated as follows: If $\mathcal{A},\mathcal{B},\mathcal{X},\mathcal{Y}\in\mathbb{B}(\mathcal{H}^{(n)})$, then

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^*\pm\mathcal{B}\mathcal{Y}\mathcal{A}^*\right)\leq\left\|\mathcal{A}\right\|\left\|\mathcal{B}\right\|\sup_{\theta\in R}\left|\left|\left(\mathcal{X}+e^{i\theta}\mathcal{Y}^*\right)\oplus\left(\mathcal{X}+e^{i\theta}\mathcal{Y}^*\right)\right|\right|_u. \tag{15}$$

An application of Theorem 2.2(a) is the following.

Corollary 2.4. *Let $\mathcal{A} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\max\left(\left|\left|Re\mathcal{A}^2\right|\right|_u, \left|\left|Im\mathcal{A}^2\right|\right|_u\right) \leq \|\mathcal{A}\| \ ||\mathcal{A} \oplus \mathcal{A}||_u.$$

Proof. In Theorem 2.2(a), replacing $\mathcal{B}, \mathcal{X}$, and $\mathcal{Y}$ by $\mathcal{I}_n$, $\mathcal{A}$, and $\mathcal{A}^*$, respectively, we have

$$\omega_u\left(\mathcal{A}^2 \pm \mathcal{A}^{*2}\right) \leq 2\|\mathcal{A}\|\,\omega_u\left(\begin{bmatrix} 0 & \mathcal{A} \\ \mathcal{A}^* & 0 \end{bmatrix}\right) \tag{16}$$

and observing that

$$\omega_u\left(\mathcal{A}^2 + \mathcal{A}^{*2}\right) = 2\left|\left|Re\mathcal{A}^2\right|\right|_u, \omega_u\left(\mathcal{A}^2 - \mathcal{A}^{*2}\right) = 2\left|\left|Im\mathcal{A}^2\right|\right|_u \tag{17}$$

and

$$\omega_u\left(\begin{bmatrix} 0 & \mathcal{A} \\ \mathcal{A}^* & 0 \end{bmatrix}\right) = ||\mathcal{A} \oplus \mathcal{A}||_u. \tag{18}$$

The result is determined by the inequality (16) and identities (17) and (18). □

Observe that if $\mathcal{A}$,$\mathcal{B}$,$\mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$, then

$$\|\mathcal{A}\mathcal{X}\mathcal{B}\|_u \leq \|\mathcal{A}\|\,\|\mathcal{B}\|\,\|\mathcal{X}\|_u. \tag{19}$$

for every unitarily invariant norm $\|\cdot\|_u$. Another consequence of our inequality given in Theorem 2.2(b) can be presented as follows.

Theorem 2.5. *Let $\mathcal{A}, \mathcal{B}, \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^* \pm \mathcal{B}\mathcal{X}\mathcal{A}^*\right) \leq 2\|\mathcal{A}\|\,\|\mathcal{B}\|\,\omega_u(\mathcal{X}).$$

In particular,

$$\omega_u\left(\mathcal{A}\mathcal{X} \pm \mathcal{X}\mathcal{A}^*\right) \leq 2\|\mathcal{A}\|\,\omega_u(\mathcal{X}).$$

Proof. We have

$$\begin{aligned}
\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^* + \mathcal{B}\mathcal{X}\mathcal{A}^*\right) &= \sup_{\theta\in R}\left|\left|Re\left(e^{i\theta}\left(\mathcal{A}\mathcal{X}\mathcal{B}^* + \mathcal{B}\mathcal{X}\mathcal{A}^*\right)\right)\right|\right|_u \\
&= \sup_{\theta\in R}\left|\left|\mathcal{A}(Re\left(e^{i\theta}\mathcal{X}\right)\mathcal{B}^* + \mathcal{B}(Re\left(e^{i\theta}\mathcal{X}\right)\mathcal{A}^*\right|\right|_u \\
&\leq \sup_{\theta\in R}\left(\left|\left|\mathcal{A}(Re\left(e^{i\theta}\mathcal{X}\right)\mathcal{B}^*\right|\right|_u + \left|\left|\mathcal{B}(Re\left(e^{i\theta}\mathcal{X}\right)\mathcal{A}^*\right|\right|_u\right) \\
&\qquad\qquad \text{(by the triangle inequality)} \\
&\leq 2\|\mathcal{A}\|\,\|\mathcal{B}\|\sup_{\theta\in R}\left|\left|Re\left(e^{i\theta}\mathcal{X}\right)\right|\right|_u \ \text{(by inequality (19))} \\
&= 2\|\mathcal{A}\|\,\|\mathcal{B}\|\,\omega_u(\mathcal{X}).
\end{aligned}$$

□

Now, we equire the following result.**?**

Lemma 2.6. *If $\mathcal{X}, \mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$ are positive with a complex number z. Then*

$$||\mathcal{X} + z\mathcal{Y}||_u \leq ||\mathcal{X} + |z|\,\mathcal{Y}||_u$$

for each unitarily invariant norm $||\cdot||_u$.

A concrete result that can be inherited from inequality (15) is the following.

Corollary 2.7. *If $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$ in which $\mathcal{X}$ and $\mathcal{Y}$ are positive. Then*

$$\omega_u\left(\mathcal{A}\mathcal{X}\mathcal{B}^* \pm \mathcal{B}\mathcal{Y}\mathcal{A}^*\right) \leq \|\mathcal{A}\| \ \|\mathcal{B}\| \ ||(\mathcal{X} + \mathcal{Y}) \oplus (\mathcal{X} + \mathcal{Y})||_u.$$

Proof. It can be seen that

$$\left|\left|\left(\mathcal{X}+e^{i\theta}\mathcal{Y}\right)\oplus\left(\mathcal{X}+e^{i\theta}\mathcal{Y}\right)\right|\right|_u \leq ||(\mathcal{X}+\mathcal{Y})\oplus(\mathcal{X}+\mathcal{Y})||_u \text{ (by Lemma 2.6).}$$

Thus,

$$\sup_{\theta\in R}\left|\left|\left|\left(\mathcal{X}+e^{i\theta}\mathcal{Y}\right)\oplus\left(\mathcal{X}+e^{i\theta}\mathcal{Y}\right)\right|\right|\right| \leq |||(\mathcal{X}+\mathcal{Y})\oplus(\mathcal{X}+\mathcal{Y})|||\,. \tag{20}$$

Through the utilization of inequalities (15) and (20) we obtain a result. □

The following result gives an explicit formula for the generalized U-numerical radii of the off-diagonal parts of a 2×2 operator matrix when the these off-diagonal parts are positive.

Proposition 2.8. *Let $\mathcal{X},\mathcal{Y}\in\mathbb{B}(\mathcal{H}^{(n)})$ be positive. Then*

$$\omega_u\left(\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right)=\frac{||(\mathcal{X}+\mathcal{Y})\oplus(\mathcal{X}+\mathcal{Y})||_u}{2}.$$

Proof. It can be inferred from the definition of $\omega_u(\cdot)$ that

$$\begin{aligned}\omega_u\left(\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right) &= \sup_{\theta\in R}\left|\left|Re\left(e^{i\theta}\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right)\right|\right|_u\\ &\geq \left|\left|Re\left(\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right)\right|\right|_u\\ &= \frac{1}{2}\left|\left|\begin{bmatrix} 0 & \mathcal{X}+\mathcal{Y}\\ \mathcal{X}+\mathcal{Y} & 0\end{bmatrix}\right|\right|_u\\ &= \frac{||(\mathcal{X}+\mathcal{Y})\oplus(\mathcal{X}+\mathcal{Y})||_u}{2}.\end{aligned}\tag{21}$$

Also,

$$\begin{aligned}\omega_u\left(\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right) &= \sup_{\theta\in R}\left|\left|Re\left(e^{i\theta}\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right)\right|\right|_u\\ &= \frac{1}{2}\sup_{\theta\in R}\left|\left|\begin{bmatrix} 0 & e^{i\theta}\mathcal{X}+e^{-i\theta}\mathcal{Y}\\ e^{-i\theta}\mathcal{X}+e^{i\theta}\mathcal{Y} & 0\end{bmatrix}\right|\right|_u\\ &= \frac{1}{2}\sup_{\theta\in R}\left|\left|\begin{bmatrix} \mathcal{X}+e^{i\theta}\mathcal{Y} & 0\\ 0 & \mathcal{X}+e^{i\theta}\mathcal{Y}\end{bmatrix}\right|\right|_u\\ &= \frac{1}{2}\sup_{\theta\in R}\left|\left|\begin{bmatrix} \mathcal{X} & 0\\ 0 & \mathcal{X}\end{bmatrix}+e^{i\theta}\begin{bmatrix} \mathcal{Y} & 0\\ 0 & \mathcal{Y}\end{bmatrix}\right|\right|_u\\ &\leq \frac{1}{2}\left|\left|\begin{bmatrix} \mathcal{X}+\mathcal{Y} & 0\\ 0 & \mathcal{X}+\mathcal{Y}\end{bmatrix}\right|\right|_u \text{ (by Lemma 2.6)}\\ &= \frac{||(\mathcal{X}+\mathcal{Y})\oplus(\mathcal{X}+\mathcal{Y})||_u}{2}.\end{aligned}\tag{22}$$

So, the result is determined by inequalities (21) and (22). □

Remark 2.9. According to Proposition 2.8, if $\mathcal{X}\in\mathbb{B}(\mathcal{H}^{(n)})$ is positive, then

$$\omega_u\left(\begin{bmatrix} 0 & \mathcal{X}\\ 0 & 0\end{bmatrix}\right)=\frac{1}{2}||\mathcal{X}\oplus\mathcal{X}||_u \tag{23}$$

and

$$\omega_u\left(\begin{bmatrix} 0 & \mathcal{X}\\ \mathcal{X} & 0\end{bmatrix}\right)=||\mathcal{X}\oplus\mathcal{X}||_u\,. \tag{24}$$

In fact, the identities (23) and (24) are not only true when $\mathcal{X}$ is positive but they are also true for general operators. This can be demonstrated as follows:

$$\begin{aligned}\omega_u\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ 0 & 0\end{array}\right]\right) &= \sup_{\theta\in R}\left\|Re\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ 0 & 0\end{array}\right]\right)\right\|_u\\ &= \frac{1}{2}\sup_{\theta\in R}\left\|\left[\begin{array}{cc}0 & e^{i\theta}\mathcal{X}\\ e^{-i\theta}\mathcal{X}^* & 0\end{array}\right]\right\|_u\\ &= \frac{1}{2}\left\|\left[\begin{array}{cc}0 & \mathcal{X}\\ \mathcal{X}^* & 0\end{array}\right]\right\|_u\\ &= \frac{1}{2}\left\|\mathcal{X}\oplus\mathcal{X}\right\|_u,\end{aligned}$$

this proves identity (23) for general operators. The proof of identity (24) is similar.

Moreover, we have obtained the following result.

Corollary 2.10. *Let $\mathcal{A},\mathcal{X},\mathcal{Y}\in\mathbb{B}(\mathcal{H}^{(n)})$ in which $\mathcal{A}$ has the Cartesian decomposition $\mathcal{A}cal$=$\mathcal{A}_1cal$+$i\mathcal{B}_1$ with real numbers $a_1,$ $a_2,$ b_1, and b_2 where $a_1\le\mathcal{A}_1\le a_2$ and $b_1\le\mathcal{A}_2\le b_2$. Then*

(a)

$$\begin{aligned}\omega_u(\mathcal{A}\mathcal{X}-\mathcal{Y}\mathcal{A}^*) &\le (a_2-a_1+b_2-b_1)\,\omega_u\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ \mathcal{Y} & 0\end{array}\right]\right)\\ &\quad+\frac{|a_1+a_2|}{2}\omega_u(\mathcal{X}-\mathcal{Y})+\frac{|b_1+b_2|}{2}\omega_u(\mathcal{X}+\mathcal{Y}).\end{aligned}$$

When considering $\mathcal{Y}=\mathcal{X}$, we see that

$$\omega_u(\mathcal{A}\mathcal{X}-\mathcal{X}\mathcal{A}^*)\le(a_2-a_1+b_2-b_1)\left\|\mathcal{X}\oplus\mathcal{X}\right\|_u+|b_1+b_2|\,\omega_u(\mathcal{X}).$$

(b)

$$\omega_u(\mathcal{A}\mathcal{X}-\mathcal{Y}\mathcal{A}^*)\le\left(a_2-a_1+b_2-b_1+\sqrt{(a_1+a_2)^2+(b_1+b_2)^2}\right)\omega_u\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ \mathcal{Y} & 0\end{array}\right]\right).$$

When considering $\mathcal{Y}=\mathcal{X}$, we see that

$$\omega_u(\mathcal{A}\mathcal{X}-\mathcal{X}\mathcal{A}^*)\le\left(a_2-a_1+b_2-b_1+\sqrt{(a_1+a_2)^2+(b_1+b_2)^2}\right)\left\|\mathcal{X}\oplus\mathcal{X}\right\|_u.$$

Proof. If $x=a+ib$, where $a=\frac{a_1+a_2}{2}$ and $b=\frac{b_1+b_2}{2}$, then

$$\|\mathcal{A}_1-a\|\le\frac{a_2-a_1}{2}\text{ and }\|\mathcal{A}_2-b\|\le\frac{b_2-b_1}{2}.$$

So,

$$\|\mathcal{A}-x\|\le\|\mathcal{A}_1-a\|+\|\mathcal{A}_2-b\|\le\frac{a_2-a_1+b_2-b_1}{2}. \tag{25}$$

Consequently,

$$\begin{aligned}\omega_u((\mathcal{A}-x)\,\mathcal{X}-\mathcal{Y}\,(\mathcal{A}-x)^*) &\le 2\,\|\mathcal{A}-x\|\,\omega_u\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ \mathcal{Y} & 0\end{array}\right]\right)\text{ (by Theorem 2.2(a))}\\ &\le (a_2-a_1+b_2-b_1)\,\omega_u\left(\left[\begin{array}{cc}0 & \mathcal{X}\\ \mathcal{Y} & 0\end{array}\right]\right) \quad (26)\\ &\qquad\text{(by inequality (25)).}\end{aligned}$$

Also, we have

$$\begin{aligned}\omega_u(\mathcal{A}\mathcal{X}-\mathcal{Y}\mathcal{A}^*) &= \omega_u((\mathcal{A}-x)\,\mathcal{X}-\mathcal{Y}\,(\mathcal{A}-x)^*+x\mathcal{X}-\bar{x}\mathcal{Y})\\ &\le \omega_u((\mathcal{A}-x)\,\mathcal{X}-\mathcal{Y}\,(\mathcal{A}-x)^*)+\omega_u(x\mathcal{X}-\bar{x}\mathcal{Y}). \quad (27)\end{aligned}$$

For part (a): Observe that,

$$\begin{aligned}\omega_u(x\mathcal{X} - \bar{x}\mathcal{Y}) &= \omega_u(a\,(\mathcal{X} - \mathcal{Y}) + ib\,(\mathcal{X} + \mathcal{Y})) \\ &\leq |a|\,\omega_u(\mathcal{X} - \mathcal{Y}) + |b|\,\omega_u(\mathcal{X} + \mathcal{Y}) \\ &= \frac{|a_1 + a_2|}{2}\omega_u(\mathcal{X} - \mathcal{Y}) + \frac{|b_1 + b_2|}{2}\omega_u(\mathcal{X} + \mathcal{Y}). \qquad (28)\end{aligned}$$

Now, part (a) follows from inequalities (26), (27), and (28).
For part (b): Observe that,

$$\begin{aligned}\omega_u(x\mathcal{X} - \bar{x}\mathcal{Y}) &\leq 2\,|x|\,\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) \text{ (by Theorem 2.2(a))} \\ &= \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\,\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right). \qquad (29)\end{aligned}$$

Hence, the result can be deduced from the inequalities (26), (27), and (29). □

Remark 2.11. Using a similar argument to the one employed in the proof of Theorem 2.5, one can have another results related to the particular cases of Corollary 2.10. These results can be stated as follows: If $\mathcal{A}, \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ in which $\mathcal{A}$ has the Cartesian decomposition $\mathcal{A}$cal=$\mathcal{A}_1$cal+$i\mathcal{B}_1$ and $a_1,$ $a_2,$ b_1, and b_2 are real numbers for which $a_1 \leq \mathcal{A}_1 \leq a_2$ and $b_1 \leq \mathcal{A}_2 \leq b_2$. Then

(a)
$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}^*) \leq (a_2 - a_1 + b_2 - b_1 + |b_1 + b_2|)\,\omega_u(\mathcal{X}).$$

(b)
$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}^*) \leq \left(a_2 - a_1 + b_2 - b_1 + \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\right)\omega_u(\mathcal{X}).$$

Corollary 2.12. *Let $\mathcal{A}, \mathcal{X}, \mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$ in which $\mathcal{A}$ is Hermitian, and there exist real numbers $a_{1,}a_2$ such that $a_1 \leq \mathcal{A} \leq a_2$. Then*

(a)
$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{Y}\mathcal{A}) \leq (a_2 - a_1)\,\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) + \frac{|a_1 + a_2|}{2}\omega_2(\mathcal{X} - \mathcal{Y}).$$

In particular, put $\mathcal{Y} = \mathcal{X}$, we obtain

$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}) \leq (a_2 - a_1)\,\|\mathcal{X} \oplus \mathcal{X}\|_u$$

and

$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}) \leq (a_2 - a_1)\,\omega_u(\mathcal{X})$$

(b)
$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{Y}\mathcal{A}) \leq 2\max{(a_2, -a_1)}\,\omega_u\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

Proof. By considering $b_1 = b_2 = 0$ in Corollary 2.10 we can deduce the results. □

Remark 2.13. By employing an argument analogous to the one utilized in the proof of Theorem 2.5, one can show the following: If $\mathcal{A}, \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ for which $\mathcal{A}$ is Hermitian and there exist real numbers $a_{1,}a_2$ such that $a_1 \leq \mathcal{A} \leq a_2$. Then

$$\omega_u(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}) \leq (a_2 - a_1)\,\omega_u\left(\mathcal{X}\right).$$

The result below is directly derived from a Corollary 2.12.

Corollary 2.14. *If $\mathcal{A}, \mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ in which $\mathcal{A}$ is positive. Then*

$$\omega_u\left(\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{A}\right) \leq \|\mathcal{A}\|\,\omega_u\left(\mathcal{X}\right).$$

Proof. Since $0 \leq \mathcal{A} \leq \|\mathcal{A}\|$, take $a_1 = 0, a_2 = \|\mathcal{A}\|$ and then apply Corollary 2.12. □

3 The generalized 2-numerical radius of operators

This section aims to give some bounds and demonstrates several novel inequalities and equalities for the generalized 2-numerical radius. First, to attain our aim, we require the following two Lemma (see[1]).

Lemma 3.1. *If $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\mathbb{B}(\mathcal{H}^{(n)})$. Then*

(a) $\omega_2\left(\begin{bmatrix}\mathcal{X} & 0\\ 0 & \mathcal{Y}\end{bmatrix}\right)\leq\sqrt{\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{Y})}$.

(b) $\omega_2\left(\begin{bmatrix}0 & \mathcal{X}\\ \mathcal{Y} & 0\end{bmatrix}\right)\leq\frac{\omega_2(\mathcal{X}+\mathcal{Y})+\omega_2(\mathcal{X}-\mathcal{Y})}{\sqrt{2}}$.

(c) $\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right)\leq\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|\mathcal{Y}\right\|_2^2}+\sqrt{\omega_2^2(\mathcal{W})+\frac{1}{2}\left\|\mathcal{Z}\right\|_2^2}$.

(d) $\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ 0 & 0\end{bmatrix}\right)=\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|\mathcal{Y}\right\|_2^2}$.

(e) $\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{bmatrix}\right)\leq\sqrt{\omega_2^2(\mathcal{X}+\mathcal{Y})+\omega_2^2(\mathcal{X}-\mathcal{Y})}$.

Lemma 3.2. *If $\mathcal{X},\mathcal{Y}\in\mathbb{B}(\mathcal{H}^{(n)})$. Then*

(a) $\omega_2(\mathcal{X})=\frac{\sqrt{\|\mathcal{X}\|_2^2+|tr\mathcal{X}^2|}}{\sqrt{2}}$.

(b) $\frac{1}{\sqrt{2}}\left\|\mathcal{X}+\mathcal{Y}\right\|_2\leq\omega_2\left(\begin{bmatrix}0 & \mathcal{X}\\ \mathcal{Y}^* & 0\end{bmatrix}\right)$.

(c) $\omega_2\left(\begin{bmatrix}0 & \mathcal{X}\\ \mathcal{X}^* & 0\end{bmatrix}\right)=\sqrt{2}\left\|\mathcal{X}\right\|_2$.

Remark 3.3.

(a) It is in clear view from Lemma 3.2(a) that if $\mathcal{X}\in\mathbb{B}(\mathcal{H}^{(n)})$, then

$$\omega_2(\mathcal{X})\geq\frac{\|\mathcal{X}\|_2}{\sqrt{2}} \tag{30}$$

with equality if and only $tr\mathcal{X}^2=0$.

(b) One of the basic facts is the following, which can be proved by direct computations: If $\mathcal{X}\in\mathbb{B}(\mathcal{H}^{(n)})$, then

$$\left\|\mathcal{X}\right\|_2^2=\left\|Re\left(\mathcal{X}\right)\right\|_2^2+\left\|Im\left(\mathcal{X}\right)\right\|_2^2. \tag{31}$$

Our first result of this section can be regarded as a refinement of inequality (30).

Theorem 3.4. *Let $\mathcal{X}\in\mathbb{B}(\mathcal{H}^{(n)})$ and suppose that $r_1=\left|\|Re\mathcal{X}\|_2^2-\frac{\|\mathcal{X}\|_2^2}{2}\right|$, $r_2=\left|\|Im\mathcal{X}\|_2^2-\frac{\|\mathcal{X}\|_2^2}{2}\right|$, $s_1=\max\left\{\|Re\mathcal{X}\|_2^2,\frac{\|\mathcal{X}\|_2^2}{2}\right\}$, and $s_2=\max\left\{\|Im\mathcal{X}\|_2^2,\frac{\|\mathcal{X}\|_2^2}{2}\right\}$. Then*

$$\omega_2(\mathcal{X})\geq\sqrt{\frac{\|\mathcal{X}\|_2^2}{2}+\frac{r_1+r_2}{4}+\frac{|s_1-s_2|}{2}}.$$

Proof. Since

$$\omega_2^2(\mathcal{X}) \geq \|Re\mathcal{X}\|_2^2, \omega_2^2(\mathcal{X}) \geq \|Im\mathcal{X}\|_2^2, \text{ and } \omega_2^2(\mathcal{X}) \geq \frac{\|\mathcal{X}\|_2^2}{2},$$

we have

$$\begin{aligned}
\omega_2^2(\mathcal{X}) &\geq \max\{s_1, s_2\} \\
&= \frac{1}{2}(s_1 + s_2) + \frac{1}{2}|s_1 - s_2| \\
&= \frac{1}{4}\left(\|\mathcal{X}\|_2^2 + \|Re\mathcal{X}\|_2^2 + \|Im\mathcal{X}\|_2^2\right) + \frac{r_1 + r_2}{4} + \frac{|s_1 - s_2|}{2} \\
&= \frac{\|\mathcal{X}\|_2^2}{2} + \frac{r_1 + r_2}{4} + \frac{|s_1 - s_2|}{2} \text{ (by identity (31))}
\end{aligned}$$

□

In the following Corollary, let r_1, r_2, s_1, and s_2 be as given in Theorem 3.4.

Corollary 3.5. *If $\mathcal{X} \in \mathbb{B}(\mathcal{H}^{(n)})$ in which $\omega_2(\mathcal{X}) = \frac{\|\mathcal{X}\|_2}{\sqrt{2}}$.Then*

(a) $\left|tr\mathcal{X}^2\right| \geq \frac{r_1+r_2}{2} + |s_1 - s_2|$.

(b) *If $\omega_2(\mathcal{X}) = \frac{\|\mathcal{X}\|_2}{\sqrt{2}}$, then $\|\mathcal{X}\|_2 = \sqrt{2}\,\|Re\mathcal{X}\|_2 = \sqrt{2}\,\|Im\mathcal{X}\|_2$.*

Proof. Part (a) can be obtained from Lemma 3.2(a) and Theorem 3.4. For part (b), observe that $\omega_2(\mathcal{X}) = \frac{\|\mathcal{X}\|_2}{\sqrt{2}}$ together with Theorem 3.4 implies that $r_1 = r_2 = 0$, and so

$$\|\mathcal{X}\|_2 = \sqrt{2}\,\|Re\mathcal{X}\|_2 = \sqrt{2}\,\|Im\mathcal{X}\|_2.$$

□

In the rest of this section, we are devoted to the generalized 2-numerical radius of 2×2 operators matrices. The text begins with a result related to the inequalities of Lemma 3.1(b) and (e).

Theorem 3.6. *If $\mathcal{X}, \mathcal{Y} \in \mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\omega_2\left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Y}^* & \mathcal{X}^* \end{bmatrix}\right) = \sqrt{\|\mathcal{X}\|_2^2 + \|\mathcal{Y}\|_2^2 + \left|2\,\|Re\mathcal{X}\|_2^2 + \|\mathcal{Y}\|_2^2 - \|\mathcal{X}\|_2^2\right|}.$$

Proof. Let $L = \begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Y}^* & \mathcal{X}^* \end{bmatrix}$. Then

$$\begin{aligned}
\|L\|_2^2 &= tr(L^*L) \\
&= tr(\mathcal{X}^*\mathcal{X} + \mathcal{Y}\mathcal{Y}^* + \mathcal{X}\mathcal{X}^* + \mathcal{Y}^*\mathcal{Y}) \\
&= 2tr(\mathcal{X}^*\mathcal{X} + \mathcal{Y}^*\mathcal{Y}) \\
&= 2(\|\mathcal{X}\|_2^2 + \|\mathcal{Y}\|_2^2) \qquad (32)
\end{aligned}$$

and

$$\begin{aligned}
\left|tr\left(L^2\right)\right| &= \left|tr\left(\mathcal{X}^2 + \mathcal{Y}\mathcal{Y}^* + \mathcal{Y}^*\mathcal{Y} + \mathcal{X}^{*2}\right)\right| \\
&= \left|tr(\mathcal{X}^2) + 2tr(\mathcal{Y}\mathcal{Y}^*) + tr(\mathcal{X}^{*2})\right| \\
&= \left|tr(\mathcal{X} + \mathcal{X}^*)^2 + 2tr(\mathcal{Y}\mathcal{Y}^*) - 2tr(\mathcal{X}\mathcal{X}^*)\right| \\
&= \left|tr(\mathcal{X} + \mathcal{X}^*)^2 + 2\,\|\mathcal{Y}\|_2^2 - 2\,\|\mathcal{X}\|_2^2\right| \\
&= 2\left|2\,\|Re\mathcal{X}\|_2^2 + \|\mathcal{Y}\|_2^2 - \|\mathcal{X}\|_2^2\right| \qquad (33)
\end{aligned}$$

Using Lemma 3.2(a), we have

$$\begin{aligned}\omega_2(L) &= \sqrt{\frac{1}{2}\|L\|_2^2+\frac{1}{2}|trL^2|}\\ &= \sqrt{\|\mathcal{X}\|_2^2+\|\mathcal{Y}\|_2^2+\left|2\|Re\mathcal{X}\|_2^2+\|\mathcal{Y}\|_2^2-\|\mathcal{X}\|_2^2\right|}\end{aligned}$$

(by relations (32) and (33)).

□

Theorem 3.7. *Suppose that $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right)\leq\sqrt{\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{W})+\frac{1}{2}(\|\mathcal{Y}\|_2+\|\mathcal{Z}\|_2)^2}.$$

Proof. Let $L=\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}$ and $M=\|\mathcal{X}\|_2^2+\|\mathcal{Y}\|_2^2+\|\mathcal{Z}\|_2^2+\|\mathcal{W}\|_2^2$. Now, by Lemma 3.2(a) we have

$$\begin{aligned}\omega_2^2(L) &= \frac{1}{2}\|L\|_2^2+\frac{1}{2}\left|trL^2\right|\\ &\leq \frac{1}{2}M+\frac{1}{2}\left|tr(\mathcal{X}^2+\mathcal{W}^2+\mathcal{Y}\mathcal{Z}+\mathcal{Z}\mathcal{Y})\right|\\ &\leq \frac{1}{2}M+\frac{1}{2}\left|tr(\mathcal{X}^2)\right|+\frac{1}{2}\left|tr(\mathcal{W}^2)\right|+\left|tr(\mathcal{Y}\mathcal{Z})\right|.\end{aligned}$$

Since $|tr(\mathcal{Y}\mathcal{Z})|\leq\|\mathcal{Y}\|_2\|\mathcal{Z}\|_2$, it follows that

$$\omega_2^2(L)\leq\frac{1}{2}M+\frac{1}{2}\left|tr(\mathcal{X}^2)\right|+\frac{1}{2}\left|tr(\mathcal{W}^2)\right|+\|\mathcal{Y}\|_2\|\mathcal{Z}\|_2$$

and so

$$\omega_2^2(L)\leq\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{W})+\frac{1}{2}\left(\|\mathcal{Y}\|_2^2+\|\mathcal{Z}\|_2^2+2\|\mathcal{Y}\|_2\|\mathcal{Z}\|_2\right).$$

Hence,

$$\omega_2^2(L)\leq\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{W})+\frac{1}{2}\left(\|\mathcal{Y}\|_2+\|\mathcal{Z}\|_2\right)^2.$$

□

Theorem 3.8. *If $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\mathbb{B}(\mathcal{H}^{(n)})$. Then*

$$\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right)\leq\min\{\zeta,\eta\},$$

where $\zeta=\sqrt{\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{W})}+\frac{1}{\sqrt{2}}\left(\omega_2\left(\mathcal{Y}+\mathcal{Z}\right)+\omega_2\left(\mathcal{Y}-\mathcal{Z}\right)\right)$ *and* $\eta=\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\|\mathcal{Y}\|_2^2}+\sqrt{\omega_2^2(\mathcal{W})+\frac{1}{2}\|\mathcal{Z}\|_2^2}$.

Proof.

$$\begin{aligned}\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right) &= \omega_2\left(\begin{bmatrix}\mathcal{X} & 0\\ 0 & \mathcal{W}\end{bmatrix}+\begin{bmatrix}0 & \mathcal{Y}\\ \mathcal{Z} & 0\end{bmatrix}\right)\\ &\leq \omega_2\left(\begin{bmatrix}\mathcal{X} & 0\\ 0 & \mathcal{W}\end{bmatrix}\right)+\omega_2\left(\begin{bmatrix}0 & \mathcal{Y}\\ \mathcal{Z} & 0\end{bmatrix}\right)\\ &\leq \sqrt{\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{W})}+\frac{1}{\sqrt{2}}\left(\omega_2\left(\mathcal{Y}+\mathcal{Z}\right)+\omega_2\left(\mathcal{Y}-\mathcal{Z}\right)\right)\\ &= \zeta.\end{aligned}$$

Also, using Lemma 3.1(c), we get

$$\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right)\leq\eta.$$

Hence, $\omega_2\left(\begin{bmatrix}\mathcal{X} & \mathcal{Y}\\ \mathcal{Z} & \mathcal{W}\end{bmatrix}\right)\leq\min\{\zeta,\eta\}$. □

This part is concluded with the following result.

Theorem 3.9.

$$\omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right)\leq \min\left\{\lambda,\vartheta,\ \mu\right\},$$

where $\lambda = 2\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|\mathcal{Y}\right\|_2^2}$, $\vartheta=\sqrt{\omega_2^2(\mathcal{X})+\omega_2^2(\mathcal{Y})+\frac{1}{2}(\|\mathcal{X}\|_2+\|\mathcal{Y}\|_2)^2}$, *and* $\mu=\sqrt{2}\left(\sqrt{\omega_2^2(\mathcal{X})}+\sqrt{\omega_2^2(\mathcal{Y})}\right)$.

Proof. If $\mathcal{U}=\left[\begin{array}{cc}0 & I\\ I & 0\end{array}\right]$. Then $\mathcal{U}$ is unitary, and so

$$\begin{aligned}
\omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right) &= \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ 0 & 0\end{array}\right]+\left[\begin{array}{cc}0 & 0\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right)\\
&\leq \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ 0 & 0\end{array}\right]\right)+\omega_2\left(\left[\begin{array}{cc}0 & 0\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right)\\
&= \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ 0 & 0\end{array}\right]\right)+\omega_2\left(\mathcal{U}^*\left[\begin{array}{cc}0 & 0\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\mathcal{U}\right)\\
&= \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ 0 & 0\end{array}\right]\right)+\omega_2\left(\left[\begin{array}{cc}\mathcal{X} & \mathcal{Y}\\ 0 & 0\end{array}\right]\right)\\
&= \sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|-\mathcal{Y}\right\|_2^2}+\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|\mathcal{Y}\right\|_2^2}\\
&= 2\sqrt{\omega_2^2(\mathcal{X})+\frac{1}{2}\left\|\mathcal{Y}\right\|_2^2}\\
&= \lambda
\end{aligned}$$

Using Theorem (3.7), we get

$$\begin{aligned}
\omega_2\left(\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{array}\right) &\leq \sqrt{\omega_2^2(-\mathcal{Y})+\omega_2^2(\mathcal{X})+\frac{1}{2}(\|\mathcal{Y}\|_2+\|\mathcal{X}\|_2)^2}\\
&= \sqrt{\omega_2^2(\mathcal{Y})+\omega_2^2(\mathcal{X})+\frac{1}{2}(\|\mathcal{Y}\|_2+\|\mathcal{X}\|_2)^2}\\
&= \vartheta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right) &= \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & 0\\ 0 & \mathcal{X}\end{array}\right]+\left[\begin{array}{cc}0 & -\mathcal{Y}\\ \mathcal{Y} & 0\end{array}\right]\right)\\
&\leq \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & 0\\ 0 & \mathcal{X}\end{array}\right]\right)+\omega_2\left(\left[\begin{array}{cc}0 & -\mathcal{Y}\\ \mathcal{Y} & 0\end{array}\right]\right)\\
&= \omega_2\left(\left[\begin{array}{cc}\mathcal{X} & 0\\ 0 & \mathcal{X}\end{array}\right]\right)+\omega_2\left(\left[\begin{array}{cc}0 & -\mathcal{Y}\\ -\mathcal{Y} & 0\end{array}\right]\right)\\
&\leq \sqrt{2}\left(\sqrt{\omega_2^2(-\mathcal{Y})}+\sqrt{2}\sqrt{\omega_2^2(\mathcal{X})}\right)\\
&= \sqrt{2}\left(\sqrt{\omega_2^2(\mathcal{Y})}+\sqrt{\omega_2^2(\mathcal{X})}\right)\\
&= \mu.
\end{aligned}$$

Hence,

$$\omega_2\left(\left[\begin{array}{cc}\mathcal{X} & -\mathcal{Y}\\ \mathcal{Y} & \mathcal{X}\end{array}\right]\right)\leq \min\left\{\lambda,\vartheta,\mu\right\}.$$

□

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Conflicts of Interest

The authors have no conflicts of interest to declare.

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