A new generalized topology coarser than the old generalized topology

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Abstract

In this research work, basic concepts and properties are considered within the context of a generalized topological space (X, μ), as tools to generate a new generalized topology $\widehat{\mu}$ by means of a μ-base formed by the μ-interiors of μ-closed sets. This leads to an exploration of the relationship between some of the properties of the generalized topologies μ and $\widehat{\mu}$, such as generalized separation axioms, generalized connectedness, generalized continuity, generalized topological sum, and generalized product topology.

Keywords: Generalized topology; μ-open; μ-base; μ-regular space

1 Introduction

General topology is of great importance in many fields of applied sciences as well as in branches of mathematics. In fact, it is used in information system, data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, quantum physics and particle physics, etc. During the last four decades, various ways of extending concepts and properties into theoretical frameworks broader than that of general topology have been presented. In particular, the notion of generalized topology is one of the most relevant extensions of general topology, which has been extensively studied by the prominent Hungarian mathematician, topologist and university professor Császár, as can be seen in the manuscripts[3]-.[9] In these works have been studied properties of a generalized topology, generalized separation axioms, generalized forms of continuity, etc. This theoretical framework has been the main motivation of a series of works where generalizations of topological notions and its applications are studied, as we can see in references[1,2] and[10]-.[21] In this paper we consider a collection of sets described in terms of the μ-interior operator of a generalized topological space (X, μ) to generate a new generalized topology $\widehat{\mu}$ on X such that $\widehat{\mu} \subseteq \mu$. Using the generalized topology $\widehat{\mu}$, we get several results, among which stand some new characterizations of generalized separation axioms in the space (X, μ).

2 Preliminaries

In this section we introduce some known definitions and results, which are fundamental for the development of the topic discussed in this manuscript.

Definition 2.1. [3] Let X be a nonempty set. A collection μ of subsets of X is called a *generalized topology* (briefly GT) on X, if $\emptyset \in \mu$ and any union of elements of μ belongs to μ.

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A *generalized topological space* (briefly GTS)is an ordered pair (X, μ) where X is a set and μ is a generalized topology on X. A subset B of X is said to be *μ-open* (resp. *μ-closed*) if $B \in \mu$ (resp. $X \setminus B \in \mu$). A generalized topology μ on X is called a *strong generalized topology*[5] (briefly strong GT) if $X \in \mu$.

Definition 2.2. [7] Let (X, μ) be a GTS. For a subset B of X, the μ-closure of B, denoted by $c_\mu(B)$, is defined as the intersection of all μ-closed sets containing B and the μ-interior of B, denoted by $i_\mu(B)$, is defined as the union of all μ-open sets contained in B.

According to,[3] if (X, μ) is a GTS, then for every $B \subseteq X$, we have $i_\mu(B) = X \setminus c_\mu(X \setminus B)$ and $c_\mu(B) = X \setminus i_\mu(X \setminus B)$. Hence, $c_\mu(X \setminus B) = X \setminus i_\mu(B)$ and $i_\mu(X \setminus B) = X \setminus c_\mu(B)$ for every $B \subseteq X$.

Definition 2.3. [8] Let (X, μ) be a GTS. A collection $\beta \subseteq \mu$ is called a base for μ (briefly *μ-base*) if $\mu = \{\bigcup \beta' : \beta' \subseteq \beta\}$.

Proposition 2.4. [11] *Let X be a nonempty set. Then:*

1. *A collection β of subsets of X is a base for a GT μ on X if and only if $\beta \subseteq \mu$ and, for each μ-open set U and each $x \in U$, there exists $B \in \beta$ such that $x \in B \subseteq U$.*
2. *Any collection β of subsets of X is a base for some GT μ on X.*

Definition 2.5. A GTS (X, μ) is called:

1. *μ-T_2*,[10] if for every pair of points $x, y \in X$, there exist two disjoint μ-open sets U and V such that $x \in U$ and $y \in V$.
2. *μ-regular*,[14] if for every μ-closed subset F of X and every point $x \in X \setminus F$, there exist disjoint μ-open sets U and V such that $x \in U$ and $F \subseteq V$.
3. *μ-connected*,[20] if there are no non-empty disjoint μ-open sets U and V such that $X = U \cup V$.

Definition 2.6. Let (X, μ) be a GTS. A subset B of X is called:

1. *μ-nowhere dense*,[10] if $i_\mu(c_\mu(B)) = \emptyset$.
2. *μ-first category*,[13] if $B = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n is a μ-nowhere dense set.
3. *μ-dense*,[10] if $c_\mu(B) = X$.

3 A generalized topology coarser than μ

Let (X, μ) be a GTS and let $\widehat{\beta} = \{i_\mu(B) : X \setminus B \in \mu\}$. According to Proposition 2.4, $\widehat{\beta}$ is a base for a GT $\widehat{\mu}$ on X and that $\widehat{\mu} \subseteq \mu$. In the fact, if $U \in \widehat{\mu}$ and $x \in U$ then there exists a μ-closed set B such that $x \in i_\mu(B) \subseteq U$, so $V = i_\mu(B)$ is a μ-open set such that $x \in V \subseteq U$, and hence $U \in \mu$.

By $c_{\widehat{\mu}}$ and $i_{\widehat{\mu}}$ we denote the closure operator and the interior operator corresponding to $\widehat{\mu}$, respectively.

Lemma 3.1. *Let (X, μ) a GTS and consider the GT $\widehat{\mu}$ on X. Then $\mu = \widehat{\mu}$ if and only if for every μ-closed subset A of X and every point $x \in X \setminus A$, there exist disjoint μ-open sets O and U such that $x \in O$, $A \subseteq c_\mu(U)$.*

Proof. Assume that $\mu = \widehat{\mu}$, A is a μ-closed subset of X and $x \in X \setminus A$. Since $\mu = \widehat{\mu}$ and $x \in X \setminus A \in \mu$, there exists a μ-closed set F such that $x \in i_\mu(F) \subseteq X - A$. Let $O = i_\mu(F)$ and $U = X \setminus F$. Then O and U are μ-open sets such that $x \in O$, $A \subseteq X \setminus i_\mu(F) = c_\mu(X \setminus F) = c_\mu(U)$ and $O \cap U = \emptyset$.

Conversely, since $\widehat{\mu} \subseteq \mu$, to show that $\mu = \widehat{\mu}$, it suffices to check that $\mu \subseteq \widehat{\mu}$. Suppose that for every μ-closed subset A of X and every point $x \in X \setminus A$, there exist disjoint μ-open sets O and U such that $x \in O$ and $A \subseteq c_\mu(U)$. Let $G \in \mu$ and $z \in G$. Then $H = X \setminus G$ is a μ-closed subset of X and $z \in X \setminus H$, which implies that there exists a pair V, W of disjoint μ-open sets such that $z \in V$ and $H \subseteq c_\mu(W)$. Put $B = X \setminus W$. Then $X \setminus B = W \in \mu$ and so $i_\mu(B) \in \widehat{\beta}$. Since $V \in \mu$ and $z \in V \subseteq X \setminus W = B$, it follows that $z \in i_\mu(B)$. We notice that $i_\mu(B) = i_\mu(X \setminus W) = X \setminus c_\mu(W) \subseteq X \setminus H = G$. This implies that $G \in \widehat{\mu}$ and hence, $\mu \subseteq \widehat{\mu}$. □

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Corollary 3.2. *If (X,μ) is a μ-regular GTS, then $\mu=\widehat{\mu}$.*

Proof. The proof follows from the definition of a μ-regular GTS and Lemma 3.1. □

In the following example we show that, in general, the converse of Corollary 3.2 is not true.

Example 3.3. Let $X=\{a_1,a_2,a_3\}$, where a_1, a_2 and a_3 are pairwise distinct elements. Suppose that $\mu=\{\emptyset,\{a_1\},\{a_2\},\{a_1,a_2\}\}$. Then (X,μ) is not μ-regular, because there are no two μ-open sets U and V that can separate a_1 and $\{a_2,a_3\}$. But, $\mu=\widehat{\beta}=\widehat{\mu}$ as we can easily see.

Theorem 3.4. *In any GTS (X,μ), we have $\widehat{\widehat{\mu}}=\widehat{\mu}$.*

Proof. Let $\widehat{A}$ be a $\widehat{\mu}$-closed set and $x\notin\widehat{A}$. By Lemma 3.1, we need only prove that there exist disjoint $\widehat{\mu}$-open sets $\widehat{U}$ and $\widehat{V}$ such that $x\in\widehat{U}$ and $\widehat{A}\subseteq c_{\widehat{\mu}}(\widehat{V})$. Note that $x\in X-\widehat{A}\in\widehat{\mu}$ and hence, there exists a μ-closed set B such that $x\in i_\mu(B)\subseteq X-\widehat{A}$. Let $\widehat{U}=i_\mu(B)$ and $\widehat{V}=i_\mu(c_\mu(X-B))$. Then $\widehat{U},\widehat{V}\in\widehat{\mu}$ and $\widehat{U}\cap\widehat{V}=i_\mu(B)\cap i_\mu(c_\mu(X-B))\subseteq(X-c_\mu(X-B))\cap c_\mu(X-B)=\emptyset$. Obviously $x\in\widehat{U}$ and $\widehat{A}\subseteq X-i_\mu(B)=c_\mu(X-B)=c_\mu(i_\mu(X-B))\subseteq c_\mu(i_\mu(c_\mu(X-B))=c_{\widehat{\mu}}(\widehat{V})$. □

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.5. *In any GTS (X,μ), the following statements are equivalent:*

1. *$\mu=\widehat{\mu}$.*
2. *There exists a GT ν on X for which $\mu=\widehat{\nu}$.*

Lemma 3.6. *Let (X,μ) be a GTS and consider the GT $\widehat{\mu}$. For any $B\subseteq X$, we have $i_\mu(c_\mu(B))\subseteq i_{\widehat{\mu}}(c_{\widehat{\mu}}(B)$.*

Proof. Since $\widehat{\mu}\subseteq\mu$, we have $i_\mu(c_\mu(B))\subseteq c_\mu(B)\subseteq c_{\widehat{\mu}}(B)$. Now, as $i_\mu(c_\mu(B))\in\widehat{\beta}\subseteq\widehat{\mu}$, it follows that $i_\mu(c_\mu(B))=i_{\widehat{\mu}}(i_\mu(c_\mu(B)))\subseteq i_{\widehat{\mu}}(c_{\widehat{\mu}}(B))$. □

Corollary 3.7. *Let (X,μ) be a GTS and consider the GT $\widehat{\mu}$. Let B be a subset of X. Then:*

1. *If B is a $\widehat{\mu}$-nowhere dense set, then B is a μ-nowhere dense set.*
2. *If B is a $\widehat{\mu}$-first category set, then B is a μ-first category set.*

Proof. The proof follows directly from Lemma 3.6. □

Proposition 3.8. *A GTS (X,μ) is μ-connected if and only if $(X,\widehat{\mu})$ is $\widehat{\mu}$-connected.*

Proof. If $(X,\widehat{\mu})$ is not connected, then we have (X,μ) is not connected because $\widehat{\mu}\subseteq\mu$. Conversely, suppose that (X,μ) is not connected. Then there exist two nonempty μ-open subsets V and W of X such that $V\cap W=\emptyset$ and $V\cup W=X$. Put $A=i_\mu(c_\mu(V))$ and $B=i_\mu(c_\mu(W))$. Observe that $A,B\in\widehat{\beta}\subseteq\widehat{\mu}$ and $X=V\cup W=i_\mu(V)\cup i_\mu(W)\subseteq i_\mu(c_\mu(V))\cup i_\mu(c_\mu(W))=A\cup B$, which implies that $A\cup B=X$. To finalize the proof, we will show that $A\cap B=\emptyset$. Suppose that $A\cap B\neq\emptyset$. Then $\emptyset\neq i_\mu(c_\mu(V))\cap i_\mu(c_\mu(W))\subseteq i_\mu(c_\mu(V))\cap c_\mu(W)$ and so, $i_\mu(c_\mu(V))\cap c_\mu(W)\neq\emptyset$. Thus, we have $\emptyset\neq i_\mu(c_\mu(V))\cap W\subseteq c_\mu(V)\cap W$ and hence, $c_\mu(V)\cap W\neq\emptyset$. Therefore, $V\cap W\neq\emptyset$, contradicting the fact that $V\cap W=\emptyset$. Consequently, $A\cap B=\emptyset$ and we conclude that $(X,\widehat{\mu})$ is not connected. □

Theorem 3.9. *Let (X,μ) be a strong GTS. Then, $\widehat{\mu}=\{\emptyset,X\}$ if and only if for each $B\subseteq X$, B is either μ-dense in X or B is μ-nowhere dense in X.*

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Proof. $\widehat{\mu} = \{\emptyset, X\}$ if and only if $\widehat{\beta} = \{\emptyset, X\}$ if and only if for each μ-closed set A, $i_\mu(A) = \emptyset$ or $i_\mu(A) = X$ if and only if for each $B \subseteq X$, $i_\mu(c_\mu(B)) = \emptyset$ or $i_\mu(c_\mu(B)) = X$ if and only if for each $B \subseteq X$, B is μ-nowhere dense or μ-dense. □

It is important to point that if μ and ν are two GT's on X and if $\mu \subseteq \nu$, then in general $\widehat{\mu} \nsubseteq \widehat{\nu}$, as we can see in the following example.

Example 3.10. Let $X = \{a, b, c\}$, where a, b and c are pairwise distinct elements. Suppose that $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\nu = \{\emptyset, \{a\}, \{a, b\}\}$. Put $B = \{a, b\}$. Then $\{a\} = i_\mu(B) \in \widehat{\mu}$. On the other hand, the collection of ν-closed sets is $\mathcal{C}_\nu = \{\{c\}, \{b, c\}, X\}$ and $i_\nu(\mathcal{C}_\nu) = \{i_\nu(B) : B \in \mathcal{C}_\nu\} = \{\emptyset, \{a, b\}\}$, which implies that $\{a\} \notin \widehat{\nu}$.

Theorem 3.11. *A GTS (X, μ) is μ-T_2 if and only if $(X, \widehat{\mu})$ is $\widehat{\mu}$-T_2.*

Proof. If $(X, \widehat{\mu})$ is $\widehat{\mu}$-T_2, then (X, μ) is μ-T_2, because $\widehat{\mu} \subseteq \mu$. Conversely, suppose that (X, μ) is μ-T_2 and let x and y be two distinct points in X. Then, there exist disjoint μ-open sets U and V such that $x \in U$ and $y \in V$, which implies that $x \notin c_\mu(V)$. Thus, $x \in X - c_\mu(V) = i_\mu(X - V)$ and $y \in i_\mu(V) \subseteq i_\mu(c_\mu(V))$. Therefore, $W = i_\mu(X - V)$ and $G = i_\mu(c_\mu(V))$ are $\widehat{\mu}$-open sets such $x \in W$, $y \in G$ and $W \cap G = i_\mu(X - V) \cap i_\mu(c_\mu(V)) \subseteq [X - c_\mu(V)] \cap c_\mu(V) = \emptyset$, i.e. $W \cap G = \emptyset$. This shows that $(X, \widehat{\mu})$ is $\widehat{\mu}$-T_2. □

Corollary 3.12. *A GTS (X, μ) is μ-T_2 if and only if for every pair of distinct points x and y in X, there exist μ-open sets U and V such that $x \notin c_\mu(V)$, $y \notin c_\mu(U)$ and $c_\mu(U) \cup c_\mu(V) = X$.*

Proof. Suppose that (X, μ) is a μ-T_2 space and let x and y be two points in X such that $x \neq y$. By Theorem 3.11, $(X, \widehat{\mu})$ is $\widehat{\mu}$-T_2 and hence there exist μ-closed sets A and B such that $x \in i_\mu(A)$, $y \in i_\mu(B)$ and $i_\mu(A) \cap i_\mu(B) = \emptyset$. Thus, $x \notin c_\mu(X - A)$, $y \notin c_\mu(X - B)$ and $X = c_\mu(X - A) \cup c_\mu(X - B)$. Putting $U = X - B$ and $V = X - A$, we get that U and V are μ-open sets such that $x \notin c_\mu(V)$, $y \notin c_\mu(U)$ and $c_\mu(U) \cup c_\mu(V) = X$. Conversely, let x and y be two distinct points in X. By hypothesis, there exist μ-open sets U and V such that $x \notin c_\mu(V)$, $y \notin c_\mu(U)$ and $c_\mu(U) \cup c_\mu(V) = X$. Then $x \in X - c_\mu(V) = i_\mu(X - V)$, $y \in X - c_\mu(U) = i_\mu(X - U)$. Therefore, $X - c_\mu(V)$ and $X - c_\mu(U)$ are $\widehat{\mu}$-open sets such that $[X - c_\mu(V)] \cap X - c_\mu(U) = \emptyset$. This means that $(X, \widehat{\mu})$ is $\widehat{\mu}$-T_2 and by Theorem 3.11, we get that (X, μ) is μ-T_2. □

Definition 3.13. Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. A function $f : X \to Y$ is called:

1. (μ_1, μ_2)-continuous,[3] if $f^{-1}(U) \in \mu_1$ for each $U \in \mu_2$.
2. (μ_1, μ_2)-open,[15] if $f(U) \in \mu_2$ for each $U \in \mu_1$.
3. (μ_1, μ_2)-closed,[14] if $f(B)$ is a μ_2-closed set in Y for each μ_1-closed subset B of X.

In some cases where more than one GT intervenes on the set X (resp. Y) we will use the terms "$(\cdot, \mu_2)$-continuous function" (resp."$(\mu_1, \cdot)$-continuous function") to refer to a function (μ_1, μ_2)-continuous.

Lemma 3.14. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces and let $f : X \to Y$ be a (μ_1, μ_2)-continuous. Then, the following properties are equivalent:*

1. *f is (μ_1, μ_2)-open.*
2. *$f^{-1}(i_{\mu_2}(B)) = i_{\mu_1}(f^{-1}(B))$ for any $B \subseteq Y$.*
3. *$f^{-1}(c_{\mu_2}(B)) = c_{\mu_1}(f^{-1}(B))$ for any $B \subseteq Y$.*

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Proof. (1) $\Longrightarrow$ (2). Assume that f is (μ_1,μ_2)-open and let $B\subseteq Y$. Then, $f^{-1}(i_{\mu_2}(B))\subseteq f^{-1}(B)$ and, $f^{-1}(i_{\mu_2}(B))\in\mu_1$ because f is (μ_1,μ_2)-continuous. This implies that $f^{-1}(i_{\mu_2}(B))\subseteq i_{\mu_1}(f^{-1}(B))$. To show the opposite inclusion, let $x\in i_{\mu_1}(f^{-1}(B))$. Then, $x\in U\subseteq f^{-1}(B)$ for some μ_1-open subset U of X. Thus, $f(x)\in f(U)\subseteq f(f^{-1}(B))\subseteq B$ and, $f(U)$ is a μ_2-open subset of Y because f is (μ_1,μ_2)-open. Thus, $f(x)\in i_{\mu_2}(B)$ and hence, $x\in f^{-1}(i_{\mu_2}(B))$.

(2) $\Longrightarrow$ (3). Let $B\subseteq Y$. Applying (2), we have

$$\begin{aligned}f^{-1}(c_{\mu_2}(B)) &= f^{-1}(Y-i_{\mu_2}(Y-B))\\ &= f^{-1}(Y)-f^{-1}(i_{\mu_2}(Y-B))\\ &= X-i_{\mu_1}(f^{-1}(Y-B))\\ &= X-i_{\mu_1}(X-f^{-1}(B))\\ &= c_{\mu_1}(f^{-1}(B)).\end{aligned}$$

(3) $\Longrightarrow$ (1). Suppose that U is any μ_1-open subset of X and let $B=Y\setminus f(U)$. Applying (3), we have $c_{\mu_1}(f^{-1}(B))=f^{-1}(c_{\mu_2}(B))$; this is, $c_{\mu_1}(f^{-1}(Y\setminus f(U)))=f^{-1}(c_{\mu_2}(Y\setminus f(U)))$. Then,

$$\begin{aligned}X\setminus i_{\mu_1}(f^{-1}(f(U))) &= c_{\mu_1}(X\setminus f^{-1}(f(U)))\\ &= c_{\mu_1}(f^{-1}(Y\setminus f(U)))\\ &= f^{-1}(c_{\mu_2}(Y\setminus f(U)))\\ &= f^{-1}(Y\setminus i_{\mu_2}(f(U)))\\ &= X\setminus f^{-1}(i_{\mu_2}(f(U))),\end{aligned}$$

which implies that $U=i_{\mu_1}(U)\subseteq i_{\mu_1}(f^{-1}(f(U)))=f^{-1}(i_{\mu_2}(f(U)))$. Thus, $f(U)\subseteq f(f^{-1}(i_{\mu_2}(f(U))))\subseteq i_{\mu_2}(f(U))$. This shows that $f(U)$ is a μ_2-open subset of Y and hence, f is (μ_1,μ_2)-open. □

Corollary 3.15. *Let (X,μ_1) and (Y,μ_2) be two generalized topological spaces. If $f:X\to Y$ is a (μ_1,μ_2)-continuous and (μ_1,μ_2)-open function, then $f^{-1}(i_{\mu_2}(c_{\mu_2}((B)))=i_{\mu_1}(c_{\mu_1}(f^{-1}(B)))$ for any $B\subseteq Y$.*

Theorem 3.16. *Let (X,μ_1) and (X,μ_2) be two generalized topological spaces. If $f:X\to Y$ is a (μ_1,μ_2)-continuous and (μ_1,μ_2)-open function, then f is $(\widehat{\mu}_1,\widehat{\mu}_2)$-continuous.*

Proof. Suppose that $U\in\widehat{\beta}_2$. Then, there exists a μ_2-closed set B such that $U=i_{\mu_2}(B)$. By Lemma 3.14, we get $f^{-1}(U)=f^{-1}(i_{\mu_2}(B))=i_{\mu_1}(f^{-1}(B))$ and $f^{-1}(B)$ is a μ_1-closed set. Thus, $f^{-1}(U)\in\widehat{\mu}_1$ and hence f is $(\widehat{\mu}_1,\widehat{\mu}_2)$-continuous. □

The following example shows that in Theorem 3.16 the condition that f is (μ_1,μ_2)-open cannot be omitted and cannot be replaced by the condition that f is (μ_1,μ_2)-closed.

Example 3.17. Let us consider $X=\{a,b\}$ and $Y=\{a,b,c\}$, where a, b and c are pairwise distinct elements. Suppose that $\mu_1=\{\emptyset,\{a\},X\}$ and $\mu_2=\{\emptyset,\{c\},\{a,c\}\}$ are generalized topologies on X and Y, respectively. Let us define a function $f:X\to Y$ by $f(x)=x$ for each $x\in\{a,b\}$. Obviously, f is (μ_1,μ_2)-continuous and (μ_1,μ_2)-closed, but it is not (μ_1,μ_2)-open. On the other hand, let us observe that $\widehat{\mu}_1=\{\emptyset,X\}$ and $\widehat{\mu}_2=\{\emptyset,\{a,c\}\}$. Since $f^{-1}(\{a,c\})=\{a\}\notin\widehat{\mu}_1$, we conclude that f is not $(\widehat{\mu}_1,\widehat{\mu}_2)$-continuous.

The following example shows that a (μ_1,μ_2)-continuous and (μ_1,μ_2)-open function, in general, is not $(\widehat{\mu}_1,\widehat{\mu}_2)$-open.

Example 3.18. Let us consider $X=\{a,b,c\}$, where a, b and c are pairwise distinct elements. Suppose that $\mu_1=\{\emptyset,\{a\},\{b\},\{a,b\}\}$ and $\mu_2=\{\emptyset,\{a\},\{a,b\}\}$ are two generalized topologies on X. Then, $\widehat{\mu}_1=\mu_1$ and $\widehat{\mu}_2=\{\emptyset,\{a,b\}\}$. Let us define a function $f:X\to X$ by $f(x)=a$ if $x\in\{a,b\}$ and $f(c)=c$. It is clear that f is (μ_1,μ_2)-continuous and (μ_1,μ_2)-open, but it is not $(\widehat{\mu}_1,\widehat{\mu}_2)$-open.

Let X be a set, $\{(X_\lambda,\mu_\lambda):\lambda\in\Lambda\}$ be a collection of generalized topological spaces and $\{f_\lambda:\lambda\in\Lambda\}$ be a collection of functions such that $f_\lambda:X\to X_\lambda$ for $\lambda\in\Lambda$. It is known from,[21] that the GT

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$$\mu = \left\{ \bigcup \mathcal{S} : \mathcal{S} \subseteq \{f_\lambda^{-1}(V_\lambda) : V_\lambda \in \mu_\lambda, \lambda \in \Lambda\} \right\}$$

is the coarsest GT for which all functions $f_\lambda : X \to X_\lambda$ are $(\cdot, \mu_\lambda)$-continuous. This GT is called the GT on X *induced by the collection* $\{f_\lambda : \lambda \in \Lambda\}$. As a particular case of the above, we obtain the GT that we describe below.

Let $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a collection of generalized topological spaces and $X = \prod_{\lambda \in \Lambda} X_\lambda$. Consider the collection of functions $\{p_\lambda : \lambda \in \Lambda\}$, where $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, is the λ-th projection function. According to,[21] the set $X = \prod_{\lambda \in \Lambda} X_\lambda$ with the GT $\mu = \prod_{\lambda \in \Lambda} \mu_\lambda$ induced by collection $\{p_\lambda : \lambda \in \Lambda\}$ is called the generalized product topology space (briefly GPTS) and μ is called the *generalized product topology* on X. Note that if $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ is a collection of strong generalized topological spaces, then each projection $p_\lambda : X \to X_\lambda$ is (μ, μ_λ)-open. Indeed, for a fixed $\lambda_0 \in \Lambda$, to prove that p_{λ_0} is (μ, μ_{λ_0})-open, consider any non-empty μ-open set W in X and a point $u \in p_{\lambda_0}(W)$. There exists $w \in W$ such that $p_{\lambda_0}(w) = u$. There exist $\lambda_1 \in \Lambda$ and $V_{\lambda_1} \in \mu_{\lambda_1}$, such that $w \in p_{\lambda_1}^{-1}(V_{\lambda_1}) \subseteq W$. Then $u \in p_{\lambda_0}\left(p_{\lambda_1}^{-1}(V_{\lambda_1})\right) \subseteq p_{\lambda_0}(W)$. Since

$$p_{\lambda_0}\left(p_{\lambda_1}^{-1}(V_{\lambda_1})\right) = \begin{cases} V_{\lambda_0} & \text{if } \lambda_1 = \lambda_0, \\ X_{\lambda_0} & \text{if } \lambda_1 \neq \lambda_0, \end{cases}$$

we conclude that $p_{\lambda_0}(W) \in \mu_{\lambda_0}$, so p_{λ_0} is (μ, μ_{λ_0})-open.

Theorem 3.19. *Let $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a collection of strong generalized topological spaces and (X, μ) be the GPTS. If $(X, \tilde{\mu})$ is the GPTS of the collection $\{(X_\lambda, \widehat{\mu}_\lambda) : \lambda \in \Lambda\}$, then $\tilde{\mu} = \widehat{\mu}$; this is, $\prod_{\lambda \in \Lambda} \widehat{\mu}_\lambda = \widehat{\prod_{\lambda \in \Lambda} \mu_\lambda}$.*

Proof. We will first show that $\tilde{\mu} \subseteq \widehat{\mu}$. For any $\lambda \in \Lambda$, we have $p_\lambda : X \to X_\lambda$ is a (μ, μ_λ)-continuous and (μ, μ_λ)-open function and so, by Theorem 3.16, $p_\lambda : X \to X_\lambda$ is $(\widehat{\mu}, \widehat{\mu}_\lambda)$-continuous. Since $\tilde{\mu}$ is the coarsest GT on X for which $p_\lambda : X \to X_\lambda$ is $(\cdot, \widehat{\mu}_\lambda)$-continuous for each $\lambda \in \Lambda$, it follows that $\tilde{\mu} \subseteq \widehat{\mu}$.

Now, we will show that $\widehat{\mu} \subseteq \tilde{\mu}$. Let $U \in \widehat{\mu}$ and $x \in \widehat{U}$. Then $x \in i_\mu(B) \subseteq U$ for some μ-closed set B in X. By definition of $\mu = \prod_{\lambda \in \Lambda} \mu_\lambda$, there exists $\lambda(x) \in \Lambda$ such that $x \in p_{\lambda(x)}^{-1}(V_{\lambda(x)}) \subseteq i_\mu(B) \subseteq U$. Applying Lemma 3.14, we have

$$x \in p_{\lambda(x)}^{-1}(i_\mu(c_\mu(V_{\lambda(x)}))) = i_\mu(c_\mu(p_{\lambda(x)}^{-1}(V_{\lambda(x)}))) \subseteq i_\mu(B) \subseteq U.$$

Since $i_\mu(c_\mu(V_{\lambda(x)})) \in \widehat{\mu}_{\lambda(x)}$, we get that $p_{\lambda(x)}^{-1}(i_\mu(c_\mu(V_{\lambda(x)}))) \in \tilde{\mu}$. This shows that $U \in \tilde{\mu}$. □

Let $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a collection of pairwise disjoint generalized topological spaces, i.e. that $X_\lambda \cap X_{\lambda'} = \emptyset$ for $\lambda \neq \lambda'$. Let us consider the union of all sets X_λ, i.e. $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ and the collection of inclusion functions $\{\varphi_\lambda : \lambda \in \Lambda\}$, where $\varphi_\lambda : X_\lambda \hookrightarrow X$ for each $\lambda \in \Lambda$. If we define a collection μ of subsets of X as

$$\mu = \{U \subseteq X : \varphi_\lambda^{-1}(U) \in \mu_\lambda, \forall \lambda \in \Lambda\},$$

then μ is a GT on X, which we will denote by $\bigoplus_{\lambda \in \Lambda} \mu_\lambda$. The GTS (X, μ) is called the *generalized topological sum* (briefly GT-sum) of $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ and is denoted by $X = \bigoplus_{\lambda \in \Lambda} X_\lambda$, see.[17–19] Observe that $\mu = \bigoplus_{\lambda \in \Lambda} \mu_\lambda$ is the finest GT for which all inclusion functions $\varphi_\lambda : X_\lambda \hookrightarrow X$ are $(\mu_\lambda, \cdot)$-continuous. Also, each inclusion function $\varphi_\lambda : X_\lambda \hookrightarrow X$ is (μ_λ, μ)-open. Indeed, if W is any μ_λ-open set in X_λ, then $\varphi_\lambda^{-1}(W) = W \cap X_\lambda = W$ and $\varphi_{\lambda'}^{-1}(W) = W \cap X_{\lambda'} = \emptyset$ for each $\lambda' \neq \lambda$ (by the pairwise disjointness); hence, W is also a μ-open set in X and as $\varphi_\lambda(W) = W$, it follows that φ_λ is (μ_λ, μ)-open.

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Theorem 3.20. *Let $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a collection of pairwise disjoint generalized topological spaces and (X, μ) be the GT-sum. If $(X, \bar{\mu})$ is the GT-sum of the collection $\{(X_\lambda, \widehat{\mu}_\lambda) : \lambda \in \Lambda\}$, then $\widehat{\mu} \subseteq \bar{\mu} \subseteq \mu$; i.e. $\widehat{\bigoplus_{\lambda\in\Lambda} \mu_\lambda} \subseteq \bigoplus_{\lambda\in\Lambda} \widehat{\mu}_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} \mu_\lambda$.*

Proof. First, let us note that

$$U \in \bar{\mu} \Leftrightarrow \varphi_\lambda^{-1}(U) \in \widehat{\mu}_\lambda, \forall \lambda \in \Lambda \Rightarrow \varphi_\lambda^{-1}(U) \in \mu_\lambda, \forall \lambda \in \Lambda \Leftrightarrow U \in \mu,$$

which tells us that $\bar{\mu} \subseteq \mu$.

To finish, we will show that $\widehat{\mu} \subseteq \bar{\mu}$. For any $\lambda \in \Lambda$, we have $\varphi_\lambda : X_\lambda \hookrightarrow X$ is a (μ_λ, μ)-continuous and (μ_λ, μ)-open function and hence, by Theorem 3.16, $\varphi_\lambda : X_\lambda \hookrightarrow X$ is $(\widehat{\mu}_\lambda, \widehat{\mu})$-continuous. As $\bar{\mu}$ is the finest GT on X for which $\varphi_\lambda : X_\lambda \hookrightarrow X$ is $(\widehat{\mu}_\lambda, \cdot)$-continuous for each $\lambda \in \Lambda$, it follows that $\widehat{\mu} \subseteq \bar{\mu}$. ☐

Theorem 3.21. Let $\{(X_\lambda, \mu_\lambda) : \lambda \in \Lambda\}$ be a collection of pairwise disjoint generalized topological spaces and (X, μ) be the GT-sum. If (X_λ, μ_λ) is μ_λ-regular for each $\lambda \in \Lambda$, then (X, μ) is μ-regular.

Proof. Let A be a μ-closed subset of the sum $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ and let $x \in X$ such that $x \notin A$. Then $A \cap X_\lambda$ is a μ_λ-closed subset of X_λ for each $\lambda \in \Lambda$ and there exists a unique $\lambda_0 \in \Lambda$ such that $x \in X_{\lambda_0}$. By the μ_{λ_0}-regularity of $(X_{\lambda_0}, \mu_{\lambda_0})$, there exist disjoint μ_{λ_0}-open subsets U_{λ_0} and V_{λ_0} of X_{λ_0} such that $x \in U_{\lambda_0}$ and $A \cap X_{\lambda_0} \subseteq V_{\lambda_0}$. Observe that

$$U_{\lambda_0} \cap X_\lambda = \begin{cases} U_{\lambda_0}, & \text{if } \lambda = \lambda_0, \\ \emptyset, & \text{if } \lambda \neq \lambda_0, \end{cases}$$

which tells us that $U_{\lambda_0} \cap X_\lambda$ is a μ_λ-open subset of X_λ for each $\lambda \in \Lambda$ and hence, U_{λ_0} is a μ-open subset of $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ containing x. Moreover, $V_{\lambda_0} \cup \left(\bigcup_{\lambda\neq\lambda_0} X_\lambda \right)$ is a μ-open subset of $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$ containing A and $U_{\lambda_0} \cap \left[V_{\lambda_0} \cup \left(\bigcup_{\lambda\neq\lambda_0} X_\lambda \right) \right] = (U_{\lambda_0} \cap V_{\lambda_0}) \cup \left[U_{\lambda_0} \cap \left(\bigcup_{\lambda\neq\lambda_0} X_\lambda \right) \right] = \emptyset$. This shows that (X, μ) is μ-regular. ☐

Corollary 3.22. *If in Theorem 3.20 we have (X_λ, μ_λ) is μ_λ-regular for each $\lambda \in \Lambda$, then $\widehat{\bigoplus_{\lambda\in\Lambda} \mu_\lambda} = \bigoplus_{\lambda\in\Lambda} \widehat{\mu}_\lambda = \bigoplus_{\lambda\in\Lambda} \mu_\lambda$.*

Proof. From Theorem 3.20 we have $\widehat{\bigoplus_{\lambda\in\Lambda} \mu_\lambda} \subseteq \bigoplus_{\lambda\in\Lambda} \widehat{\mu}_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} \mu_\lambda$. Now, if each (X_λ, μ_λ) is μ_λ-regular, then by Theorem 3.21 it follows that (X, μ) is μ-regular and by Corollary 3.2, $\widehat{\bigoplus_{\lambda\in\Lambda} \mu_\lambda} = \bigoplus_{\lambda\in\Lambda} \mu_\lambda$. Therefore, $\widehat{\bigoplus_{\lambda\in\Lambda} \mu_\lambda} = \bigoplus_{\lambda\in\Lambda} \widehat{\mu}_\lambda = \bigoplus_{\lambda\in\Lambda} \mu_\lambda$. ☐

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