# On Refined Netrusophic Fractional Calculus

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**Abstract**

Depending on the geometric isometry (AH-Isometry), it has been proven that every Neutrosophic real function is equivalent to three real functions. Then, the foundation of the Refined Netrusophic calculus was established, where new definitions of Refined Netrusophic integration and Refined Netrusophic differentiation were introduced, along with some illustrative examples. Following that, definitions for the Refined Netrusophic gamma function and Refined Netrusophic beta function were presented to pave the way towards achieving the desired goal, which is Refined Netrusophic Fractional calculus.

**Keywords:** Refined Neutrosophic real function; Refined Neutrosophic Integration; Refined Neutrosophic Derivative; Refined Netrusophic Fractional integral and Refined Netrusophic Fractional Derivative.

## 1.Introduction

Fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order $\boldsymbol{n}$ could be extended to still be valid when $\boldsymbol{n}$ is not an integer. This question was first raised by L'Hopital on September 30th, 1695. On that day, in a letter to Leibniz, he posed a question about $\frac{d^n x}{dx^n}$ , Leibniz's notation for the derivative of the linear function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$. L'Hopital curiously asked what the result would be if $\boldsymbol{n} = \frac{1}{2}$ Leibniz responded that it would be "an apparent paradox, from which one day useful consequences will be drawn," [16]

Neutrosophy is a new branch of philosophy concerns with the indeterminacy in all areas of life and science. It has become a useful tool in generalizing many classical systems such as equations [1,9], number theory [2,3], topology [4,5], linear spaces [6,10], modules [4,5], and ring of matrices [7,8].

In the literature, we find many studies about neutrosophic calculus, where some definitions and properties were presented about neutrosophic real functions and numbers [10].

The neutrosophic real functions with one variable were defined only in a special case [11], as follows:

Recently, Abobala et.al, have presented the concept of two-dimensional AH-isometry to study the correspondence between neutrosophic plane $\boldsymbol{R}(\boldsymbol{I}) \times \boldsymbol{R}(\boldsymbol{I})$ and the classical module $\boldsymbol{R}^2 \times \boldsymbol{R}^2$. Also, the one-dimensional AH-isometry between R(I) and $\boldsymbol{R} \times \boldsymbol{R}$. This isometry was useful in defining inner products and norms [10], ordering [9], and neutrosophic geometrical shapes [10].

In this work, we use the one-dimensional AH-isometry to turn the general case of refined neutrosophic real functions with one variable into three classical real functions so we will go from $\boldsymbol{R}(\boldsymbol{I_1}, \boldsymbol{I_2})$ space into $\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$ space. The



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definitions of Refined Netrusophic integration and Refined Netrusophic differentiation were introduced. Following that, definitions for the Refined Netrusophic gamma function and Refined Netrusophic beta function were presented to pave the way towards Refined Netrusophic Fractional calculus.

## 2. Terminologies

We present here some basic definitions and axioms of neutrosophic logic and refined neutrosophic logic.

**Definition 2.1.** [19]: Let $X$ be a non-empty fixed set. A neutrosophic set $A$ is an object having the form $\{x,(\mu A(x),\delta A(x),\gamma A(x)): x\in X\}$, where $\mu A(x)$, $\delta A(x)$ $and$ $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$ to the set $A$ .

**Definition 2.2.** [20]: Let $K$ be a field, the neutrosophic file generated by $\langle K\cup I\rangle$ which is denoted by $K(I)=\langle K\cup I\rangle$.

**Definition 2.3.** [21]: Classical neutrosophic number has the form $a+bI$ where a,b are real or complex numbers and $I$ is the indeterminacy such that $0\cdot I=0$ and $I^2=I$ which results that $I^n=I$ for all positive integers $n$.

**Definition 2.4.** [14] Let $R(I)=\{a+bI\,; a,b\in R\}$ where $I^2=I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [49]

$$T: R(I)\to R\times R\,;\; T(a+bI)=(a,a+b)$$

**Remark 2.5.** [14]

$T$ is an algebraic isomorphism between two rings, it has the following properties:

1) $T$ is bijective.
2) $T$ preserves addition and multiplication, i.e.:
3) Since $T$ is bijective, then it is invertible by:

$$T^{-1}: R\times R\to R(I)\,;\; T^{-1}(a,b)=a+(b-a)I$$

4) $T$ preserves distances, i.e.:

$$\|T(AB)\|=T(\|AB\|)$$

**Definition 2.6.** [15]

Let $f: R(I)\to R(I); f=f(X)$ and $X=x+yI\in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable.

a neutrosophic real function $f(X)$ written as follows:

$$f(X)=f(x+yI)=f(x)+I[f(x+y)-f(x)]$$

**Definition 2.7.** [17]: Let $(X(I_1,I_2),+,\cdot)$ be any refined neutrosophic algebraic structure where $+$ and $\cdot$ are ordinary addition and multiplication respectively. $(I_1$ $and$ $I_2)$ are the split components of the indeterminacy factor $I$ that is $I=\alpha I_1+\beta I_2$ with $\alpha,\beta\in R$ $or$ $C$. Also $I_1 and$ $I_2$ are taken to properties ${I_1}^2=I_1$ , ${I_2}^2=I_2$ $and$ $I_1I_2=I_2I_1=I_1$ . The refined neutrosophic real number has the form $a+bI_1+cI_2$.

For any two elements, we define

1. $X(I_1,I_2)+Y(I_1,I_2)=a+bI_1+cI_2+d+eI_1+fI_2=a+d+(b+e)I_1+(c+f)I_2$
2. $X(I_1,I_2)\cdot Y(I_1,I_2)=(a+bI_1+cI_2)\cdot(d+eI_1+fI_2)=a\cdot d+(ae+bd+be+bf+ce)I_1+(af+cd+cf)I_2$.
3. $\frac{X(I_1,I_2)}{Y(I_1,I_2)}=\frac{a+bI_1+cI_2}{d+eI_1+fI_2}$
4. $a+bI_1+cI_2\le d+eI_1+fI_2$ if and only if $a\le d, a+b+c\le d+e+f, a+c\le d+f$

**Definition 2.8.** [18] Let $R(I_1,I_2)=\{a+bI_1+cI_2\,; a,b,C\in R\}$ where ${I_1}^2=I_1$ , ${I_2}^2=I_2$ $and$ $I_1I_2=I_2I_1=I_1$ be the refined neutrosophic field of reals. The refined one-dimensional isometry (AH-Isometry) is defined as follows: [49]

$$T: R(I_1,I_2)\to R\times R\times R\,;\; T(a+bI_1+cI_2)=(a,a+b+c,a+c)$$

**Remark 2.9.** [18]

$T$ is an algebraic isomorphism between two rings, it has the following properties:

5) $T$ is bijective.
6) $T$ preserves addition and multiplication.
7) Since $T$ is bijective, then it is invertible by:

$$T^{-1}: R\times R\times R\to R(I_1,I_2)\,;\; T^{-1}(a,b,c)=a+(b-c)I_1+(c-a)I_2$$

8) $T$ preserves distances, i.e.:

$$\|T(AB)\|=T(\|AB\|)$$



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## 3. Refined Neutrosophic Calculus

### 3.1 Refined Neutrosophic Real Function

**Definition 3.1.1.** [15] Let $f: R(I_1, I_2) \to R(I_1, I_2); f = f(X(I_1, I_2))$ and $X = x + yI_1 + zI_2 \in R(I_1, I_2)$ the $f$ is called a refined neutrosophic real function with one refined neutrosophic variable, written as follows:

$$f(X(I_1, I_2)) = f(x + yI_1 + zI_2) = f(x) + (f(x + y + z) - f(x + z))I_1 + (f(x + z) - f(x))I_2$$

**Theorem 3.1.2.** any refined neutrosophic real function into three classical real functions, i.e., to the classical Euclidean plane $R \times R \times R$.

**Proof.**

Let $f(X(I_1, I_2)) = f(x + yI_1 + zI_2) = f(x) + (f(x + y + z) - f(x + z))I_1 + (f(x + z) - f(x))I_2$ a refined neutrosophic real function.

Now, Using the one-dimensional AH-isometry, we have.

$T\left(f(X(I_1, I_2))\right) = T(f(x) + (f(x + y + z) - f(x + z))I_1 + (f(x + z) - f(x))I_2)$, then.

$(f_1, f_2, f_3) = (f(x), f(x + y + z) - f(x + z), f(x + z) - f(x))$, then, we have.

$$\begin{cases} f_1 = f(x) \\ f_2 = f(x + y + z) - f(x + z) \\ f_3 = f(x + z) - f(x) \end{cases}$$

the functions $f(x), f(x + y + z) - f(x + z), f(x + z) - f(x)$ are three real functions.

**Exemple 3.1.3.** Let $R(I_1, I_2)$ be the refined neutrosophic field of reals, we have:

1. $f(X(I_1, I_2)) = e^{x+yI_1+zI_2} = e^x + (e^{x+y+z} - e^{x+z})I_1 + (e^{x+z} - e^x)I_2$
2. $f(X(I_1, I_2)) = \ln(x + yI_1 + zI_2) = ln(x) + (ln(x + y + z) - ln(x + z))I_1 + (ln(x + z) - ln(x))I_2$, where $x + yI_1 + zI_2 > 0 + 0I_1 + 0I_2$.
3. $f(X(I_1, I_2)) = \sqrt{x + yI_1 + zI_2} = \sqrt{x} + (\sqrt{x + y + z} - \sqrt{x + z})I_1 + (\sqrt{x + z} - \sqrt{x})I_2$, where $x + yI_1 + zI_2 \geq 0 + 0I_1 + 0I_2$.
4. $f(X(I_1, I_2)) = sin(x + yI_1 + zI_2) = sin(x) + (sin(x + y + z) - sin(x + z))I_1 + (sin(x + z) - sin(x))I_2$

### 3.2 The Derivative of Refined Neutrosophic Functions on $R(I_1, I_2)$.

**Definition 3.2.1.** Let $f(X(I_1, I_2))$ a refined neutrosophic function on $R(I_1, I_2)$, the we define a derivative of a refined neutrosophic function $f(X(I_1, I_2))$ as follows**:**

$$f'_{X(I_1,I_2)}(X(I_1, I_2)) = f'_x(x) + \left(f'_{x+y}(x + y + z) - f'_{x+z}(x + z)\right)I_1 + (f'_{x+z}(x + z) - f'_x(x))I_2$$

**Examples 3.2.2.**

1. $e^{X(I_1,I_2)} = e^{x+yI_1+zI_2} = e^x + (e^{x+y+z} - e^{x+z})I_1 + (e^{x+z} - e^x)I_2$ we have:
   $f'_{X(I_1,I_2)}(X(I_1, I_2)) = e^x + (e^{x+y+z} - e^{x+z})I_1 + (e^{x+z} - e^x)I_2 = e^{x+yI_1+zI_2}$.
2. $f(X(I_1, I_2)) = ln(x + yI_1 + zI_2) = ln(x) + (ln(x + y + z) - ln(x + z))I_1 + (ln(x + z) - ln(x))I_2$, where $x + yI_1 + zI_2 > 0 + 0I_1 + 0I_2$ .
   We have.

$$f'_{X(I_1,I_2)}(X(I_1, I_2)) = \frac{1}{x} + \left(\frac{1}{x + y + z} - \frac{1}{x + z}\right)I_1 + \left(\frac{1}{x + z} - \frac{1}{x}\right)I_2 = \frac{1}{x + yI_1 + zI_2} = \frac{1}{X(I_1, I_2)}$$

   Notice that: $\left(\frac{1}{x} + \left(\frac{1}{x+y+z} - \frac{1}{x+z}\right)I_1 + \left(\frac{1}{x+z} - \frac{1}{x}\right)I_2\right)(x + yI_1 + zI_2) = 1 + 0I_1 + 0I_2$
3. $f(X(I_1, I_2)) = sin(x + yI_1 + zI_2) = sin(x) + (sin(x + y + z) - sin(x + z))I_1 + (sin(x + z) - sin(x))I_2$.



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We have.

$$f'_{X(I_1,I_2)}\big(X(I_1,I_2)\big) = cos(x) + \big(cos(x+y+z) - cos(x+z)\big)I_1 + \big(cos(x+z) - cos(x)\big)I_2 = cosX(I_1,I_2).$$

### 3.3 The Integral of Refined Neutrosophic Functions on $R(I_1, I_2)$.

**Definition 3.3.1.** Let $f\big(X(I_1,I_2)\big)$ a refined neutrosophic function on $R(I_1,I_2)$, the we define a integration of a refined neutrosophic function $f\big(X(I_1,I_2)\big)$as follows:

$$\int f\big(X(I_1,I_2)\big)\,dX(I_1,I_2) = F\big(X(I_1,I_2)\big) = F(x) + [F(x+y+z) - F(x+z)]I_1 + [F(x+z) - F(x)]I_2.$$

**Examples 3.3.2.**

1. $f\big(X(I_1,I_2)\big) = e^{X(I_1,I_2)}$. We have.

$$\int e^{x+yI_1+zI_2}\,dX = \int e^x\,dx + \left[\int e^{x+y+z}\,d(x+y+z) - \int e^{x+z}\,d(x+z)\right]I_1 + \left[\int e^{x+z}\,d(x+z) - \int e^x\,dx\right]I_1$$

$= e^x + (e^{x+y+z} - e^{x+z})I_1 + (e^{x+z} - e^x)I_2 + a + bI_1 + cI_2 = e^{x+yI_1+zI_2} + a + bI_1 + cI_2$ where $a, b, c$ are conests.

2. $f\big(X(I_1,I_2)\big) = cos(x+yI_1+zI_2)$ we have

$$\int cos(x+yI_1+zI_2)\,dX(I_1,I_2)$$
$$= \int cos(x)\,dx + \left[\int cos(x+y+z)\,d(x+y+z) - \int cos(x+z)d(x+z)\right]I_1 + \left[\int cos(x+z)\,d(x+z) - \int cos(x)\,dx\right]I_1$$
$$= sin(x) + \big(sin(x+y+z) - sin(x+z)\big)I_1 + \big(sin(x+z) - sin(x)\big)I_2 + a + bI_1 + cI_2 = sin(x+yI_1+zI_2) + a + bI_1 + cI_2$$

where $a, b, c$ are conests.

### 3.4 The Definite Integration of Refined Neutrosophic Functions on $R(I_1, I_2)$.

**Definition 2.4.1.**

Let $f\big(X(I_1,I_2)\big)$ a refined neutrosophic function on $R(I_1,I_2)$, we define the definite integration of a refined neutrosophic function $f\big(X(I_1,I_2)\big)$ as follows:

$$\int_{a+bI_1+cI_2}^{d+eI_1+fI_2} f\big(X(I_1,I_2)\big)dX(I_1,I_2) = \int_a^d f(x)dx + \left[\int_{a+b+c}^{d+e+f} f(x+y+z)d(x+y+z) - \int_{a+c}^{d+f} f(x+z)d(x+z)\right]I_1 + \left[\int_{a+c}^{d+f} f(x+z)d(x+z) - \int_a^d f(x)dx\right]I_2$$

**Example 3.4.2.**

1. $J = \int_{0+0I_1+0I_2}^{1+I_1+I_2} e^{X(I_1,I_2)}dX(I_1,I_2) = \int_0^1 e^x dx + \left[\int_0^3 e^{(x+y+z)}d\big((x+y+z)\big) - \int_1^2 e^{(x+z)}d(x+z)\right]I_1 + \left[\int_1^2 e^{(x+z)}d\big((x+y+z)\big) - \int_0^1 e^x dx\right]I_2 = [e^x]_0^1 + \big[[e^{(x+y+z)}]_0^3 - [e^{(x+z)}]_1^2\big]I_1 + \big[[e^{(x+z)}]_1^2 - [e^x]_0^1\big]I_2$

$J = (e-1) + \big((e^3-1) - (e^2-e)\big)I_1 + \big((e^2-e) - (e-1)\big)I_2.$



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**4. Refined Neutrosophic Fractional Calculus**

**4.1 Useful Refined Neutrosophic Functions**

**Definition 4.1.1.** The most basic interpretation of the refined neutrosophic Gamma function is simply the generalization of the refined neutrosophic factorial for all refined neutrosophic real numbers.

We define a refined neutrosophic Gamma function as follows:

$$\Gamma\big(Z(I_1,I_2)\big) = \Gamma(z_1 + z_2 I_1 + z_3 I_2) = \Gamma(z_1) + \big(\Gamma(z_1+z_2+z_3) - \Gamma(z_1+z_3)\big)I_1 + \big(\Gamma(z_1+z_3) - \Gamma(z_1)\big)I_2$$
$$= \int_0^\infty (t_1+t_2I_1+t_3I_2)^{(z_1+z_2I_1+z_3I_2)-1}\, e^{-(t_1+t_2I_1+t_3I_2)}\, d(t_1+t_2I_1+t_3I_2)$$

**Remark 4.1.2.**

1. $\Gamma(1+0I_1+0I_2) = \Gamma(1) + \big(\Gamma(1)-\Gamma(1)\big)I_1 + \big(\Gamma(1)-\Gamma(1)\big)I_2 = 1$

And we can proof in this way,

$$\Gamma(1) = \Gamma(1+0I_1+0I_2) = \int_0^\infty (t_1+t_2I_1+t_3I_2)^{1-1}\, e^{-(t_1+t_2I_1+t_3I_2)}\, d(t_1+t_2I_1+t_3I_2)$$
$$= \int_0^\infty e^{-(t_1+t_2I_1+t_3I_2)}\, d(t_1+t_2I_1+t_3I_2)$$
$$= \int_0^\infty e^{-t_1} d(t_1) + \left[\int_0^\infty e^{-(t_1+t_2+t_3)} d(t_1+t_2+t_3) - \int_0^\infty e^{-(t_1+t_3)} d\big((t_1+t_3)\big)\right] I_1$$
$$+ \left[\int_0^\infty e^{-(t_1+t_3)} d(t_1+t_3) - \int_0^\infty e^{-t_1} d(t_1)\right] I_2$$

$$\Gamma(1) = [-e^{-t_1}]_0^\infty + \left(\left[-e^{-(t_1+t_2+t_3)}\right]_0^\infty - \left[-e^{-(t_1+t_3)}\right]_0^\infty\right) I_1 + \left(\left[-e^{-(t_1+t_3)}\right]_0^\infty - [-e^{-t_1}]_0^\infty\right) I_2$$

$$\Gamma(1) = 1 + 0I_1 + 0I_2 = 1$$

In the same way we can proof,

1. $\Gamma(Z(I_1,I_2)+1) = \big(Z(I_1,I_2)\big)\Gamma\big(Z(I_1,I_2)\big)$
2. $\Gamma(n+mI_1+pI_2) = (n+mI_1+pI_2)!\,; n+mI_1+pI_2 \in N(I_1,I_2)$

**Definition 4.1.3.**

Like the refined neutrosophic Gamma function, the refined neutrosophic Beta function is defined by a definite refined neutrosophic integral. Its definition is given by

$$B\big(P(I_1,I_2), Q(I_1,I_2)\big) = B(p_1+p_2I_1+p_3I_2, q_1+q_2I_1+q_3I_2)$$
$$= \int_0^1 (t_1+t_2I_1+t_3I_2)^{(p_1+p_2I_1+p_3I_2)-1} \big(1-(t_1+t_2I_1+t_3I_2)\big)^{(q_1+q_2I_1+q_3I_2)-1}\, d(t_1+t_2I_1+t_3I_2)$$

The Beta function can also be defined in terms of the Gamma function:



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$$B\big(P(I_1,I_2),Q(I_1,I_2)\big)=\frac{\Gamma\big(P(I_1,I_2)\big).\Gamma\big(Q(I_1,I_2)\big)}{\Gamma\Big(\big(P(I_1,I_2)\big)+\big(Q(I_1,I_2)\big)\Big)}$$

**4.2 Definition of the Riemann-Liouville Refined Neutrosophic Fractional Integral**

We can start by introducing a succinct notation that will be frequently used. From now on, $\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}f\big(X(I_1,I_2)\big)}$ will denote the refined neutrosophic fractional integration of a refined neutrosophic function $\boldsymbol{f\big(X(I_1,I_2)\big)}$ to an arbitrary order $\boldsymbol{v(I_1,I_2)}$ and $\boldsymbol{v(I_1,I_2)}$ is a refined neutrosophic positive real number .

**Definition 4.2.1.** Let $\boldsymbol{v(I_1,I_2)}$ be a refined neutrosophic positive real number. Let $\boldsymbol{f\big(X(I_1,I_2)\big)}$ be integrable and piecewise continuous on $(\boldsymbol{0},\infty)$ . Then for $(\boldsymbol{I_1},\boldsymbol{I_2})>\boldsymbol{0}+\boldsymbol{0I_1}+\boldsymbol{0I_2}$ , the Riemann-Liouville refined neutrosophic fractional integral of $\boldsymbol{f\big(X(I_1,I_2)\big)}$ of order $\boldsymbol{v(I_1,I_2)}$ is:

$$\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}f\big(X(I_1,I_2)\big)=\frac{1}{\Gamma\big(v(I_1,I_2)\big)}\int_0^{X(I_1,I_2)}\big(X(I_1,I_2)-t(I_1,I_2)\big)^{v(I_1,I_2)-1}f\big(t(I_1,I_2)\big)\,d\big(t(I_1,I_2)\big)}$$

**Remark 4.2.3. :** Let $\big(X(I_1,I_2)\big)$ , $\boldsymbol{X=x+yI_1+zI_2\in R(I_1,I_2)}$ be a refined neutrosophic real function with one refined neutrosophic variable, then as a result of the **Theorem 3.1.2.** we can write:

$$\begin{aligned}\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}f\big(X(I_1,I_2)\big)}&=\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}\big(f(x)+[f(x+y+z)-f(x+z)]I_1+\big[\big(f(x+z)\big)-f(x)\big]I_2\big)}\\&=\boldsymbol{D_x^{-v(I_1,I_2)}f(x)+\Big[D_{x+y+z}^{-v(I_1,I_2)}f(x+y+z)-D_{x+z}^{-v(I_1,I_2)}f(x+z)\Big]I_1}\\&\quad+\boldsymbol{\Big[\Big(D_{x+z}^{-v(I_1,I_2)}f(x+z)\Big)-D_x^{-v(I_1,I_2)}f(x)\Big]I_2}\end{aligned}$$

**Example 4.2.4. :** Let's evaluate $\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}X^{w(I_1,I_2)}(I_1,I_2)}$ by definition,

$$\begin{aligned}&\boldsymbol{D_{X(I_1,I_2)}^{-v(I_1,I_2)}X^{w(I_1,I_2)}(I_1,I_2)}\\&=\boldsymbol{\frac{1}{\Gamma\big(v(I_1,I_2)\big)}\int_0^{X(I_1,I_2)}\big(X(I_1,I_2)-t(I_1,I_2)\big)^{v(I_1,I_2)-1}\big(t(I_1,I_2)\big)^{w(I_1,I_2)}d\big(t(I_1,I_2)\big)}\\&=\boldsymbol{\frac{1}{\Gamma\big(v(I_1,I_2)\big)}\int_0^{X(I_1,I_2)}\left(1-\frac{t(I_1,I_2)}{X(I_1,I_2)}\right)^{v(I_1,I_2)-1}X(I_1,I_2)^{v(I_1,I_2)-1}\big(t(I_1,I_2)\big)^{w(I_1,I_2)}d\big(t(I_1,I_2)\big)}\\&=\boldsymbol{\frac{1}{\Gamma\big(v(I_1,I_2)\big)}\int_0^1\big(1}\\&\boldsymbol{-u(I_1,I_2)\big)^{v(I_1,I_2)-1}X(I_1,I_2)^{v(I_1,I_2)-1}\big(X(I_1,I_2)u(I_1,I_2)\big)^{w(I_1,I_2)}d\big(u(I_1,I_2)\big)}\end{aligned}$$

Where $\left(\boldsymbol{u(I_1,I_2)=\frac{t(I_1,I_2)}{X(I_1,I_2)}}\right)$

$$=\boldsymbol{\frac{1}{\Gamma\big(v(I_1,I_2)\big)}X(I_1,I_2)^{v(I_1,I_2)+u(I_1,I_2)}\int_0^1\big(1-u(I_1,I_2)\big)^{v(I_1,I_2)-1}\big(u(I_1,I_2)\big)^{w(I_1,I_2)}d\big(u(I_1,I_2)\big)}$$



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$$= \frac{1}{\Gamma\big(v(I_1, I_2)\big)} X(I_1, I_2)^{w(I_1,I_2)+v(I_1,I_2)} B\big(w(I_1, I_2) + 1, v(I_1, I_2)\big)$$

$$= \frac{\Gamma(w(I_1, I_2) + 1)}{\Gamma(w(I_1, I_2) + v(I_1, I_2) + 1)} X(I_1, I_2)^{w(I_1,I_2)+v(I_1,I_2)}$$

We refer to the above example as the Refined Neutrosophic Power Rule. The Refined Neutrosophic Power Rule tells us that the Refined Neutrosophic fractional integral of a constant of order $v(I_1, I_2)$ is

$$D_{X(I_1,I_2)}^{-v(I_1,I_2)} X^{w(I_1,I_2)}(I_1, I_2) = \frac{\Gamma(w(I_1, I_2) + 1)}{\Gamma(w(I_1, I_2) + v(I_1, I_2) + 1)} X(I_1, I_2)^{w(I_1,I_2)+v(I_1,I_2)}$$

And in particular, if $v(I_1, I_2) = \frac{1}{2} + 0I_1 + 0I_2$,

$$D_{X(I_1,I_2)}^{-\frac{1}{2}+0I_1+0I_2,} [x + xI_1 + x^2 I_2]^0 = \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} [x + xI_1 + x^2 I_2]^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \sqrt{x + xI_1 + x^2 I_2}$$

$$= \frac{2}{\sqrt{\pi}} \left(\sqrt{x} + \left[\sqrt{x + x + x^2} - \sqrt{1 + x^2}\right] I_1 + \left[\sqrt{x + x^2} - \sqrt{x}\right] I_2\right)$$

$$D_{X(I_1,I_2)}^{-\frac{1}{2}+0I_1+0I_2,} [x + xI_1 + x^2 I_2]^{1+I_1+0I_2} = \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} [x + xI_1 + x^2 I_2]^{\frac{3}{2}++I_1+0I_2} = \frac{4}{3\sqrt{\pi}} \sqrt{(x + xI_1 + x^2 I_2)^3}$$

$$D_{X(I_1,I_2)}^{-\frac{1}{2}+0I_1+0I_2,} [x + xI_1 + x^2 I_2]^2 = \frac{\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)} [x + xI_1 + x^2 I_2]^{\frac{3}{2}} = \frac{16}{15\sqrt{\pi}} \sqrt{x + xI_1 + x^2 I_2}$$

**Theorem 4.2.5.** Let $\big(X(I_1, I_2)\big)$ , $X = x + yI_1 + zI_2 \in R(I_1, I_2)$ be a refined neutrosophic real function with one refined neutrosophic variable. Then for all $v(I_1, I_2), w(I_1, I_2) > 0 + 0I_1 + 0I_2$ ,

$$D_{X(I_1,I_2)}^{-v(I_1,I_2)} \left[D_{X(I_1,I_2)}^{-w(I_1,I_2)} f\big(X(I_1, I_2)\big)\right] = D_{X(I_1,I_2)}^{-(v+w)(I_1,I_2)} f\big(X(I_1, I_2)\big) = D_{X(I_1,I_2)}^{-w(I_1,I_2)} \left[D_{X(I_1,I_2)}^{-v(I_1,I_2)} f\big(X(I_1, I_2)\big)\right]$$



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**Proof:**

**By definition of the** refined neutrosophic **fractional integral we have**

$$D_{X(I_1,I_2)}^{-v(I_1,I_2)}\left[D_{X(I_1,I_2)}^{-w(I_1,I_2)} f\big(T(I_1,I_2)\big)\right]$$

$$= \frac{1}{\Gamma\big(v(I_1,I_2)\big)} \int_0^{T(I_1,I_2)} \big(T(I_1,I_2) - X(I_1,I_2)\big)^{v(I_1,I_2)-1} \left[D_{X(I_1,I_2)}^{-w(I_1,I_2)} f\big(T(I_1,I_2)\big)\right] d\big(X(I_1,I_2)\big)$$

$$= \frac{1}{\Gamma\big(v(I_1,I_2)\big)} \int_0^{T(I_1,I_2)} \big(T(I_1,I_2) - X(I_1,I_2)\big)^{v(I_1,I_2)-1} \left[\frac{1}{\Gamma\big(w(I_1,I_2)\big)} \int_0^{X(I_1,I_2)} \big(X(I_1,I_2) - Y(I_1,I_2)\big)^{W(I_1,I_2)-1} f\big(Y(I_1,I_2)\big) d\big(Y(I_1,I_2)\big)\right] d\big(X(I_1,I_2)\big)$$

$$= \frac{1}{\Gamma\big(v(I_1,I_2)\big)\Gamma\big(w(I_1,I_2)\big)} \int_0^{T(I_1,I_2)} \big(T(I_1,I_2) - X(I_1,I_2)\big)^{v(I_1,I_2)-1} d\big(X(I_1,I_2)\big) \int_0^{X(I_1,I_2)} \big(X(I_1,I_2) - Y(I_1,I_2)\big)^{w(I_1,I_2)-1} f\big(Y(I_1,I_2)\big) d\big(Y(I_1,I_2)\big)$$

$$= \frac{1}{\Gamma\big(v(I_1,I_2) + w(I_1,I_2)\big)} \int_0^{T(I_1,I_2)} \big(T(I_1,I_2) - Y(I_1,I_2)\big)^{v(I_1,I_2)+w(I_1,I_2)-1} f\big(Y(I_1,I_2)\big) d\big(Y(I_1,I_2)\big)$$

$$= B\big(v(I_1,I_2), w(I_1,I_2)\big) \int_0^{T(I_1,I_2)} \big(T(I_1,I_2) - Y(I_1,I_2)\big)^{v(I_1,I_2)+w(I_1,I_2)-1} f\big(Y(I_1,I_2)\big) d\big(Y(I_1,I_2)\big)$$

**4.3 Definition of the Riemann Liouville Refined Neutrosophic Fractional Derivative**

The refined neutrosophic fractional derivative can be defined using the definition of the refined neutrosophic fractional integral.

**Definition 4.3.1.** Suppose that $v(I_1,I_2) = n(I_1,I_2) - w(I_1,I_2) : n(I_1,I_2) = [\![w(I_1,I_2)]\!]$ Then, the refined neutrosophic fractional derivative of $f\big(X(I_1,I_2)\big)$ of order $w(I_1,I_2)$ is

$$D_{X(I_1,I_2)}^{u(I_1,I_2)} f\big(X(I_1,I_2)\big) = D_{X(I_1,I_2)}^{n(I_1,I_2)}\left[D_{X(I_1,I_2)}^{-v(I_1,I_2)} f\big(X(I_1,I_2)\big)\right]$$

**Example 4.3.2.** suppose we wish to find the refined neutrosophic fractional derivative of $X^{m(I_1,I_2)}(I_1,I_2)$ of order $v(I_1,I_2)$ we just need to interchange $w(I_1,I_2) = n(I_1,I_2) - v(I_1,I_2)$ , $n(I_1,I_2) = 1$ and

$w(I_1,I_2) = 1 - v(I_1,I_2)$ . so,

$$D_{X(I_1,I_2)}^{v(I_1,I_2)} f\big(X(I_1,I_2)\big) = D_{X(I_1,I_2)}^{1}\left[D_{X(I_1,I_2)}^{-(1-v(I_1,I_2))} f\big(X(I_1,I_2)\big)\right] = D_{X(I_1,I_2)}^{1}\left[D_{X(I_1,I_2)}^{-(1-v(I_1,I_2))} X^{m(I_1,I_2)}(I_1,I_2)\right]$$



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$$= D^{1}_{X(I_1,I_2)}\left[\frac{\Gamma(m(I_1,I_2)+1)}{\Gamma\big((m(I_1,I_2)-v(I_1,I_2)+1)+1\big)}X(I_1,I_2)^{m(I_1,I_2)-v(I_1,I_2)+1}\right]$$

$$= (m(I_1,I_2)-v(I_1,I_2)+1)\frac{\Gamma(m(I_1,I_2)+1)}{(m(I_1,I_2)-v(I_1,I_2)+1)\Gamma\big((m(I_1,I_2)-v(I_1,I_2)+1)\big)}X(I_1,I_2)^{m(I_1,I_2)-v(I_1,I_2)}$$

$$= \frac{\Gamma(m(I_1,I_2)+1)}{\Gamma\big((m(I_1,I_2)-v(I_1,I_2)+1)\big)}X(I_1,I_2)^{m(I_1,I_2)-v(I_1,I_2)}$$

In particular, we will find the $\left(\frac{1}{2}+0I_1+0I_2\right)^{th}$ order derivative of $f(X(I_1,I_2)) = (X(I_1,I_2))^{u(I_1,I_2)}$ :

$$D^{\frac{1}{2}+0I_1+0I_2}_{X(I_1,I_2)}(X(I_1,I_2))^{u(I_1,I_2)} = D^{1+0I_1+0I_2}_{X(I_1,I_2)}\left[D^{-\frac{1}{2}+0I_1+0I_2}_{X(I_1,I_2)}(X(I_1,I_2))^{u(I_1,I_2)}\right]$$
$$= \frac{\Gamma(u(I_1,I_2)+1)}{\Gamma\left(\left(u(I_1,I_2)-\frac{1}{2}+1\right)\right)}X(I_1,I_2)^{u(I_1,I_2)-\frac{1}{2}}$$

**Example 4.3.3.**

1) $D^{\frac{1}{2}+0I_1+0I_2}_{X(I_1,I_2)}(x+xI_1+x^2I_2)^{0+0I_1+0I_2} = D^{1+0I_1+0I_2}_{X(I_1,I_2)}\left[D^{-\frac{1}{2}+0I_1+0I_2}_{X(I_1,I_2)}(x+xI_1+x^2I_2)^{0+0I_1+0I_2}\right] =$
$D^{1+0I_1+0I_2}_{X(I_1,I_2)}\left[\frac{2}{\sqrt{\pi}}\sqrt{x+xI_1+x^2I_2}\right] = \frac{2}{\sqrt{\pi}}D^{1+0I_1+0I_2}_{X(I_1,I_2)}\left(\sqrt{x}+\left[\sqrt{x+x+x^2}-\sqrt{1+x^2}\right]I_1+\left[\sqrt{x+x^2}-\sqrt{x}\right]I_2\right) =$
$\frac{2}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{x}}+\left[\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x+x+x^2}}-\frac{1}{2\sqrt{x+x^2}}+\frac{1}{2\sqrt{x+x^2}}-\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x+x^2}}\right]I_1+\left[\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x+x^2}}-\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}\right]I_2\right) =$
$\frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{x}}+\left[\frac{1}{\sqrt{x+x+x^2}}-\frac{1}{\sqrt{x+x^2}}\right]I_1+\left[\frac{1}{\sqrt{x+x^2}}-\frac{1}{\sqrt{x}}\right]I_2\right)$

2) $D^{\frac{1}{2}+0I_1+0I_2}_{X(I_1,I_2)}(x+xI_1+x^2I_2)^{0+0I_1+0I_2} = \frac{\Gamma(0+0I_1+0I_2+1)}{\Gamma\left(\left(0+0I_1+0I_2-\frac{1}{2}+1\right)\right)}(x+xI_1+x^2I_2)^{0+0I_1+0I_2-\frac{1}{2}} = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}(x+xI_1+$
$x^2I_2)^{0+0I_1+0I_2-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{x+xI_1+x^2I_2}} = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{x}+\left[\sqrt{x+x+x^2}-\sqrt{x+x^2}\right]I_1+\left[\sqrt{x+x^2}-\sqrt{x}\right]I_2}$

Notice that 1) and 2) is iqual.

**3. Conclusion**

Refined neutrosophic fractional calculus is a more generalized form of calculus. Unlike the refined neutrosophic integer order calculus where operations are centered mainly at the refined neutrosophic integers, fractional calculus considers every real refined neutrosophic number, $v(I_1,I_2)$ . And as it has been briefly noted in this paper , the meaning and applications of this new type of calculus are quite comparable to those of the ordinary calculus, especially when gets closer and closer to a refined neutrosophic integer. In the future, studying the neutrosophic fractional differential equations became possible thanks to the definitions mentioned in the paper .

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