Solving Initial Value Problem in Composite Materials for Heat Equation

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Abstract

In this paper, we display the definition and arrangement of the beginning esteem issue in composite materials for warm condition. The issue includes finding the starting temperature conveyance when as it were the temperature spreading at time t=T>0 is given. Typically, a challenging issue since it has a place to a course of numerically unsteady issues that are ill-posed. To characterize this issue, we have to be present work spaces and unravel the coordinate issue to decide them. The method of division of factors is commonly utilized to fathom the coordinate issue, but it isn't reasonable for the due to the expansive blunders and disparate arrangement it produces. Ivanov V.K. proposed a strategy to get a steady inexact arrangement by supplanting the coming about arrangement with a fractional whole that depends on δ, N=N(δ). Another approach is the Picard strategy that employments a family of administrators $\{P_N\}$ to map the space L_2 into itself and get a regularized inexact arrangement. We show the comes about of computational tests and assess the viability of the Picard strategy.

Keywords: inverse problem; Picard method; ill-posed problem; composite material;

1. Introduction

Numerous down to earth issues are defined as reverse issues of numerical material science, which by and large drop beneath the category of ill-posed issues. The reverse introductory esteem issue for the warm condition is recognized as an ill-posed issue [1]. With the headway of high-speed individual computers, numerical strategies have ended up more helpful to utilize in tackling converse issues. Tikhonov A.N., Lavrentiev M.M., Ivanov V.K., and their understudies and successors have broadly examined hypothetical concepts and computational usage related to the determination of reverse and ill-posed issues [2], [3].

Various works have inspected the utilize of discretization calculations as a critical implies of lessening necessarily conditions to frameworks of straight arithmetical conditions [4]–[7].. The assessed mistake related with the discretization of fundamentally conditions [8][9], as well as other strategies such as the Fourier change strategy [10], has been explored. The finite-dimensional guess space, in conjunction with remaining strategies for regularization arrangements, has been considered in profundity in [7], [8],[13]–[16]. There exist a plenty of papers which have elucidated and utilized regularization strategies for solving converse issues, especially when dealing with converse issues

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within the fundamentally condition of the primary kind frame [17] [18], [19]. The victory of these regularization strategies is predicated on astute and exhaustive examination of the scientific converse issues related with the issue articulations, as well as on the classification of express issues in fathoming them [20]– [25].

The vital point of this composition is to recover the source work of the dissemination condition through work of the Picard strategy. The partition of factors strategy was utilized in settling the coordinate issue for the partial differential condition of the warm condition of Composite Materials. All of the previously mentioned strategies should be executed within the diverse sections of this talk. Within the to begin with segment, the numerical detailing of this pickle was scrutinized, with an arrangement being looked for within the pretense of a cluster of eigenfunctions comparing to the Sturm-Liouville issue. Segment 2 is sanctified to the reverse issue for the warm conditions and diagrams the known information and administrator. A short time later, the Picard strategy was illustrated with the help of a regularizing family of administrators $\{P_N\}$ that map the space L_2 into itself. A case was displayed in Area 3 to assess the accuracy of our estimation arrangement. At long last, the clarification of the proposed method has been summarized within the concluding area with proposed future work.

2. Solution of direct problem

The coordinate issue is to decide the thermal field $z(x,t)$ in composite materials at any minute $t \in [0,T]$ from the pole temperature at the introductory minute $z(x,0)$. The numerical definition of this issue is portrayed by the framework of differential equations:

$$\frac{\partial z_1(x,t)}{\partial t} = a_1^2 \frac{\partial^2 z_1(x,t)}{\partial x^2}; x \in (0,x_0); t \in (0,T], \quad (1)$$

$$\frac{\partial z_2(x,t)}{\partial t} = a_2^2 \frac{\partial^2 z_2(x,t)}{\partial x^2}; x \in (x_0,1); t \in (0,T], \quad (2)$$

$$z_1(x,0) = z_1(x); 0 \le x \le x_0, z_2(x,0) = z_2(x); x_0 \le x \le 1, \quad (3)$$

$$z_1(0) = z_1'(0) = z_2(0) = z_2'(0) = 0, \quad (4)$$

where $z_1(x) \in \left(C[0,x_0] \cap C^2[0,x_0]\right); z_2(x) \in \left(C[x_0,1] \cap C^2[x_0,1]\right)$

$$\frac{\partial z_1(0,t)}{\partial x} = 0; t \in [0,T], \quad (5)$$

$$z_2(1,t) = 0, t \in [0,T], \quad (6)$$

$$z_1(x_0,t) = z_2(x_0,t), t \in [0,T], \quad (7)$$

$$a_1 \frac{\partial z_1(x_0,t)}{\partial x} = a_2 \frac{\partial z_2(x_0,t)}{\partial x}; t \in [0,T], \quad (8)$$

In the direct problem (1) – (8) it is required to find the functions $\tilde{z}(x,t)$.

$$\tilde{z}(x,t) = \begin{cases} z_1(x,t), x \in [0,x_0], t \in [0,T] \\ z_2(x,t), x \in [x_0,1], t \in [0,T] \end{cases}, \quad (9)$$

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where $\tilde{z}(x,t)\in C\left([0,1]\times[0,T]\right)\cap C^{2,1}\left(\left[(0,x_0)\cup(x_0,1)\right]\times[0,T]\right)$

$$\tilde{z}(x,t)\to z(x),t\to 0 \tag{10}$$

where the notation is used: x – is the spatial coordinate, t – is the time coordinate, and a_1,a_2 – are constants, called thermal diffusivity. Cases of constants for a few materials are given in Table. 1, and they illustrate that these constants can contrast altogether from each other.

Table 1 : Values a_1, a_2 are for some common materials

Material	$a^2 = m^2/sec$
Silver	1.65×10^{-4}
Copper	1.11×10^{-4}
Steel	$1,88 \times 10^{-5}$
Aluminum	9.7×10^{-5}
Glass	3.4×10^{-7}

Theorem 1. For any function $\tilde{z}(x,t)$, satisfying (4), ∃ is the only solution of the direct problem satisfying (1) – (4), (5), (8), (9).

Proof. The uniqueness of the arrangement takes after from the greatest guideline given in [26] and connected in [27]. We are going seek for a formal arrangement of framework (1) – (8) within the frame of a arrangement in terms of eigenfunctions comparing to the Sturm-Liouville issue $\{S_n(x)\}_{n=1}^{\infty}$, given in [28],

$$\tilde{z}(x,t)=\sum_{n=1}^{\infty} z_n e^{-\lambda_n^2 t} S_n(x), \tag{11}$$

where function $S_n(x)$ defined by $S_n(x)=\begin{cases} S_n^1(x), x\in(0,x_0) \\ S_n^2(x), x\in(x_0,1)\end{cases}$,

$$S_n(x)=\begin{cases} \cos\left(\frac{\lambda_n x}{a_1}\right)\sin\left(\frac{\lambda_n(1-x_0)}{a_2}\right), x\in[0,x_0] \\ \sin\left(\frac{\lambda_n(1-x)}{a_2}\right)\cos\left(\frac{\lambda_n x_0}{a_1}\right), x\in[x_0,1]\end{cases}, \tag{12}$$

$$z_n=\frac{\alpha_n}{\lambda_n}\left[\int_0^{x_0} z(x)S_n^1(x)dx+\int_{x_0}^{1} z(x)S_n^2(x)dx\right] \tag{13}$$

where $\alpha_n=\beta_n\cos\left(\frac{\lambda_n x_0}{a_2}\right),$,

$$\beta_n^2=\frac{2a_1a_2}{a_2x_0\sin\left(\frac{\lambda_n(1-x)}{a_2}\right)+a_1(1-x_0)\cos\left(\frac{\lambda_n x_0}{a_1}\right)} \tag{14}$$

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$$\lambda_n = \frac{\pi a_1 a_2 (2n-1)}{2(a_2 x_0 + a_1(1-x_0))}, n = 1,2,3,.. \quad (15)$$

Let us assume that the function $f(x)$ is known. This function represents the solution to the direct problem for $t = T$,

$$\tilde{z}(x,t) = f(x) \quad (16)$$

where $f'(x) \in C([0,x_0)) \cup C((x_0,1])$ and $\tilde{z}(x,T) = \sum_{n=1}^{\infty} z_n e^{-\lambda_n^2 T} S_n(x),$. The exact meaning of the function $f_0(x)$ is not known, but a pair is given instead $f_\delta(x), \delta$, where $f_\delta(x) \in L_2[0,1]$ *and* $\delta > 0$

$$\|f_\delta(x) - f_0(x)\|_{L_2} \le \delta \quad (17)$$

It is required, using the initial data of the problem $f_\delta(x)$ *and* δ , to determine the approximate solution $z_\delta(x) \in L_2[0,1]$, to prove the convergence of $z_\delta(x)$ into $z_0(x)$ for $\delta \to 0$ in the metric $L_2[0,1]$.

4. Solution of the inverse problem

The Picard method uses a regularizing family of operators $\{P_n\}$, that map the space $L_2[0,1]$ into itself and are defined by the formula,

$$R_N f(x) = \sum_{n=1}^{N} f_n e^{\lambda_n^2 T} \quad (18)$$

where $f_n = \frac{\alpha_n}{\lambda_n}\left[\int_0^{x_0} f(x) S_n^1(x) d\frac{x}{a_1} + \int_{x_0}^{1} f(x) S_n^2(x) d\frac{x}{a_2}\right].$

Let us present a number of properties of the family $\{P_n\}$, formulated in the form of lemmas.

Lemma 1. For any N the operator P_N , defined by formula (18), is linear bounded with the norm $\| P_N \| = e^{\lambda_N^2 T}$

Proof. The linearity of the operator P_N follows from formula (18).

Now let's prove that

$$\| P_N \| = e^{\lambda_N^2 T}, \quad (19)$$

from definition $\| P_N \|$ we get that

$$\| P_N \| = \sup\{\| P_N f \| : \| f \| \le 1\}, \quad (20)$$

and

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$$\| P_N f \|^2 = \sum_{n=1}^{N} f_n^2 e^{2\lambda_N T}, \qquad (21)$$

and

$$\sum_{n=1}^{\infty} f_n^2 \leq 1, \qquad (22)$$

from (21) and (22) следует, что

$$\| P_N \| \leq e^{\lambda_N^2 T}. \qquad (23)$$

Consider the element $f_N(x) = S_N(x)$.

Since $f_N \in L_2[0,1]$ and $\| f_N \| = 1$, then acting on it by the operator P_N, we get the following

$$\| P_N f \|^2 = e^{2\lambda_N T}, \qquad (24)$$

from (20) and (24) follow that

$$\| P_N \| \geq e^{\lambda_N T}, \qquad (25)$$

from (23) and (25) follow that

$$\| P_N \| = e^{\lambda_N T}, \qquad (26)$$

Lemma 2. The family of operators $\{P_n\}$, defined by (18), regularizes the inverse problem (1), (2), (5) – (8), (9), (16), (17).

Proof. It follows from the definition of a regularizing family of operators that for any element of $z(x) \in L_2[0,1]$ The relation $\forall u^0(x)) \in L_2[0,1]$.

$$P_N\left[\sum_{n=1}^{\infty} z_n e^{\lambda_N^2 T} S_n(x)\right] \to z(x), \text{ for } N \to \infty, \text{ in metric } L_2[0,1],$$

where $z(x) = \{z_1(x); 0 \leq x \leq x_0, z_2(x); x_0 \leq x \leq 1\}$ and

$$z_n = \beta_n^2 \left[\int_0^{x_0} z(x) S_n^1(x) dx + \int_{x_0}^{1} z(x) S_n^2(x) dx\right].$$

Since $P_N\left[\sum_{n=1}^{\infty} z_n e^{\lambda_N^2 T} S_n(x)\right] - z(x) = \sum_{n=N+1}^{\infty} z_n S_n(x),$ then

$$\left\| P_N\left[\sum_{n=1}^{\infty} z_n e^{\lambda_N^2 T} S_n(x)\right] - z(x) \right\|^2 = \sum_{n=N+1}^{\infty} z_n^2. \qquad (27)$$

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Due to the fact that $z(x)) \in L_2[0,1]$ follows the convergence of the series

$$\sum_{n=1}^{\infty} z_n^2.$$

Thus,

$$\sum_{n=N+1}^{\infty} z_n^2 \to 0 \, for \, N \to \infty.$$

Thus, the lemma is proved. The regularized solution $z_\delta^N(x)$ of the inverse problem is defined by the formula

$$z_\delta^N(x) = P_N f_\delta(x). \quad (28)$$

Now let's move on to determining the dependence $N(\delta)$, for this we will make an estimate

$$z(x) - z_\delta^N(x) \le z(x) - z_0^N(x) + z_0^N(x) - z_\delta^N(x), \quad (29)$$

from (29) follow that, if $N(\delta)$ choose from conditions $N(\delta) \to \infty$, and $\delta e^{-\lambda_{N(\delta)}^2 T} \to 0$ for $\delta \to 0,$ then

$$z_\delta^{N(\delta)}(x) \to z(x) \, for \, \delta \to 0. \quad (30)$$

The element $z_\delta^{N(\delta)}(x)$ will be called an inexact arrangement of the reverse issue. In specific, we will select $N(\delta)$ from the condition $\delta e^{-\lambda_{N(\delta)}^2 T} \to 0$

$$e^{-\lambda_{N(\delta)}^2 T} = \frac{1}{\sqrt{\delta}} \quad (31)$$

$$N(\delta) = \left[\sqrt{\frac{1}{2T} \ln\left(\frac{1}{\delta}\right)} \right] + 1. \quad (32)$$

From the Picard method described above, it can be seen that this method is effective for finding an estimated solution. Picard's method uses a regularizing family of $\{P_n\}$, operators that map the space $L_2[0,1]$ into itself.

5. Numerical example

we ought to be choose the darken function for the beginning temperature (1) – (8), we have to be decide the obscure function for the starting temperature $u(x,0) = u(x)$, according to the known

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function $\tilde{u}(x,T)=f(x)$, for illustration use as the initial temperature $\cos\left(\pi x/2\right)$ (see figure 1).

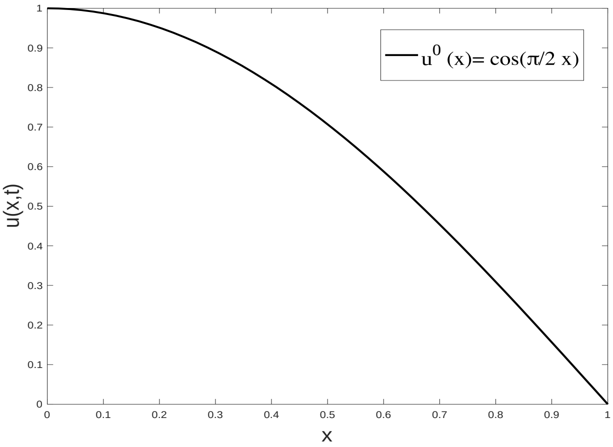
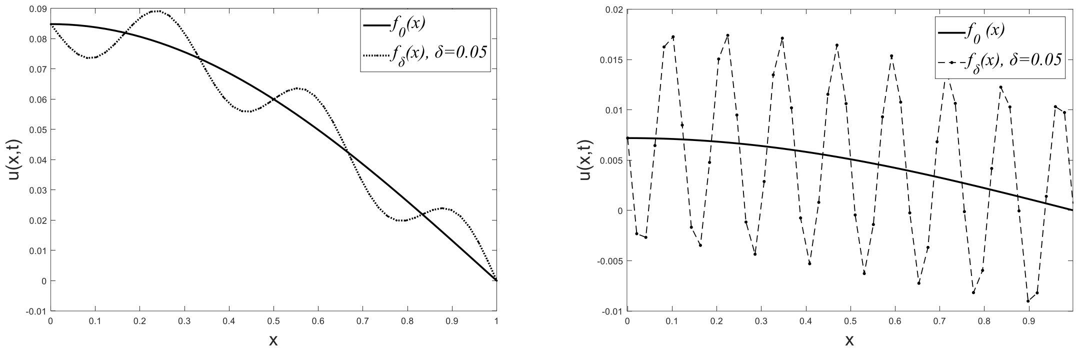


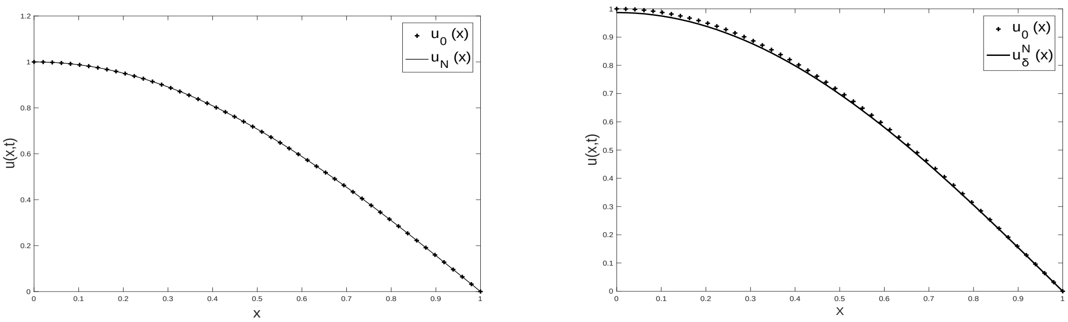
Figure 1: Initial condition u(x)

Example 1 understanding for temperature at particular time focuses (T=1, T=2) and $a_1 = a_2 = 1$ can be upgraded by presenting a clamor flag to the accessible information $f_0(x)$ for ensuing issue examination. The application of the reverse issue calculation, as characterized in (28), can be utilized to get arrangements. Typically outlined in figures 2 and 3.



a) T=1 b) T=2

Figure 2: Direct solution for temperature measurement $f_o(x)$ *and* $f_\delta(x)$



a) T=1, $\delta = 0$ b) T=1, $\delta = 0.05$

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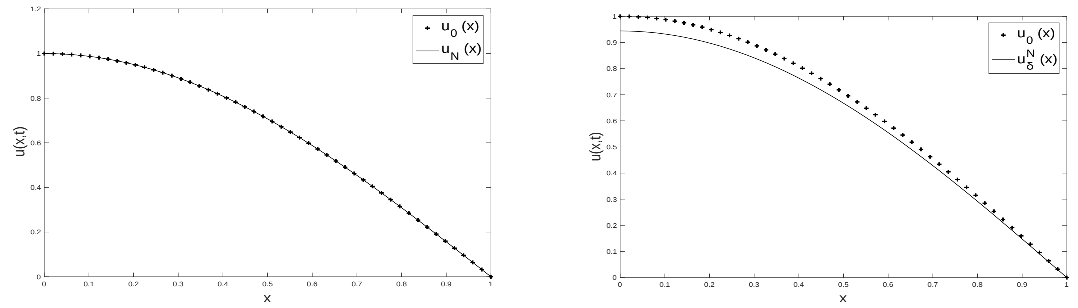
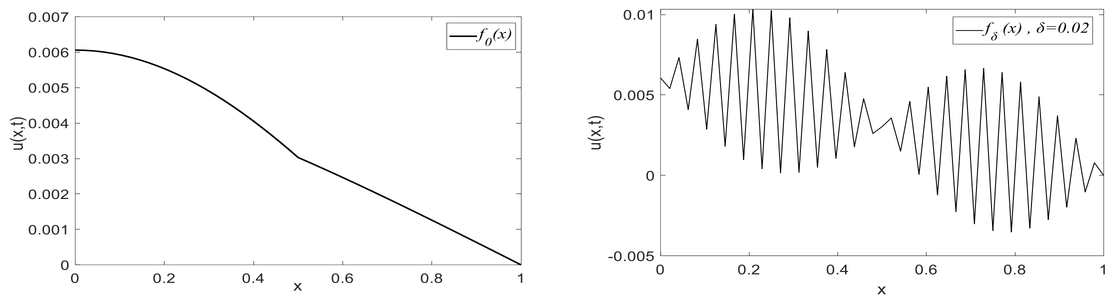


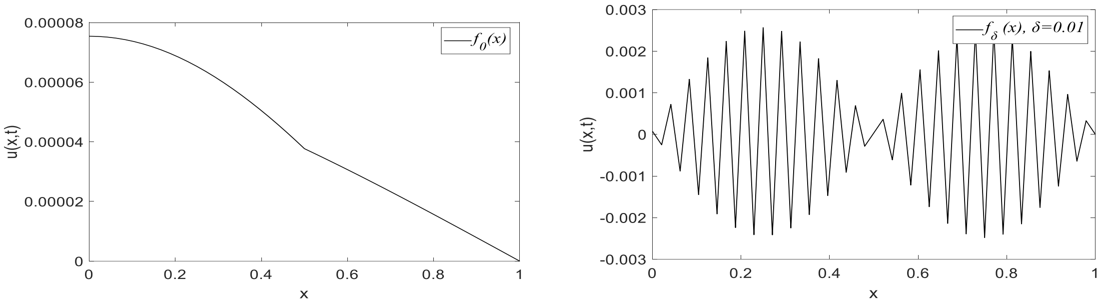
c) T=2, $\delta = 0$ d) T=2, $\delta = 0.05$

Figure 3 : Reverse solution $u_0(x)$, $u_N(x)$ and $u_\delta^N(x)$

Example 2. Direct solution for temperature, where time is (T=1 and T=2), $a_1 = 1,$ and $a_2 = 2$. We can add a noise signal to the given $f_0(x)$ for its use in problem analysis, see figure 4 and figure 5.

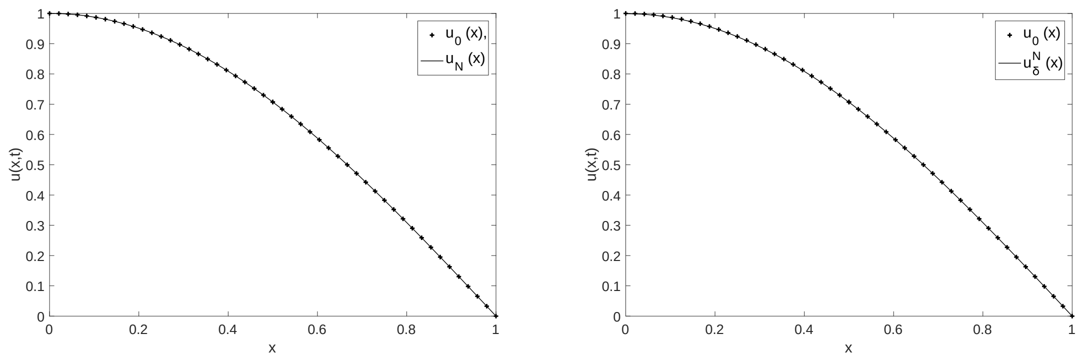


a) T=1



b) T=2

Figure 4: Direct solution for temperature measurement $f_0(x)$ *and* $f_\delta(x)$



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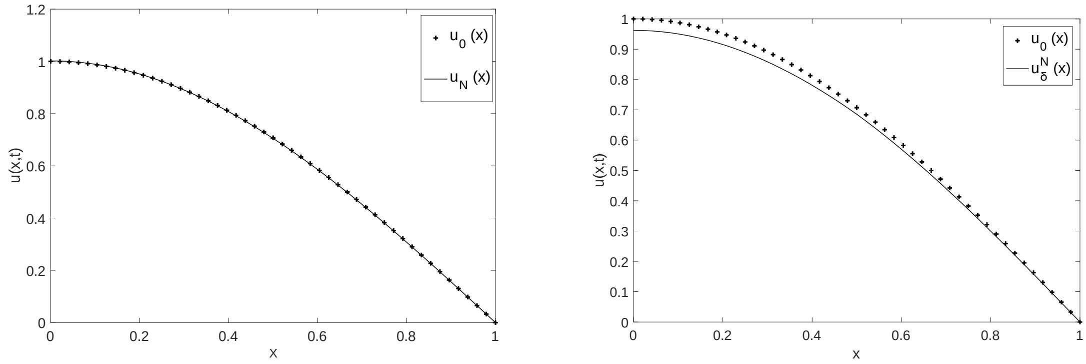
a) T=1, $\delta = 0$ b) T=1, $\delta = 0.02$



c) T=2, $\delta = 0$ d) T=2, $\delta = 0.01$

Figure 5: Reverse solution $u_0(x)$, $u_N(x)$ and $u_\delta^N(x)$

6. Conclusion

This work is committed to the algorithmic determination of the switch warm conductivity issue in composite materials. The issue at hand is of the ill-posed Cauchy assortment, and a specialized strategy utilizing the strategy of partition of factors is utilized to fathom the condition overseeing turn around warm conductivity in composite materials. In arrange to fathom the issue of warm conduction starting esteem issue by utilizing Picard's hypothesis, numerical examinations are viably utilized. The calculation is made successful by selecting a fitting parameter to control the assessed arrangement. It is noteworthy to note that the calculation has shown adequacy within the estimation of the beginning temperature, utilizing a given estimation temperature and a known clamor level δ.

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References

[1] S I Kabanikhin, Inverse and ill-posed problems: theory and applications, 55. Walter De Gruyter, 2011.
[2] S I Kabanikhin, Inverse and ill-posed problems. de Gruyter, 2011.
[3] S I Kabanikhin, INVERSE AND ILL-POSED PROBLEMS. Sib. Elektron. Mat. Izv., 7, 2010.
[4] A G Goncharskii, AV Leonov, A S, Yagola, Finite-Difference Approximation of Linear Ill-Posed Problems. Zh. Vych. Mat. Mat. Fiz, 14(4), 1022–1027, 1974.
[5] V K. Ivanov, V V Vasin, T V P, Theory of Linear Ill-Posed Problem and Application. Nauok Moscow, 1978.
[6] V P Tanana, Projection methods and finite-difference approximation of linear incorrectly formulated problems. Sib. Math. J., 16(6), 999–1004, 1975.
[7] V V Vasin, Discrete convergence and finite-dimensional approximation of regularizing algorithms. USSR Comput. Math. Math. Phys., 19(1), 8–19, 1979.
[8] V P Tanana and A. I. Sidikova, On Estimating the Error of an Approximate Solution Caused by the Discretization of an Integral Equation of the First Kind. Proc. Steklov Inst. Math., 299(1), 217–224, 2017.
[9] V P Tanana, E Y Vishnyakov, A I Sidikova, An approximate solution of a Fredholm integral equation of the first kind by the residual method. Numer. Anal. Appl., 9(1), 74–81, 2016.
[10] B T Al-Nuaimi, H K Al-Mahdawi, Z Albadran, H Alkattan, M Abotaleb, E-S M El-kenawy, Solving of the Inverse Boundary Value Problem for the Heat Conduction Equation in Two Intervals of Time. Algorithms, 16(1), 33, 2023.
[11] G G Skorik,V V Vasin, Regularized Newton type method for retrieval of heavy water in atmosphere by IR spectra of the solar light transmission. Eurasian J. Math. Comput. Appl., 7(2), 79–88, 2019.

Doi: https://doi.org/10.54216/JAIM.060101
Received: February 08, 2023 Revised: May 07, 2023 Accepted: September 21, 2023

[12] L D Menikhes, V P Tanana, The finite-dimensional approximation for the Lavrent'ev method. Sib. Zhurnal Vychislitel'noi Mat., 1(1), 59–66, 1998.
[13] A I Sidikova, H K Al-Mahdawi, The solution of inverse boundary problem for the heat exchange for the hollow cylinder. in AIP Conference Proceedings, 2398(1) 2022.
[14] H K I Al-Mahdawi, M Abotaleb, H Alkattan, A M Z Tareq, A Badr, A Kadi, Multigrid Method for Solving Inverse Problems for Heat Equation. Mathematics, 10(15), 2802, 2022.
[15] H K I Al-Mahdawi, H Alkattan, M Abotaleb, A Kadi, E-S M El-kenawy, Updating the Landweber Iteration Method for Solving Inverse Problems," Mathematics, 10(15), 2798, 2022.
[16] H K Al-Mahdawi, A I Sidikova, H Alkattan, M Abotaleb, A Kadi, E-S M El-kenawy, Parallel Multigrid Method for Solving Inverse Problems. MethodsX, 101887, 2022.
[17] S A Noaman, H K Al-Mahdawi, B T Al-Nuaimi, A I Sidikova, Iterative method for solving linear operator equation of the first kind," MethodsX, 10, 102210, 2023.
[18] R Plato, P Mathé, B Hofmann, Optimal rates for Lavrentiev regularization with adjoint source conditions. Math. Comput., 87(310), 785–801, 2018.
[19] R Plato, The Product Midpoint Rule for Abel-Type Integral Equations of the First Kind with Perturbed Data. in New Trends in Parameter Identification for Mathematical Models, Springer, 195–225,2018.
[20] V B Glasko, N I Kulik, I N Shklyarov, A N Tikhonov, An inverse problem of heat conductivity. Zhurnal Vychislitel'noi Mat. i Mat. Fiz., 19(3), 768–774, 1979.
[21] A S Belonosov, M A Shishlenin, Continuation problem for the parabolic equation with the data on the part of the boundary. Siber. Electron. Math. Rep., 11, 22–34, 2014.
[22] S I Kabanikhin, A Hasanov, A V Penenko, A gradient descent method for solving an inverse coefficient heat conduction problem, Numer. Anal. Appl., 1(1), 34–45, 2008.
[23] A G Yagola, I E Stepanova, Y Van, V N Titarenko, Obratnye zadachi i metody ikh resheniya. Prilozheniya k geofizike. Inverse Probl. Methods their Solut. Appl. to Geophys. Moscow Binom. Lab. znanii, 2014.
[24] S I Kabanikhin, O I Krivorot'ko, M A Shishlenin, A numerical method for solving an inverse thermoacoustic problem," Numer. Anal. Appl., 6(1), 34–39, 2013.
[25] V P Tanana, On the order-optimality of the projection regularization method in solving inverse problems. Sib. Zhurnal Ind. Mat., 7(2), 117–132, 2004.
[26] A N Tikhonov, A A Samarskii, Equations of mathematical physics. Courier Corporation, 2013.
[27] H K Al-Mahdawi, M Abotaleb, H Alkattan, E-S M El-Kenawy, E M Mohamed, Solving the Inverse Initial Value Problem for the Heat Conductivity Equation by Using the Picard Method.
[28] V P Tanana, A A Ershova, On the solution of an inverse boundary value problem for composite materials. Vestn. Udmurt. Univ. Mat. Mekhanika. Komp'yuternye Nauk., 28(4), 474–488, 2018.

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