Clean Graphs over Rings of Order P^2

Heba Adel Abdelkarim[1], **Edris Rawashdeh**[2], **Eman Rawshdeh**[3,*]
[1]Department of Mathematics, Irbid National University, Irbid, Jordan
[2]Department of Mathematics, Yarmouk University, Irbid, Jordan
[3]Department of Basic Scientific Sciences, Al-Huson University College, Al-Balqa Applied University, Irbid, Jordan

Emails: dr.heba@inu.edu.jo; edris@yu.edu.jo; eman.rw@bau.edu.jo

Abstract

Assume R is a commutative ring with unity. The clean graph $CL(R)$ is defined in which every vertex has the form (a, v), where a is an idempotent in R and v is a unit. In $CL(R)$, two distinct vertices (a_1, v_1) and (a_2, v_2) are adjacent if $a_1a_2 = a_2a_1 = 0$ or $v_1v_2 = v_2v_1 = 1$. In this paper, we show that the clean graph $CL(R)$ over a ring of order p^2 can be defined only if R is one of the rings: Z_{p^2}, $Z_p \oplus Z_p$, $Z_p(+)Z_p$ and $GF(p^2)$. Then, we study the spectrum, the biclique partition number, and the eigensharp property for the these clean graphs.

Keywords: Commutative Ring; Clean Graph; Spectrum of graph; Biclique partition number; Eigensharp graph

1 Introduction

The construction of graphs that are related with algebraic structures is a fundamental area in a modern graph theory. In fact, many properties can be better understood when studied theoretically with the graph that represents this algebraic structure. In particular, the study of graphs related to commutative rings is one of the most dynamic research areas in this field as it plays a crucial role as an algebraic structure in mathematics, for example one can see,[3],[4] and.[5]

One of the interesting concept is the zero divisor graph of rings which was introduced by Beck[7] in 1988. Linking a ring to a graph in the analysis of zero-divisor graphs offers a glimpse into the algebraic properties of rings, focusing on the zero-divisor set's structure.

Akbari et al.[2] introduced the idempotent graph, $I(R)$, with non-trivial idempotents of the ring R as vertices and two vertices are adjacent in $I(R)$ if and only if their product is zero. An element of a ring is said to be clean if it can be written as the sum of an idempotent element and a unit. A ring is called a clean ring if all the elements of the ring are clean. For a ring R, the clean graph $CL(R)$ is defined to be the graph in which every vertex has the form (u, v) where, u is an idempotent in the ring R and v is a unit. Two distinct vertices (u_1, v_1) and (u_2, v_2) in $CL(R)$ are adjacent if and only if $u_1u_2 = u_2u_1 = 0$ or $v_1v_2 = v_2v_1 = 1$.

Nicholson[16] in 1977 was the first to introduce the clean rings. Habibi et al.[13] presented the innovative notion of a clean graph denoted as $Cl(R)$ for a given ring R and determined the clique number, the chromatic number and the domination number of the clean graph $Cl(R)$ for some classes of rings. Investigating the placement of clean graphs representing commutative rings on different surfaces has been studied in.[19] In[17] it has been proved that $Cl(R)$ is connected if and only if R is additively generated by its idempotents. More literature and contribution on clean graphs can be seen in.[19]

An important concept in graph theory is the use of subgraphs from a specific graph family to cover a graph. There has been extensive research on various types of graph covering. Including, for instance tree covering, cycle covering, edge covering and star covering, see[10] and.[6]

A biclique is a maximal induced complete bipartite subgraph of a graph. A biclique partition covering of a graph G is a collection $\mathcal{L}_G = \{W_1, W_2, ..., W_j\}$ of complete bipartite subgraphs of G such that for every $i = 1, 2, ..., j$, there is only one $W_i \in \mathcal{L}_G$ such that $e \in E(W_i)$ for each edge $e \in E(G)$. The smallest cardinality of any biclique partition of a graph G is called the biclique partition number of G, and is denoted by $bp(G)$. The biclique partition number has many uses in different areas of applied science such as network addressing,[12] immunology[15] and automata theory.[8]

The adjacency matrix of a graph G with vertex set $V(G)$ and edge set $E(G)$, denoted by $A(G)$, is a square matrix of order $|V(G)|$ where the ij-th entry equals to 1 in the case where v_iv_j is an edge of G and to 0 in the other case.

Since the adjacency matrix $A(G)$ of the graph G is symmetric; then it's eigenvalues are real. the number of positive, negative and zero eigenvalues are denoted by $r_+(A(G))$, $r_-(A(G))$ and $r_0(A(G))$, respectively.

It is clear that for anon-null graph G, $r_+(A(G)) > 0$ and $r_-(A(G)) > 0$. It has been proved (see, for example,[12]) that

$$bp(G) \geq \max\{r_+(A(G)),\ r(A(G))\}.$$

A graph G is called an eigensharp when $bp(G) = \max\{r_+(A(G)),\ r_-(A(G))\}$. The eigensharp graphs include specific groups of graphs, such as complete graphs K_n, complete bipartite graphs $K_{n,m}$, and cycle graphs C_n with $n \neq 4k$, $k \geq 2$. In fact $bp(K_{n,m}) = 1$, $bp(K_n) = n-1$, and $bp(C_n) = \lceil \frac{n}{2} \rceil$, see,[11],[12] ,[14] and.[18]

The number of linearly independent eigenvectors that are associated with an eigenvalue λ_i is known as its multiplicity. If $\lambda_i, 1 \leq i \leq l$, are the distinct eigenvalues of the adjacency matrix $A(G)$ with multiplicity q_i, then $\sigma(A(G)) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_l \\ q_1 & q_2 & ... & q_l \end{pmatrix}$ is called the *spectrum* of G.

In this paper, we study the clean graphs over rings of order p^2. In particular, we find the spectrum of the adjacency matrix of these graphs and we show that they are eigensharp.

2 Preliminaries

Let R be a ring of order p^2. By the fundamental theorem of finitely generated abelian groups, the additive abelian group related to R is a direct product of cyclic groups $C_{p^j} = \langle a; p^j a = 0 \rangle$, where $1 \leq j \leq 2$ and $\sum j = 2$. Thus, if R is a finite ring of order p^2, its additive group is isomorphic to C_{p^2} or $C_p \times C_p$.

A commutative ring with unity is said to be local if it has a unique maximal ideal. For any prime p, the ring Z_{p^n} is a local ring having the unique maximal ideal containing the set of all multiples of p. An element e in a ring R is said to be an idempotent if $e^2 = e$. It is generally known that, the only idempotent elements in a local ring are 0 and 1.

Throughout the paper, $Z_p \oplus Z_p$ is the ring defined as the ring of all ordered 2-tuples (a, b) from Z_p for which addition and multiplication are defined componentwise $(\text{mod} p)$. The ring $Z_p(+)Z_p$ is defined to be the set of all ordered 2-tuples from Z_p with componentwise addition $(\text{mod} p)$, and multiplication as:

$$(a, b)(c, d) = (ac(\text{mod} p), ad + bc(\text{mod} p)).$$

The set of units in the ring Z_p is $Z_p\backslash\{0\}$. The set $\{(a,b) : a,b \in Z_p\backslash\{0\}\}$ represents the set of units in the ring $Z_p \oplus Z_p$. In $Z_p(+)Z_p$, the unity is $(1,0)$, and (a,b) is a unit if and only if a is a unit in Z_p, where $(a,b)^{-1} = (a^{-1}, -ba^{-2})$. Therefore, the set of units in the ring $Z_p(+)Z_p$ is $\{(a,b) : a \in Z_p\backslash\{0\}\}$.

Waterhouse[21] classified the rings of order m with an additive cyclic group C_m. The result is shown in Proposition 2.1. Fine[9] extended the result introduced in[21] and demonstrated that there are precisely 11 rings of order p^2 up to isomorphism.

Proposition 2.1. [21] *If R is a ring with an additive cyclic group C_m, then up to isomorphism, for each divisor d of m there is a ring $R_d = \langle a; ma = 0, a^2 = da\rangle$ where a is an additive generator of C_m.*

Therefore, if R is a ring with a cyclic additive group C_{p^2}, by Proposition 2.1, R is isomorphic to $R_1 = \langle a; p^2a = 0, a^2 = a\rangle$, or to $R_p = \langle a; p^2a = 0, a^2 = pa\rangle$, or to $R_{p^2} = \langle a; p^2a = 0, a^2 = 0\rangle$.

Remark 2.2. The ring $R_1 = \langle a; p^2a = 0, a^2 = a\rangle$ is isomorphic to the ring Z_{p^2}.

Assume R is a ring with the additive group $C_p \times C_p$. Then R has the representation:

$$R = \langle a, b; pa = pb = 0, ab = \alpha_1 a + \beta_1 b, ba = \alpha_2 a + \beta_2 b\rangle,$$

for $\alpha_i, \beta_i \in Z_p$ and $i = 1, 2$.

Next, we describe the most essential 3 rings that will form as a result.

Remark 2.3. Let R be a ring with the additive group $C_p \times C_p$.

1. If $ab = ba = 0$, then R is isomorphic to the ring $Z_p \oplus Z_p$ with respect to the homomorphism φ defined by, $\varphi(a) = (1,0)$ and $\varphi(b) = (0,1)$.

2. If $R = \langle a, b; pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = a\rangle$, then, an isomorphism ψ can be defined from R into the ring $Z_p(+)Z_p$ such that $\psi(a) = (0,1)$ and $\psi(b) = (1,0)$. Hence, R is isomorphic to the ring $Z_p(+)Z_p$.

3. If $R = \begin{cases} \langle a, b; 2a = 2b = 0, a^2 = a, b^2 = a + b, ab = b, ba = b\rangle, p = 2 \\ \langle a, b; pa = pb = 0, a^2 = a, b^2 = ja, ab = b, ba = b\rangle, \text{ if } j \neq x^2 \forall x \in Z_p, p \neq 2 \end{cases}$, then, by,[9] R is isomorphic to the finite field $GF(p^2)$.

4. The remaining five p^2 rings that were discovered by[9] are:
   $R = \langle a, b; pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$
   $R = \langle a, b; pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$
   $R = \langle a, b; pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0\rangle$
   $R = \langle a, b; pa = pb = 0, a^2 = b, ab = 0\rangle$
   $R = \langle a, b; pa = pb = 0, a^2 = b^2 = 0\rangle$

In fact, the only p^2 rings exhibiting clean associated graphs are Z_{p^2}, $Z_p(+)Z_p$, $GF(p^2)$ and $Z_p \oplus Z_p$. The remaining rings lack a unity, making it impossible to identify clean graphs associated with them. See Remark 3 in.[1]

3 The spectrum of the clean graphs $CL\left(Z_{p^2}\right), CL\left(Z_p(+)Z_p\right), CL\left(GF(p^2)\right)$,and $CL(Z_p \oplus Z_p)$

In this section, we find the adjacency matrix and the spectrum of the clean graphs $CL(Z_{p^2}), CL(Z_p(+)Z_p), CL(GF(p^2))$ and $CL(Z_p \oplus Z_p)$.

Throughout, let s be an even positive integer greater than or equal to 5, J_s be the all-1 matrix of order s, $A(K_s)$ be the adjacency matrix of the complete graph K_s, F_s and Q_s be matrices of order s defined by

$$(F_s)_{i,j} = \begin{cases} 0, & \text{if} \quad i \ \text{ or } \ j \in \{1,2\}, \\ 0, & \text{if} \quad i = j, \\ 1, & \text{if} \quad j = i+1 \ \text{ and } \ j \geq 4 \ \text{ is even,} \\ 1, & \text{if} \quad j = i-1 \ \text{ and } \ j \geq 3 \ \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

and

$$(Q_s)_{i,j} = \begin{cases} 0, & \text{if} \quad i \ \text{ or } \ j \in \{1,2,3,4\}, \\ 0, & \text{if} \quad i = j, \\ 1, & \text{if} \quad j = i+1 \ \text{ and } \ j \geq 6 \ \text{ is even,} \\ 1, & \text{if} \quad j = i-1 \ \text{ and } \ j \geq 5 \ \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The following lemma will be used frequently through the remainder of the paper.

Lemma 3.1. *Suppose the matrix F_s is defined by (1). The spectrum of the the block matrix*

$$H = \begin{bmatrix} A(K_s) & J_s \\ J_s & F_s \end{bmatrix}$$

is given by

$$\sigma(H) = \begin{pmatrix} 0 & 1 & -1 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \frac{s-4}{2} & \frac{3s-4}{2} & 1 & 1 & 1 \end{pmatrix} \tag{3}$$

where λ_1, λ_2, and λ_3 are the roots of the polynomial

$$p(x) = x^3 - sx^2 - (s^2 - s + 1)x + 2s.$$

Proof. Let X be an eigenvector of the matrix H, then if we look deeply to the construction of the matrix H, we may consider the entries of the vector X to be

$$x_i = \begin{cases} a_i, & \text{if} \quad i = 1,2,\ldots,s, \\ b_{i-s}, & \text{if} \quad i = s+1, s+2, \\ c_{i-s-2}, & \text{if} \quad i = s+3, s+4, \ldots, 2s. \end{cases}.$$

Since, we have to find λ so that $HX = \lambda X$, then for $j = 1,2,\ldots,s$

$$\sum_{i=1, i\neq j}^{s} a_i + b_1 + b_2 + \sum_{i=1}^{s-2} c_i = \lambda a_j. \tag{4}$$

The $s+1-$th and $s+2-$th entries of X have to satisfy

$$\sum_{i=1}^{s} a_i = \lambda b_1 = \lambda b_2. \tag{5}$$

In the same manner, for $j = 2,4,\ldots,s-2$, we get

$$\sum_{i=1}^{s} a_i + c_j = \lambda c_{j-1} \tag{6}$$

and

$$\sum_{i=1}^{s} a_i + c_{j-1} = \lambda c_j.l \tag{7}$$

It is clear that if $\lambda \neq -1$, then from equations (4), we get $a_1 = a_2 = \cdots = a_s = a$.

Case 1. $\lambda \notin \{0, 1, -1\}$.

By subtracting equation (7) from equation (6), we obtain $c_{j-1} = c_j$ for $j = 2, 4, \ldots, s-2$. From equation (5), $b_1 = b_2$ and $b_1 = \frac{s_a}{\lambda}$. Now, by subtracting equation (5) from (6), we get

$$(\lambda - 1)c_j = \lambda b_1,$$

and so

$$\sum_{j=1}^{s-2} c_j = \frac{s(s-2)}{\lambda - 1} a.$$

Thus, equation (4) becomes

$$(s-1)a + 2\frac{sa}{\lambda} + \frac{s(s-2)}{\lambda - 1} a = \lambda a. \tag{8}$$

It is clear that a has to be a nonzero constant so that X is an eigenvector corresponding to $\lambda \notin \{0, 1, -1\}$. Thus from equation (8), λ must satisfy

$$p(x) = x^3 - sx^2 - (s^2 - s + 1)x + 2s.$$

This means the matrix H has at least three eigenvalues of multiplicity one. It is not difficult to show that the polynomial $p(x)$ has three distinct roots, with exactly one of them being negative.

Case 2. $\lambda = 0$.

Equations (5) implies $a_1 = a_2 = \cdots = a_s = 0$, and so from equation (6), we get $c_1 = c_2 = \cdots = c_{s-2} = 0$. Thus equation (4) gives $b_2 = -b_1$.

Hence, the eigenspace for $\lambda = 0$ has bases only one vector which can be

$$(0, 0, \underset{s}{\ldots}, 0, 1, -1, 0, 0, \underset{s-2}{\ldots}, 0)^T.$$

Case 3. $\lambda = 1$.

By subtracting equation (7) from equation (6), we obtain $c_{j-1} = c_j$ for $j = 2, 4, \ldots, s-2$. Therefore, equation (7) and (6) give $a_1 = a_2 = \cdots = a_s = 0$ and $b_1 = b_2 = 0$, respectively. Finally from equation (4), we conclude that $\sum_{i=1}^{s-2} c_i = 0$. Thus, the eigenspace for $\lambda = 1$ has dimension $\frac{s-2}{2} - 1 = \frac{s-4}{2}$.

Case 4. $\lambda = -1$.

From equation (7), we obtain

$$\sum_{i=1}^{s} a_i = -\frac{2}{s-2} \sum_{i=1}^{s-2} c_i.$$

Thus equation (5) implies $b_1 = b_2 = \frac{2}{s-2} \sum_{i=1}^{s-2} c_i$ and equation (4) gives

$$\frac{-2}{s-2} \sum_{i=1}^{s-2} c_i + \frac{4}{s-2} \sum_{i=1}^{s-2} c_i + \sum_{i=1}^{s-2} c_i = \frac{s}{s-2} \sum_{i=1}^{s-2} c_i = 0.$$

Thus for $s > 2$, we get $\sum_{i=1}^{s-2} c_i = b_1 = \sum_{i=1}^{s} a_i = 0$ and so from equation (6), we have $c_j = -c_{j-1}$ for $j = 2, 4, \ldots, s-2$. Therefore, the eigenspace for $\lambda = -1$ has dimension $s - 1 + \frac{s-2}{2} = \frac{3s-4}{2}$. □

Now, consider the clean graph $CL(Z_{p^2})$. Since Z_{p^2} is local ring, the idempotent elements are 0 and 1. Furthermore, as $Z_{p^2} = \{i + jp : i, j \in Z_p\}$, the set of units is $U = Z_{p^2} \setminus \{ip : 0 \leq i \leq p-1\}$ with $(i + jp) = (i + jp)^{-1}$ only if $i = 1$ or $i = p-1$ and $j = 0$.

Let $G = CL(Z_{p^2})$ and $V_0 = \{(0, x) : x \in U\}$ and $V_1 = \{(1, x) : x \in U\}$ be a partition of $V(G)$. Let G_0 and G_1 denote the induced subgraphs corresponding to the vertex sets V_0 and V_1, respectively. Since each vertex in V_0 has 0 as its initial component, G_0 is a complete graph isomorphic to K_{p^2-p}. Also, each vertex in V_0 is adjacent with each vertex in V_1. Moreover, two distinct vertices in V_1, are adjacent if they have x and x^{-1} as second component. But the two vertices $(1, 1)$ and $(1, p-1)$ are not adjacent to any vertex in V_1. Therefore, and by using Lemma 3.1, we have the following lemma.

Lemma 3.2. *The adjacency matrix of the clean graph $CL(Z_{p^2})$ can be expressed as*

$$A(CL(Z_{p^2})) = \begin{bmatrix} A\left(K_{p^2-p}\right) & J_{p^2-p} \\ J_{p^2-p} & F_{p^2-p} \end{bmatrix}.$$

Moreover, the spectrum of the matrix $A(CL(Z_{p^2}))$ is given by equation (3) with $s = p^2 - p$.

Now, we consider the clean graphs $CL(Z_p(+)Z_p)$ and $CL(GF(p^2))$. As indicated previously, we shall define the adjacency matrix for both. Before that, we will clarify some notes related to each ring.

Remark 3.3. In $Z_p(+)Z_p$, the set of idempotents is $\{(0, 0), (1, 0)\}$ and (x, y) is a unit if and only if $x \neq 0 \pmod{p}$.

Remark 3.4. If (x, y) is a unit in $Z_p(+)Z_p$, then $(x, y)^{-1} \neq (x, y)$ unless, $x = 1$ or $x = p-1$ and $y = 0 \pmod{p}$.

The ring $GF(p^2)$, according to Remark 2.3 (part (3)), is defined as

$$GF(p^2) = \begin{cases} \langle a, b; 2a = 2b = 0, a^2 = a, b^2 = a + b, ab = b, ba = b\rangle, \ p = 2 \\ \langle a, b; pa = pb = 0, a^2 = a, b^2 = ja, ab = b, ba = b\rangle, \text{ if } j \neq x^2 \ \forall x \in Z_p, p \neq 2 \end{cases}.$$

Since $GF(p^2)$ is a finite field, we have the following well-known result.

Remark 3.5. The unity for the ring $GF(p^2)$ is a, and hence the idempotent elements are 0 and a. Furthermore, every nonzero element in $GF(p^2)$ is a unit with $x \neq x^{-1}$ except for $x = a$ or $x = -a$.

Now, according to Remarks 3.3 and 3.5, we only have two distinct idempotents 0 and 1, and hence, the vertex set can be represented as a union of a two basic sets V_0 and V_1, where V_0 is the set in which the initial component for each vertex is the zero idempotent element and V_1 is the set in which the non-zero idempotent element, say 1, is the initial component. Analogous to the clean graph $CL(Z_{p^2})$, consider the induced subgraphs G_0 and G_1 corresponding to the vertex sets V_0 and V_1 respectively. Then, G_0 is isomorphic to a complete graph. Also, it is clear that, every vertex in G_0 is adjacent with every vertex in G_1. For the graph G_1, recall that a vertex has the form $(1, x)$, where x is a unit. Thus two distinct vertices in G_1 are adjacent if their second components are x and x^{-1} respectively. Referring to Remarks 3.4 and 3.5, two elements exists, say 1 and -1, that are the inverses for themselves. Let s to be the number of units in each ring. Then, the related vertices $(1, 1)$ and $(1, -1)$ are not adjacent with any vertex in G_1, and so, G_1 is a disconnected graph decomposed as the union of $\frac{s-2}{2}$ paths of order 2, namely $[(1, x), (1, x^{-1})]$, $x \notin \{0, 1, -1\}$. Therefore, the adjacency matrices for the graphs G_0 and G_1 are $A(G_1) = A(K_s)$ and $A(G_2) = F_s$. Thus, and by using Lemma 3.1, we have the following lemma.

Lemma 3.6. *The adjacency matrix of the clean graphs $CL(Z_p(+)Z_p)$ and $CL(GF(p^2))$ can be expressed as*

$$A(CL(Z_p(+)Z_p)) = \begin{bmatrix} A(K_{p^2-p}) & J_{p^2-p} \\ J_{p^2-p} & F_{p^2-p} \end{bmatrix} \textit{ and } A(CL(GF(p^2))) = \begin{bmatrix} A(K_{(p^2-1)}) & J_{(p^2-1)} \\ J_{(p^2-1)} & F_{(p^2-1)} \end{bmatrix},$$

respectively. Moreover, the spectrum of the matrices $A(CL(Z_p(+)Z_p))$ and $A(CL(GF(p^2)))$ are given by equation (3) with $s = p^2 - p$ and $s = p^2 - 1$, respectively.

Remark 3.7. Since the adjacency matrix of $CL\left(Z_{p^2}\right)$ and $CL(Z_p(+)Z_p)$ are equal, the graphs $CL(Z_p(+)Z_p)$ and $CL\left(Z_{p^2}\right)$ are isomorphic.

Now we are in a position to find the adjacency matrix of the clean graph $CL(Z_p \oplus Z_p)$.

Let $R = Z_p \oplus Z_p$. Consider the clean graph $G = CL(R)$. The vertex set $V(G)$ is the set of all ordered pairs $((a, b), (c, d))$ where, (a, b) is an idempotent in R and (c, d) is a unit. By Remark 3.8, we review some of the basic properties of $V(G)$.

Remark 3.8. The element (c, d) is a unit in R if and only if c and d are units in Z_p, i.e. if $a, b \in Z_p \setminus \{0\}$. Note that, $(c, d)^{-1} \neq (c, d)$ unless $c, d \in \{1, p-1\}$. The set of idempotent elements in R is $I = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Since the number of units in R is $(p-1)^2$, so the order of $V(G)$ is $4(p-1)^2$.

Lemma 3.9. *The adjacency matrix of the clean graph $CL(Z_p \oplus Z_p)$ is given by*

$$A(CL(R)) = \begin{bmatrix} A(K_{(p-1)^2}) & J_{(p-1)^2} & J_{(p-1)^2} & J_{(p-1)^2} \\ J_{(p-1)^2} & Q_s & J_{(p-1)^2} & Q_s \\ J_{(p-1)^2} & J_{(p-1)^2} & Q_s & Q_s \\ J_{(p-1)^2} & Q_s & Q_s & Q_s \end{bmatrix}, \tag{9}$$

where Q_s is defined by (2).

Proof. Consider R to be $Z_p \oplus Z_p$. Let $G = CL(R)$ and let I to be the set of idempotent elements in R. Then, the vertex set $V(G)$ can be partitioned as a union of the sets:

$$V_i = \{((a, b), (c, d)) : (a, b) \in I, c, d \in Z_p \setminus \{0\}\}, \ \ 1 \leq i \leq 4,$$

where $(0, 0)$ is the initial component of each vertex in V_1, $(1, 0)$ is the initial component of each vertex in V_2 and so on. Now, let G_i be the induced subgraph corresponding to the vertex sets V_i. Since G_1 is isomorphic to the complete graph $K_{(p-1)^2}$, then $A(G_1) = A(K_{(p-1)^2})$. Additionally, every vertex in G_1 is adjacent to every vertex in G_2, G_3 and G_4. Also, since the initial components in V_2 and V_3 are $(1, 0)$ and $(0, 1)$ respectively, every vertex in G_2 is adjacent to every vertex in G_3. Hence, $A(G)$ containing the block matrix $J_{(p-1)^2}$ corresponding to the adjacency between the vertices of V_1 and V_j for $2 \leq j \leq 3$, and corresponding to the adjacency between a vertices of V_2 and V_3. In the other cases, for $i = j \geq 2$ and for $i = 2$ and $3 \leq j \leq 4$, the adjacency between the vertices of V_i and V_j depends on the second component. Since $(1, 1), (1, p-1), (p-1, 1)$ and $(p-1, p-1)$ are the inverses for themselves, thus the block matrix Q describes the adjacency related to the subgraphs that will be induced. Therefore, the adjacency matrix for the graph $CL(R)$ is given by (9). ☐

Remark 3.10. The spectrum of the matrix $A(CL(= Z_p \oplus Z_p))$ has been investigated in.[20]

4 The eigensharp property for the clean graphs $CL\left(Z_{p^2}\right)$, $CL\left(Z_p(+)Z_p\right)$, $CL\left(GF(p^2)\right)$, and $CL(Z_p \oplus Z_p)$

In this section, we show that the clean graphs $CL\left(Z_{p^2}\right)$, $CL\left(Z_p(+)Z_p\right)$, $CL\left(GF(p^2)\right)$, and $CL(Z_p \oplus Z_p)$ are eigensharps.

Theorem 4.1. *The clean graph $CL(Z_{p^2})$ is eigensharp.*

Proof. Let $G = CL(Z_{p^2})$ and $U = Z_{p^2} \setminus \{ip : 0 \leq i \leq p-1\}$. Recall that $V(G) = \{(e, u) : e \in \{0, 1\}, u \in U\}$ has an order $2\left(p^2 - p\right)$.

Now, from Lemma 3.2, the adjacency matrix of the graph G is

$$A(G) = \begin{bmatrix} A(K_{(p^2-p)}) & J_{(p^2-p)} \\ J_{(p^2-p)} & F_{(p^2-p)} \end{bmatrix}.$$

Therefore, by Lemma 3.1 we have

$$\sigma(A(G)) = \begin{pmatrix} 0 & 1 & -1 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \frac{p^2-p-4}{2} & \frac{3\left(p^2-p\right)-4}{2} & 1 & 1 & 1 \end{pmatrix},$$

where λ_1, λ_2 and λ_3 are the roots of the polynomial

$$p\left(\lambda\right) = \lambda^3 - (p^2-p)\lambda^2 - \left(\left(p^2-p\right)^2 - \left(p^2-p\right) + 1\right)\lambda + 2\left(p^2-p\right)\right).$$

Thus $bp(G) \geq \max\left\{r_-\left(A(CL(Z_{p^2}))\right), r_+\left(A(CL(Z_{p^2}))\right)\right\} = \frac{3\left(p^2-p\right)-2}{2}$. On the other hand, the vertex set $V(G)$ can be partitioned into the sets $V_0 = \{(0,x) : x \in U\}$ and $V_1 = \{(1,x) : x \in U\}$. Let G_0 and G_1 be the induced subgraph corresponding to the vertex sets V_0 and V_1, respectively. Then G_0 is clearly isomorphic to the complete graph K_{p^2-p} and thus by,[12] G_0 can be covered by p^2-p-1 bicliques. Let $B\left(V_0, V_1\right)$ be the biclique subgraph related to the sets V_0 and V_1. Then, $B\left(V_0, V_1\right) \simeq K_{p^2-p,p^2-p}$ and $B\left(V_0, V_1\right)$ covers all edges with endpoints from the independent sets V_0 and V_1.

Consider the graph G_1. Since the vertices $(1,1)$ and $(1,p-1)$ are not adjacent to any other vertex in the set V_1; the graph G_1 is disconnected. Also, the vertex $(1,x)$ such that $x \notin \{1, p-1\}$ is adjacent to a vertex $(1, x^{-1})$. Thus G_1 contains $\frac{p^2-p-2}{2}$ components that isomorphic to the line graph P_2, namely $P_{xx^{-1}}$, which are bicliques. So, we have a biclique partition of $CL\left(Z_{p^2}\right)$ with cardinality $\frac{3\left(p^2-p\right)-2}{2}$. Hence, $bp(G) = \frac{3\left(p^2-p\right)-2}{2}$ and so $CL(Z_{p^2})$ is eigensharp. □

In Remark 3.7, we have noticed that the two graphs $CL\left(Z_{p^2}\right)$ and $CL(Z_p(+)Z_p)$ are isomorphic. Hence we conclude the following.

Corollary 4.2. *The clean graph $CL(Z_p(+)Z_p)$ is eigensharp.*

Now, consider the clean graph $G = CL(GF(p^2))$. Recall that, a is the unity of the ring ring $GF(p^2)$, and so a is the only nonzero idempotent element in this ring, see Remark 3.5. Also, by Lemma 3.6, we have determined the adjacency matrix of G.

In the following theorem, we show that G is an eigensharp graph.

Theorem 4.3. *The clean graph $CL\left(GF(p^2)\right)$ is eigensharp.*

Proof. Let $G = CL(GF(p^2))$. By Lemma 3.1, we have

$$bp(G) \geq \max\left\{r_-\left(CL\left(GF(p^2)\right)\right), r_+\left(CL\left(GF(p^2)\right)\right)\right\} = \frac{3p^2-5}{2}.$$

On the other hand, G has $2\left(p^2-1\right)$ vertices such that $V(G)$ can be partitioned into the sets $V_1 = \{(0,x) : x \in GF(p^2), x \neq 0\}$ and $V_2 = \{(a,y) : y \in GF(p^2), x \neq 0\}$. Let G_1 and G_2 be the induced subgraphs according to the vertex sets V_1 and V_2, respectively. Then $G_1 \simeq K_{p^2-1}$, and so by[12] G_1 can be covered by p^2-2 bicliques. Moreover, let $B\left(V_1, V_2\right)$ be the biclique subgraph induced by the sets of vertices V_1 and V_2, respectively. In fact $B\left(V_1, V_2\right) \simeq K_{p^2-1,p^2-1}$ and $B\left(V_1, V_2\right)$ cover all edges between the independent sets V_1 and V_2.

Now, consider the graph G_2. Since a and $-a$ are the inverses for themselves, it is clearly that, the vertices (a,a) and $(a,-a)$ are not adjacent to any vertex in V_2. Thus G_2 is a disconnected graph decomposed as the union of $\frac{p^2-3}{2}$ paths of order 2, namely $[(a,y),(a,y^{-1})]$ and $y \notin \{0, a, -a\}$. Thus, G_2 can be covered by $\frac{p^2-3}{2}$ bicliques. Therefore, $CL\left(GF(p^n)\right)$ has a biclique partition of cardinality $\frac{3p^2-5}{2}$ and hence, $CL\left(GF(p^2)\right)$ is eigensharp. □

Theorem 4.4. *The clean graph $CL\left(Z_p \oplus Z_p\right)$ is eigensharp.*

Proof. Let $G = CL\left(Z_p \oplus Z_p\right)$. Then G has $4\left(p-1\right)^2$ vertices such that each vertex is defined by an idempotent element from the set $I = \{(0,0),(0,1),(1,0),(1,1)\}$ and a unit from the set $U = \{(a,b) : a,b \in Z_p \diagdown \{0\}\}$. See Remark 3.8.

From Lemma 3.9, we determined the adjacency matrix of the graph G. Hence, by[20] we conclude that $bp(G) \geq max\left\{r_{-}(A(G)), r_{+}\left(A(G)\right)\right\} = 5\left(\frac{(p-1)^2-2}{2}\right)$.

We introduce a bicluique partition for G with cardinality equals to $5\left(\frac{(p-1)^2-2}{2}\right)$. The vertex set $V(G)$ can be partitioned as a union of the sets:

$$V_z = \{((a,b),(c,d)) : (a,b) \in I, c,d \in Z_p \diagdown \{0\}\}, 1 \leq z \leq 4,$$

where $(0,0)$ is the initial component of each vertex in V_1, $(1,0)$ is the initial component of each vertex in V_2 and so on. Now, let G_i be the induced subgraph corresponding to the vertex sets V_z.

For $(i,j) \neq (0,0)$, we can assume a partition for $V_z : 2 \leq z \leq 4$ as follows:

$$\widetilde{B}_{ij} = \{((i,j),(x,y)) : x,y \in \{1,p-1\}\},$$

$$B_{ij} = \{((i,j),(s,t)) : s,t \notin \{1,p-1\}, ((i,j),(s^{-1},t^{-1})) \notin B_{ij}\},$$

$$B_{ij}^{-1} = \{((i,j),(s^{-1},t^{-1})) : ((i,j),(s,t)) \in B_{ij}\},$$

where $\widetilde{B}_{ij}$ has order 4, and B_{ij}, B_{ij}^{-1} have order $\frac{(p-1)^2-4}{2}$. Actually, the subgraph G_1 is isomorphic to the complete graph $K_{(p-1)^2}$, as every two different vertices in $V(G_1)$ are adjacent. Thus G_1 is covered by at least $(p-1)^2 - 1$ disjoint bicliques; namely $\Pi = \{D_t, 1 \leq t \leq (p-1)^2 - 1\}$. Since, every vertex in V_1 is adjacent to every vertex in V_2, V_3 and V_4, we get the biclique subgraph $B\left(V_1, W\right)$, where $W = V_2 \cup V_3 \cup V_3$ is induced by the sets of vertex V_1 and W. In fact $B\left(V_1, W\right) \simeq K_{(p-1)^2, 3(p-1)^2}$. Another complete bipartite graph can be funded as follows: define $\widetilde{B}_1 = \{((0,1),(x,y)) : x,y \notin \{1,p-1\}\}$ and $\widetilde{B}_2 = \{((1,0),(x,y)) : x,y \notin \{1,p-1\}\}$ as subsets of G_2 and G_3 respectively. Because every vertex in $\widetilde{B}_1$ is adjacent to every vertex in $\widetilde{B}_2$, the subgraph $B(\widetilde{B}_1, \widetilde{B}_2)$ is a biclique graph isomorphic to the graph $K_{(p-1)^2-4,(p-1)^2-4}$. Furthermore, if we take $\pounds = \{S_{ij}(u) : (i,j) \neq (0,0), u \in B_{ij}\}$ to be the set of all induced stars associated with a vertex u, through every vertex in B_{ij} and among all the vertex sets $V(G_i)$, then $\pounds$ is a family of $3\left(\frac{(p-1)^2-4}{2}\right)$ distinct stars. Now, assume that $u, v \in W$ such that $u, v \notin \widetilde{B}_{ij}$. Since u and v are adjacent, then $u = ((i_0,j_0),(x,y)) \in B_{i_0j_0}$ and $v = ((i_1,j_1),(x^{-1},y^{-1})) \in B_{i_1j_1}$ with $(i_0,j_0) \neq (0,0) \neq (i_1,j_1)$. Thus $uv \in S_{i_0j_0}(u)$. Therefore, $\Gamma = \left\{\Pi, K(G_1,W), B(\widetilde{B}_1,\widetilde{B}_2), \pounds\right\}$ is a bicluiqe partition of G with cardinality $5\left(\frac{(p-1)^2-2}{2}\right)$ and hence, G is eigensharp. □

AMS 2010 Subject Classification: $13M99, 05C25, 05C50, 05C70$.

References

[1] H. A. Abdelkarim, The Geodetic Number for the Unit Graphs Associated with Rings of Order P and P^2, Symmetry, 15, 1799 (2023).

[2] S. Akbari, M.Habibi, A. Majidinya, and R. Manaviyat, On the Idempotent Graph of a Ring, Journal of Algebra and Its Applications, 12 (6), 1350003 (2013).

[3] S. Akbari, D. Kiani, F. Mohammadi and S. Moradi, The Total Graph and Regular Graph of a Commutative Ring, Journal of Pure and Applied Algebra, 213: 2224-2228 (2009).

[4] D. F. Anderson and M. Naseer, "Beck's Coloring of a Commutative Ring," Journal of Algebra, vol. 159, no. 2, pp. 500–514, (2021)

[5] D. Anderson and A. Badawi, The Total Graph of a Commutative Ring, Journal of Algebra, 320: 2706-2719 (2008).

[6] R. G. Artes and Jr. R. D. Dignos, Tree Covers of Graphs, Applied Mathematical Sciences, 8 (150): 7469 - 7473 (2014).

[7] I. Beck. Coloring of Commutative Rings. Journal of Algebra, 116(1): 208-226 (1988).

[8] M. Dutta, S. Kalita , H. K. Saikia, Graphs in Automata, Electronic Journal of Mathematical Analysis and Applications Vol. 10(2) Jul: 105-114 (2022).

[9] B. Fine, Classification of Finite Rings of Order p^2, Mathematics Magazine, 66(4): 248-252 (1993).

[10] G. Fan, Covering Graphs by Cycles, SIAM Journal on Discrete Mathematics, 5 (4): 491–496 (1992).

[11] E. Ghorbani and H.R. Maimani,. On Eigensharp and Almost Eigensharp Graphs, Linear Algebra and its Applications429: 2746-2753 (2008).

[12] R.L. Graham and H.O. Pollak, On the Addressing Problem for Loop Switching, Bell System Technical Journal, 50: 2495-2519 (1971).

[13] M. Habibi, E. Y. Celikel, and C. Abdioglu, Clean Graph of a Ring, Journal of Algebra and Its Applications, 20(9): 2150156 (2021).

[14] T. Kratzke and B, Reznick, West, D. Eigensharp graphs: Decomposition into Complete Bipartite Subgraphs.Transactions of the AMS - American Mathematical Society, 308,: 637-653 (1988) .

[15] D.S. Nau, G. Markowski, M. A. Woodbury and D. B. Amos A Mathematical Analysis of Human Leukocyte Antigen Serology, Mathematical Biosciences 40: 243-270 (1978).

[16] W. K. Nicholson, Lifting Idempotents and Exchange Rings, Transactions of the AMS - American Mathematical Society, 229: 269-278 (1977).

[17] Z. Z. Petrovi´c, and Z. Pucanovi´c, The Clean Graph of a Commutative Ring, Ars Combinatoria, 134: 363-378 (2017).

[18] T. Pinto. Biclique Covers and Partitions, The Electronic Journal of Combinatoric 21: 1-19 (2014).

[19] V. Ramanathan, C. Selvaraj, A. Altaf, and S. Pirzada. Classification of Rings Associated with the Genus of Clean Graphs, Algebra Colloquium, 31: 451-466 (2024).

[20] E. Rawashdeh, H.A. Abdelkarim, E. Rawshdeh, The Spectrum of Certain Large Block Matrix, Euorpean Journal of pure and applied mathematics, 17 (4): 2550-2561 (2024).

[21] W. C. Waterhouse, Rings with Cyclic Additive Group, The American Mathematical Monthly, 71: 449-450 (1964).