Modified Compact Finite Difference Methods for Solving Fuzzy Time Fractional Wave Equation in Double Parametric Form of Fuzzy Number

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Abstract

Fuzzy fractional partial differential equations have become a powerful approach to handle uncertainty or imprecision in real-world modeling problems. In this article, two compact finite difference schemes, the compact Crank-Nicolson and the compact center time center space methods, were developed and used to obtain a numerical solution for fuzzy time fractional wave equations in the double parametric form. The principles of fuzzy set theory are utilized to perform a fuzzy analysis and formulate the proposed numerical schemes. The Caputo formula is used to define the time-fractional derivative considered. The stability of the proposed schemes is analyzed by means of the Von Neumann method. To illustrate the practicality of the numerical methods, a specific numerical instance was performed. The outcomes were showcased through tables and figures, revealing the efficacy of the schemes in terms of accuracy and their ability to decrease computational expenses.

Keywords: Compact finite difference methods; Fuzzy Caputo formula; Double parametric form; Fuzzy time fractional wave equation

1. Introduction

Partial differential equations of fractional order are often used in fields like physics, engineering, finance, and medical sciences. They provide more accurate and detailed models than traditional integer-order differential equations [1-4]. Recently, many research studies have been focused on the fractional wave equation, which is relevant in acoustics, electromagnetism, and seismic analysis. This equation also describes the movement of objects like strings, wires, and fluid surfaces [5-7]. Any wave or motion can be represented as a combination of sine waves [8-9]. Solving the fractional wave equation analytically is often difficult, so researchers used numerical or approximation methods. Jafari and Daftardar-Gejji (2006) [10] utilized the Adomian decomposition method to obtain approximate solutions for both nonlinear and linear fractional wave equations. Jafari and Momani (2007) [11] used the Homotopy perturbation method to solve these equations. Odibat and Momani (2006) [12] also used the Adomian decomposition technique for the time fractional wave equation (TFWE) with boundary conditions. They described the fractional derivative using the Caputo sense and found that the Adomian decomposition method to be an effective approach for solving the TFWE.

One commonly used numerical approach is finite difference schemes, which many researchers have discussed [13-17]. This method is important for solving fractional wave equations because it allows us to

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break the equation down and solve it numerically, accurately handling the system's non-local and memory effects. Ghode et al. (2021) [16] developed an explicit finite difference method to solve the TFWE. Liu et al. (2022) [17] introduced a method for solving the initial boundary value problems of variable-order TFWE by combining central differences in space with H2N2 estimation in time. The H2N2 is a technique for approximating solutions to differential equations over time where it involves discretizing the time domain to create a sequence of approximate solutions They used an energy analysis method to discuss the convergence of the proposed method and numerical examples have been presented to show its effectiveness. The Compact finite difference methods offer several advantages over high-order finite difference methods. High-order methods require more grid dots, which increases computational efforts. Compact FDM solve this problem by using derivatives of function values at the nodes of the independent variable. This approach provides accurate and highly efficient solutions compared to classical FDM. In numerical methods of modeling processes with fractional wave equations, the variables and parameters are considered exact. However, due to experimental and measurement errors, these parameters can be unclear and uncertain. This has led to the use of fuzzy fractional wave equations. In the past few years, there has been a rise in focus on studying fuzzy fractional wave equations, with various contributions documented in prior research studies [18-21].

By reviewing the literature and to the best of our knowledge, it was found that there are no research studies that solve the fuzzy time fractional wave equation (FTFWE) using the compact finite difference methods. The aim of this paper is to find a numerical solution for the FTFWE. In the solution process, two different compact finite methods are developed and applied for solving the FTFWE under the double parametric form.

2. Preliminaries

We introduce in this section the relevant theorems and definitions that will be utilized throughout the paper.

Definition 1: r-level set [10]

The fuzzy r-level set $\widetilde{U}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e. $\widetilde{U}_r = \{x \in X | \mu_{\tilde{u}} > r, r \in [0,1]\}$.

Definition 2: Fuzzy numbers [10]

Fuzzy numbers are a specific subset of real numbers that represent uncertain values and are associated with degrees of membership within a set. A fuzzy number μ is termed a triangular fuzzy number if it is defined by three parameters $a < b < c$, where the graph of $\mu(x)$ forms a triangle with the base spanning the interval $[a, c]$ and the vertex at $x = b$. The membership function of such a triangular fuzzy number is given by:

$$\mu_{\tilde{u}_r} = \begin{cases} 0, & if\ x < a \\ \dfrac{x-a}{b-a}, & if\ a \leq x \leq b \\ \dfrac{c-x}{c-b}, & if\ b \leq x \leq c \\ 0, & if\ x > c \end{cases}$$

where the r-level sets of triangular fuzzy numbers are

$$[\mu]_r = [a + r(b-a), c - r(c-b)], r \in [0, 1]$$

Definition 3: Double parametric form of fuzzy numbers [10]

Using the single parametric form, we have $\widetilde{U} = [\underline{u}(r), \overline{u}(r)]$. Now this may can written as crisp number using double parametric form

$\widetilde{U}(r, \beta) = \beta[\overline{u}(r) - \underline{u}(r)] + \underline{u}(r)$ where r and $\beta \in [0,1]$.

3. Wave Equation with Time Fractional Derivative in Fuzzy Form

In this section, the overall structure of the FTFWE is presented based on Hukuhara derivative using a fuzzy technique called the double parametric form.

Take into account the representation of FTFWE, and by incorporating the given boundary and initial conditions. [19]

$$\frac{\partial^\alpha \tilde{u}(x,t,\alpha)}{\partial^\alpha t} = \tilde{k}(x,t)\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + \tilde{b}(x,t) \quad , 1 < \alpha \leq 2, \qquad (x,t) \epsilon\, \Omega = [0, L] \times [0, T]$$

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$$\tilde{u}(x,0)=\widetilde{\emptyset_1}(x), \frac{\partial\tilde{u}}{\partial t}(x,0)=\widetilde{\emptyset_2}(x), \tilde{u}(0,t)=\tilde{v},\ \tilde{u}(l,t)=\tilde{y}, \quad (1)$$

In accordance with the singular parametric form of Hukuhara derivatives, we can express Eq. (1) in the following manner:

$$\left[\frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}, \frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right] = [\ \underline{\mathrm{k}}(x,t;r), \overline{\mathrm{k}}(x,t;r)] \left[\frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}, \frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2}\right] + [\ \underline{b}(x,t;r), \overline{b}(x,t;r)]$$

(2)

Experiencing the effects of uncertain boundaries and initial conditions:

$$[\underline{u}(x,0;r), \overline{u}(x,0;r)] = [\underline{\emptyset_1}(x,0;r), \overline{\emptyset_1}(x,0;r)]$$

$$\left[\frac{\partial \underline{u}}{\partial t}(x,0;r), \frac{\partial \overline{u}}{\partial t}(x,0;r)\right] = [\underline{\emptyset_2}(x,0;r), \overline{\emptyset_2}(x,0;r)]$$

$$[\underline{u}(0,t;r), \overline{u}(0,t;r)] = [\underline{v}(0,t;r), \overline{v}(0,t;r)]$$

$$[\underline{u}(l,t;r), \overline{u}(l,t;r)] = [\underline{y}(l,t;r), \overline{y}(l,t;r)]$$

Now, by using the double parametric form in [19], the Eq.(2) is rewritten as the following:

$$\left[\beta\left(\frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t} - \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right) + \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right] = [\ \beta\ [\overline{\mathrm{k}}(x,t,r) - \underline{\mathrm{k}}(x,t,r)] + \underline{\mathrm{k}}(x,t,r)] \left[\beta\left(\frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2} - \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}\right) + \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}\right] + \left[\beta\ \left(\overline{b}(x,t;r) - \underline{b}(x,t;r)\right) + \underline{b}(x,t;r)\right]$$

(3)

Subjected to fuzzy initial and boundary conditions

$$\left(\beta\ \left(\overline{u}(x,0;r) - \underline{u}(x,0;r)\right) + \underline{u}(x,0;r)\right) = \left(\beta\ \left(\overline{\emptyset_1}(x;r) - \underline{\emptyset_1}(x;r)\right) + \underline{\emptyset_1}(x;r)\right)$$

$$\left(\beta\left(\frac{\partial \overline{u}}{\partial t}(x,0;r) - \frac{\partial \underline{u}}{\partial t}(x,0;r)\right) + \frac{\partial \underline{u}}{\partial t}(x,0;r)\right) = \left(\beta\ \left(\overline{\emptyset_2}(x;r) - \underline{\emptyset_2}(x;r)\right) + \underline{\emptyset_2}(x;r)\right)$$

$$\left(\beta\ \left(\overline{u}(0,t;r) - \underline{u}(0,t;r)\right) + \underline{u}(0,t;r)\right) = \left(\beta\ \left(\overline{v}(x;r) - \underline{v}(x;r)\right) + \underline{v}(x;r)\right)$$

$$\left(\beta\ \left(\overline{u}(l,t;r) - \underline{u}(l,t;r)\right) + \underline{u}(l,t;r)\right) = \left(\beta\ \left(\overline{y}(x;r) - \underline{y}(x;r)\right) + \underline{y}(x;r)\right)$$

where $\beta \in [0,1]$.

Now the fuzzy functions are converted to a system of equations as follows:

$$\frac{\partial^\alpha \tilde{u}(x,t;r,\beta)}{\partial^\alpha t} = \beta\left(\frac{\partial^\alpha \overline{u}(x,t,\alpha;r)}{\partial^\alpha t} - \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}\right) + \frac{\partial^\alpha \underline{u}(x,t,\alpha;r)}{\partial^\alpha t}$$

$$\frac{\partial^2 \tilde{u}(x,t,r,\beta)}{\partial x^2} = \left(\beta\ \left(\frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2} - \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}\right) + \frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2}\right)$$

$$\tilde{\mathrm{k}}(x,t;r,\beta) = \left(\overline{\mathrm{k}}(x,t,r) - \underline{\mathrm{k}}(x,t,r)\right) + \underline{\mathrm{k}}(x,t,r)$$

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$$\tilde{b}(x,t;r,\beta) = \left(\beta\ \left(\bar{b}(x,t;r) - \underline{b}(x,t;r)\right) + \underline{b}(x,t;r)\right)$$

$$\tilde{u}(x,0,r,\beta) = \left(\beta\ \left(\bar{u}(x,0;r) - \underline{u}(x,0;r)\right) + \underline{u}(x,0;r)\right)$$

$$\frac{\partial\tilde{u}}{\partial t}(x,0;r,\beta) = \left(\beta\ \left(\frac{\partial\bar{u}}{\partial t}(x,0;r) - \frac{\partial\underline{u}}{\partial t}(x,0;r)\right) + \frac{\partial\underline{u}}{\partial t}(x,0;r)\right)$$

$$\widetilde{\emptyset_1}(x,r,\beta) = \left(\beta\ \left(\overline{\emptyset_1}(x;r) - \underline{\emptyset_1}(x;r)\right) + \underline{\emptyset_1}(x;r)\right)$$

$$\widetilde{\emptyset_2}(x,r,\beta) = \left(\beta\ \left(\overline{\emptyset_2}(x;r) - \underline{\emptyset_2}(x;r)\right) + \underline{\emptyset_2}(x;r)\right)$$

$$\tilde{u}(0,t,r,\beta) = \left(\beta\ \left(\underline{u}(0,t;r) - \bar{u}(0,t;r)\right) + \underline{u}(0,t;r)\right)$$

$$\tilde{v}(x,r,\beta) = \left(\beta\ \left(\bar{v}(x;r) - \underline{v}(x;r)\right) + \underline{v}(x;r)\right)$$

$$\tilde{u}(l,t,r,\beta) = \left(\beta\ \left(\underline{u}(l,t;r) - \bar{u}(l,t;r)\right) + \underline{u}(l,t;r)\right)$$

$$\tilde{y}(x,r,\beta) = \left(\beta\ \left(\bar{y}(x;r) - \underline{y}(x;r)\right) + \underline{y}(x;r)\right)$$

By plugging these values into Eq. (3), we obtian:

$$\frac{\partial^{\alpha}\tilde{u}(x,t,\alpha;r,\beta)}{\partial^{\alpha}t} = \tilde{k}(x,t;r,\beta)\frac{\partial^2\tilde{u}(x,t;r,\beta)}{\partial x^2} + \tilde{b}(x,t;r,\beta) \quad , \qquad 0 \le r \le 1, 0 \le \beta \le 1$$

$$\tilde{u}(x,0;r,\beta) = \widetilde{\emptyset_1}(x,r,\beta)\ , \frac{\partial\tilde{u}}{\partial t}(x,0;r,\beta) = \widetilde{\emptyset_2}(x,r,\beta),\quad \tilde{u}(0,t,\beta) = \tilde{v}(x,r,\beta) \quad , \quad \tilde{u}(l,t,\beta) = \tilde{y}(x,r,\beta)$$
(4)

The single parametric form allows us to determine the upper and lower bounds of the solutions by assuming $\beta = 1$ and $\beta = 0$, respectively. This can be expressed as follows

$\tilde{u}(x,t;r,1) = \overline{u}(x,t;r)$ *and* $\tilde{u}(x,t;r,0) = \underline{u}(x,t;r)$.

4. Taylor Series and Derivatives Approximation

We can assume $u_{i+1}^{n}$and u_{i+1}^{n} about (x_i, t_n) by Taylor series to derive the FCFD approximations for the spatial derivatives.

$$\left.\begin{aligned} u_{i+1}^{n} &= u_i^n + h\ \left(\frac{\partial u}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \cdots \\ u_{i-1}^{n} &= u_i^n - h\ \left(\frac{\partial u}{\partial x}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \cdots \end{aligned}\right\} \tag{5}$$

The first derivatives of u_{i+1}^{n} and u_{i-1}^{n} is

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_{i+1}^n &= \left(\frac{\partial u}{\partial x}\right)_i^n + h\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \cdots \\ \left(\frac{\partial u}{\partial x}\right)_{i-1}^n &= \left(\frac{\partial u}{\partial x}\right)_i^n - h\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \cdots \end{aligned}\right\} \tag{6}$$

The second derivatives of u_{i+1}^{n} and u_{i-1}^{n} is

$$\left.\begin{aligned} \left(\frac{\partial^2 u}{\partial x^2}\right)_{i+1}^n &= \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + h\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \frac{h^3}{6}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + \cdots \\ \left(\frac{\partial^2 u}{\partial x^2}\right)_{i-1}^n &= \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - h\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \frac{h^2}{2}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n - \frac{h^3}{6}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + \cdots \end{aligned}\right\} \tag{7}$$

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By using Eq.(5-7) the first and second spatial derivatives are approximated respectively to obtain:

$$\left(\frac{\partial u}{\partial x}\right)_i^n = \frac{\delta_x/2h}{(1+\frac{1}{6}\delta^2{}_x)}u_i^n + \frac{h^4}{180}\left(\frac{\partial^5 u}{\partial x^5}\right)_i^n + O(h^5) \quad (8)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n = \frac{\delta^2{}_x/h^2}{(1+\frac{1}{12}\delta^2{}_x)}u_i^n + \frac{h^4}{240}\left(\frac{\partial^6 u}{\partial x^6}\right)_i^n + O(h^6) \quad (9)$$

where $\delta_x = u_{i+1}^n - u_{i-1}^n$ and $\delta^2{}_x = u_{i+1}^n - 2u_i^n + u_{i-1}^n$ for $u_i^n |\, 0 \le i \le M, 0 \le n \le N$

$$(1 + \tfrac{1}{6}\,\delta^2{}_x)u_i^n = \tfrac{1}{6}(u_{i+1}^n + 4u_i^n + u_{i-1}^n),\ 1 \le i \le M-1 \quad (10)$$

$$(1 + \tfrac{1}{12}\,\delta^2{}_x)u_i^n = \tfrac{1}{12}(u_{i+1}^n + 10u_i^n + u_{i-1}^n),\ 1 \le i \le M-1 \quad (11)$$

5. CCTCS Scheme for the Solution of FTFWE

we employ in this section the double parametric form of a fuzzy number to implement the compact centre time centre space (CCTCS) scheme for solving the FTFWE. The Caputo formula is utilized for the time fractional derivative, and a fourth-order accurate compact approximation is applied for the second-order spatial derivative at time level 'n'.

To numerically solve the FTFWE using the CCTCS scheme, we discretize the time fractional derivative given in Eq. (4) through the application of the Caputo formula as presented in [22]. Similarly, the partial derivatives in the governing equation are discretized based on Eq. (9) to yield the desired outcome.

$$\frac{k^{-\alpha}}{\Gamma(3-\alpha)}\left[\underline{u}_i^{n+1} - 2\underline{u}_i^n + \underline{u}_i^{n-1} + \sum_{j=1}^{n-1} b_j\left(\underline{u}_i^{n-j+1} - 2\underline{u}_i^{n-j} + \underline{u}_i^{n-j-1}\right)\right] = \tilde{a}(x)\frac{\delta^2{}_x/h^2}{\left(1+\frac{1}{12}\,\delta^2{}_x\right)}\tilde{u}_i^n + \tilde{b}(x) \quad (12)$$

$$\left(1+\frac{1}{12}\,\delta^2{}_x\right)\left(\frac{k^{-\alpha}}{\Gamma(3-\alpha)}\left[\underline{u}_i^{n+1} - 2\underline{u}_i^n + \underline{u}_i^{n-1} + \sum_{j=1}^{n-1} b_j\left(\underline{u}_i^{n-j+1} - 2\underline{u}_i^{n-j} + \underline{u}_i^{n-j-1}\right)\right]\right) = \frac{\tilde{D}(x)\delta^2{}_x}{h^2}\tilde{u}_i^n + (1+\frac{1}{12}\,\delta^2{}_x)(\tilde{b}(x)) \quad (13)$$

From Eq. (12) and Eq. (13) we obtain:

$$\frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \times \frac{1}{12}\Bigg([(\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1}) - 2(\tilde{u}_{i+1}^{n} + 10\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n}) + (\tilde{u}_{i+1}^{n-1} + 10\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1} + 10\tilde{u}_i^{n-j+1} + \tilde{u}_{i-1}^{n-j+1}\right) - 2\left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right) + \left(\tilde{u}_{i+1}^{n-j-1} + 10\tilde{u}_i^{n-j-1} + \tilde{u}_{i-1}^{n-j-1}\right)\right]\Bigg) = \tilde{D}(x)\left[\frac{\tilde{u}_{i+1}^n - 2\tilde{u}_i^n + \tilde{u}_{i-1}^n}{h^2}\right] + \frac{1}{12}(\tilde{b}_{i+1}^n + 10\tilde{b}_i^n + \tilde{b}_{i-1}^n) \quad (14)$$

$$[(\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1}) - 2(\tilde{u}_{i+1}^{n} + 10\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n}) + (\tilde{u}_{i+1}^{n-1} + 10\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1} + 10\tilde{u}_i^{n-j+1} + \tilde{u}_{i-1}^{n-j+1}\right) - 2\left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right) + \left(\tilde{u}_{i+1}^{n-j-1} + 10\tilde{u}_i^{n-j-1} + \tilde{u}_{i-1}^{n-j-1}\right)\right] = \frac{12\,\tilde{D}(x)\,k^{\alpha}\,\Gamma(3-\alpha)}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + k^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^n + 10\tilde{b}_i^n + \tilde{b}_{i-1}^n\right)$$

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(15)

Now we let $\tilde{p}(r)=\frac{12\,\tilde{D}(x,r)K^{\alpha}\Gamma(3-\alpha)}{h^2}$, and from Eq. (15) we obtain:

$$
\begin{aligned}
&[(\tilde{u}_{i+1}^{n+1}+10\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1})-2(\tilde{u}_{i+1}^{n}+10\tilde{u}_i^{n}+\tilde{u}_{i-1}^{n})+(\tilde{u}_{i+1}^{n-1}+10\tilde{u}_i^{n-1}+\tilde{u}_{i-1}^{n-1})\\
&\quad+\sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1}+\ 10\tilde{u}_i^{n-j+1}+\tilde{u}_{i-1}^{n-j+1}\right)-2\left(\tilde{u}_{i+1}^{n-j}+10\tilde{u}_i^{n-j}+\tilde{u}_{i-1}^{n-j}\right)\right.\\
&\quad\left.+\left(\tilde{u}_{i+1}^{n-j-1}+10\tilde{u}_i^{n-j-1}+\tilde{u}_{i-1}^{n-j-1}\right)\right]\\
&\quad=(\,p\,u_{i+1}^{n}-2\,p\,u_i^{n}+\ p\,u_{i-1}^{n})+k^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^{n}+10\tilde{b}_i^{n}+\tilde{b}_{i-1}^{n}\right)
\end{aligned}
$$

(16)

By simplifying Eq. (16), we get the general formula of CCTCS for FTFWE

$$
\begin{aligned}
&[(\tilde{u}_{i+1}^{n+1}+10\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1})+(\tilde{u}_{i+1}^{n-1}+10\tilde{u}_i^{n-1}+\tilde{u}_{i-1}^{n-1})\\
&\quad+\sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1}+\ 10\tilde{u}_i^{n-j+1}+\tilde{u}_{i-1}^{n-j+1}\right)-2\left(\tilde{u}_{i+1}^{n-j}+10\tilde{u}_i^{n-j}+\tilde{u}_{i-1}^{n-j}\right)\right.\\
&\quad\left.+\left(\tilde{u}_{i+1}^{n-j-1}+10\tilde{u}_i^{n-j-1}+\tilde{u}_{i-1}^{n-j-1}\right)\right]\\
&\quad=(s+2)\tilde{u}_{i+1}^{n}+(20-2s)\,\tilde{u}_i^{n}+(s+2)\tilde{u}_{i-1}^{n}+k^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^{n}+10\tilde{b}_i^{n}+\tilde{b}_{i-1}^{n}\right)
\end{aligned}
$$

(17)

6. Compact Crank-Nicholson for Solution of the FTFWE

In this section, we employ the double parametric form of fuzzy numbers in a compact Crank-Nicholson scheme. We utilize the Caputo formula for time fractional derivatives and a fourth-order accuracy compact approximation at time level $n+1/2$ for the second-order space derivative to effectively solve the FTFWE.

To numerically solve the FTFWE using the compact Crank-Nicholson scheme, we discretize the time fractional derivative in Eq. (4) with the Caputo formula presented in [22]. Additionally, we discretize the second partial derivatives of the same equation at time level $n+1/2$, employing Eq. (9) to obtain the desired results.

$$
\begin{aligned}
&\frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}\times\frac{1}{12}\Bigg([(\tilde{u}_{i+1}^{n+1}+10\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1})-2(\tilde{u}_{i+1}^{n}+10\tilde{u}_i^{n}+\tilde{u}_{i-1}^{n})+(\tilde{u}_{i+1}^{n-1}+10\tilde{u}_i^{n-1}+\tilde{u}_{i-1}^{n-1})\\
&\quad+\sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1}+\ 10\tilde{u}_i^{n-j+1}+\tilde{u}_{i-1}^{n-j+1}\right)-2\left(\tilde{u}_{i+1}^{n-j}+10\tilde{u}_i^{n-j}+\tilde{u}_{i-1}^{n-j}\right)\right.\\
&\quad\left.+\left(\tilde{u}_{i+1}^{n-j-1}+10\tilde{u}_i^{n-j-1}+\tilde{u}_{i-1}^{n-j-1}\right)\right]\Bigg)\\
&\quad=\frac{\tilde{D}(x)}{2}\left[\frac{\tilde{u}_{i+1}^{n+1}-2\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1}}{h^2}+\frac{\tilde{u}_{i+1}^{n}-2\tilde{u}_i^{n}+\tilde{u}_{i-1}^{n}}{h^2}\right]+\frac{1}{12}(\tilde{b}_{i+1}^{n}+10\tilde{b}_i^{n}+\tilde{b}_{i-1}^{n})
\end{aligned}
$$

(18)

$$
\begin{aligned}
&[(\tilde{u}_{i+1}^{n+1}+10\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1})-2(\tilde{u}_{i+1}^{n}+10\tilde{u}_i^{n}+\tilde{u}_{i-1}^{n})+(\tilde{u}_{i+1}^{n-1}+10\tilde{u}_i^{n-1}+\tilde{u}_{i-1}^{n-1})\\
&\quad+\sum_{j=1}^{n-1} b_j\left[\left(\tilde{u}_{i+1}^{n-j+1}+\ 10\tilde{u}_i^{n-j+1}+\tilde{u}_{i-1}^{n-j+1}\right)-2\left(\tilde{u}_{i+1}^{n-j}+10\tilde{u}_i^{n-j}+\tilde{u}_{i-1}^{n-j}\right)\right.\\
&\quad\left.+\left(\tilde{u}_{i+1}^{n-j-1}+10\tilde{u}_i^{n-j-1}+\tilde{u}_{i-1}^{n-j-1}\right)\right]\\
&\quad=\frac{12\,\tilde{D}(x)\,\Delta t^{\alpha}\,\Gamma(3-\alpha)}{2h^2}(\tilde{u}_{i+1}^{n+1}-2\tilde{u}_i^{n+1}+\tilde{u}_{i-1}^{n+1}+u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})\\
&\quad+\Delta t^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^{n}+10\tilde{b}_i^{n}+\tilde{b}_{i-1}^{n}\right)
\end{aligned}
$$

(19)

Now we let $\tilde{s}(r)=\frac{6\,\tilde{D}(x,r)K^{\alpha}\Gamma(3-\alpha)}{h^2}$, and from Eq. (19) we obtain:

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$$[(\tilde{u}_{i+1}^{n+1} + 10\tilde{u}_i^{n+1} + \tilde{u}_{i-1}^{n+1}) - 2(\tilde{u}_{i+1}^{n} + 10\tilde{u}_i^{n} + \tilde{u}_{i-1}^{n}) + (\tilde{u}_{i+1}^{n-1} + 10\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \sum_{j=1}^{n-1} b_j \left[\left(\tilde{u}_{i+1}^{n-j+1} + 10\tilde{u}_i^{n-j+1} + \tilde{u}_{i-1}^{n-j+1}\right) - 2\left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right) + \left(\tilde{u}_{i+1}^{n-j-1} + 10\tilde{u}_i^{n-j-1} + \tilde{u}_{i-1}^{n-j-1}\right)\right] = (s\,\tilde{u}_{i+1}^{n+1} - 2s\,\tilde{u}_i^{n+1} + s\,\tilde{u}_{i-1}^{n+1} + s\,u_{i+1}^{n} - 2s\,u_i^{n} + s\,u_{i-1}^{n}) + \Delta t^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^{n} + 10\tilde{b}_i^{n} + \tilde{b}_{i-1}^{n}\right)$$

(20)

Now if simplifying Eq. (20) we get the formula of compact Crank-Nicholson method for FTFWE

$$[\,(1-s)\,\tilde{u}_{i+1}^{n+1} + (10+2s)\,\tilde{u}_i^{n+1} + (1-s)\,\tilde{u}_{i-1}^{n+1} + (\tilde{u}_{i+1}^{n-1} + 10\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \sum_{j=1}^{n-1} b_j \left[\left(\tilde{u}_{i+1}^{n-j+1} + 10\tilde{u}_i^{n-j+1} + \tilde{u}_{i-1}^{n-j+1}\right) - 2\left(\tilde{u}_{i+1}^{n-j} + 10\tilde{u}_i^{n-j} + \tilde{u}_{i-1}^{n-j}\right) + \left(\tilde{u}_{i+1}^{n-j-1} + 10\tilde{u}_i^{n-j-1} + \tilde{u}_{i-1}^{n-j-1}\right)\right] = (s+2)\,u_{i+1}^{n} + (20-2s)\,u_i^{n} + (s+2)\,u_{i-1}^{n} + \Delta t^{\alpha}\,\Gamma(3-\alpha)\left(\tilde{b}_{i+1}^{n} + 10\tilde{b}_i^{n} + \tilde{b}_{i-1}^{n}\right)$$

(21)

7. Numerical Example

Consider FTFWE [23]

$$\frac{\partial^{\alpha}\tilde{u}(x,t,\alpha)}{\partial t^{\alpha}} = \frac{\partial^{2}\tilde{u}(x,t)}{\partial x^{2}}, \qquad 1<\alpha<2, (x,t)\in\Omega=[0,L]\times[0,T]$$

(22)

Based on the given boundary conditions $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$ and given initial condition

$$\tilde{u}(x,0) = \tilde{\mu}\,[\sin(5\pi x) + 2\,Sin(7\pi x)\,]$$

(23)

The fuzzy number will remain identical when expressed in a single parametric form as follows:

$$\tilde{\mu}(r) = \left[\underline{\mu}(r), \overline{\mu}(r)\right] = [r-1\,, 1-r] \text{ for all } r \in [0,1]$$

The analytical solution of Eq. (22) was given in [23]:

$$\begin{cases} \underline{u}(x,t,\alpha;r) = \underline{\mu}[\sin(5\pi x)\,Cos(5\pi t) + 2\,Sin(7\pi x)Cos(7\pi t)\,] \\ \overline{u}(x,t,\alpha;r) = \overline{\mu}[\sin(5\pi x)\,Cos(5\pi t) + 2\,Sin(7\pi x)Cos(7\pi t)\,] \end{cases}$$

(24)

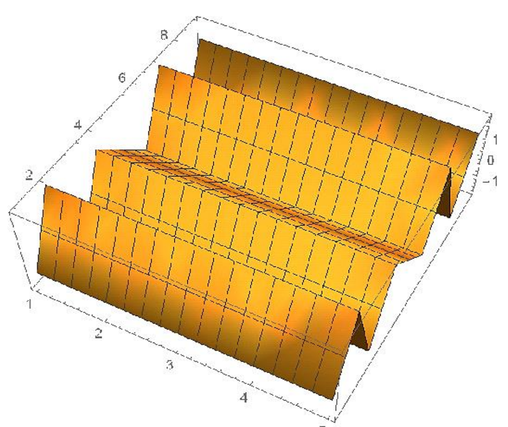

Figure 1. The analytical lower solution of Equation (22) at $t = 0.005, x = 1.8\ and\ r = 0$

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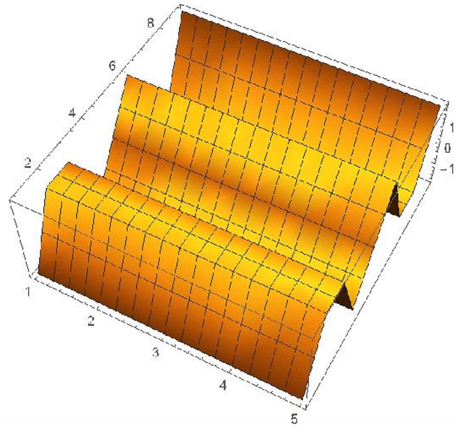

Figure 2. The analytical upper solution of Equation (22) at $t = 0.005, x = 1.8\ and\ r = 0$

The definition of the error for the fuzzy solution of Eq. (22) can be stated as

$$[\tilde{E}]_r = \left|\tilde{U}(t,x;r) - \tilde{u}(t,x;r)\right| = \begin{cases} [\underline{E}]_r = \left|\underline{U}(t,x;r) - \underline{u}(t,x;r)\right| \\ [\overline{E}]_r = \left|\overline{U}(t,x;r) - \overline{u}(t,x;r)\right| \end{cases} \quad (25)$$

At $\Delta x = h = 0.2$ and $\Delta t^{\alpha} = (0.001)^{1.5}$ we have the following results:

The fuzzy number, when expressed in double parametric form, remains unchanged and can be represented as follows $\tilde{\mu}(r) = [\beta(2-2r) + (r-1)]$

Table 1: Numerical solution of Eq. (22) by CCTCS and Compact C-N at $t = 0.05$ and $x = 1.8$ for all $r, \beta \in [0,1]$

		Compact CTCS		Compact C-N	
β	r	$\tilde{u}(1.8,0.05;r,\beta)$	$\tilde{E}(1.8,0.05;r,\beta)$	$\tilde{u}(1.8,0.05;r,\beta)$	$\tilde{E}(1.8,0.05;r,\beta)$
$\beta = 0$ Lower solution	0	-1.8914107694	7.84682×10^{-4}	-1.8915505552	9.24468×10^{-4}
	0.1	-1.7022696925	7.06214×10^{-4}	-1.7023954997	8.32021×10^{-4}
	0.3	-1.32398753845	5.49278×10^{-4}	-1.3240853886	6.47128×10^{-4}
	0.5	-0.9457053847	3.92341×10^{-4}	-0.94577527764	4.62234×10^{-4}
	0.7	-0.5674232308	2.35405×10^{-4}	-0.5674651666	2.7734×10^{-4}
	0.9	-0.1891410769	7.84682×10^{-5}	-0.1891550555	9.24468×10^{-5}
	1	0	0	0	0
	0	1.8914107694	7.84682×10^{-4}	1.8915505552	9.24468×10^{-4}
	0.1	1.7022696925	7.06214×10^{-4}	1.7023954997	8.32021×10^{-4}

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$\beta = 1$ Upper solution	0.3	1.32398753845	5.49278×10^{-4}	1.3240853886	6.47128×10^{-4}
	0.5	0.9457053847	3.92341×10^{-4}	0.94577527764	4.62234×10^{-4}
	0.7	0.5674232308	2.35405×10^{-4}	0.5674651666	2.7734×10^{-4}
	0.9	0.1891410769	7.84682×10^{-5}	0.1891550555	9.24468×10^{-5}
	1	0	0	0	0

Table 2: Numerical solution of Eq. (22) by CCTCS and Compact C-N at $t = 0.005$ and $x = 1.8$ for all $r, \beta \in [0,1]$

		Compact CTCS		Compact C-N	
β	r	$\tilde{u}(1.8, 0.05; r, \beta)$	$\tilde{E}(1.8, 0.05; r, \beta)$	$\tilde{u}(1.8, 0.05; r, \beta)$	$\tilde{E}(1.8, 0.05; r, \beta)$
$\beta = 0.4$ Lower solution	0	-0.3782821539	1.56936×10^{-4}	-0.3783101110	1.84894×10^{-4}
	0.1	-0.3404539385	1.41243×10^{-4}	-0.3404790999	1.66404×10^{-4}
	0.3	-0.2647975077	1.09856×10^{-4}	-0.2648170777	1.29426×10^{-4}
	0.5	-0.1891410776	7.84682×10^{-5}	-0.1891550555	9.24468×10^{-5}
	0.7	-0.1134846462	4.70809×10^{-5}	-0.1134930330	5.54681×10^{-5}
	0.9	-0.0378282154	1.56936×10^{-5}	-0.0378310111	1.56936×10^{-5}
	1	0	0	0	0
$\beta = 0.6$ Upper solution	0	0.3782821539	1.56936×10^{-4}	0.3783101110	1.84894×10^{-4}
	0.1	0.3404539385	1.41243×10^{-4}	0.3404790999	1.66404×10^{-4}
	0.3	0.2647975077	1.09856×10^{-4}	0.2648170777	1.29426×10^{-4}
	0.5	0.1891410776	7.84682×10^{-5}	0.1891550555	9.24468×10^{-5}
	0.7	0.1134846462	4.70809×10^{-5}	0.1134930330	5.54681×10^{-5}
	0.9	0.0378282154	1.56936×10^{-5}	0.0378310111	1.56936×10^{-5}
	1	0	0	0	0

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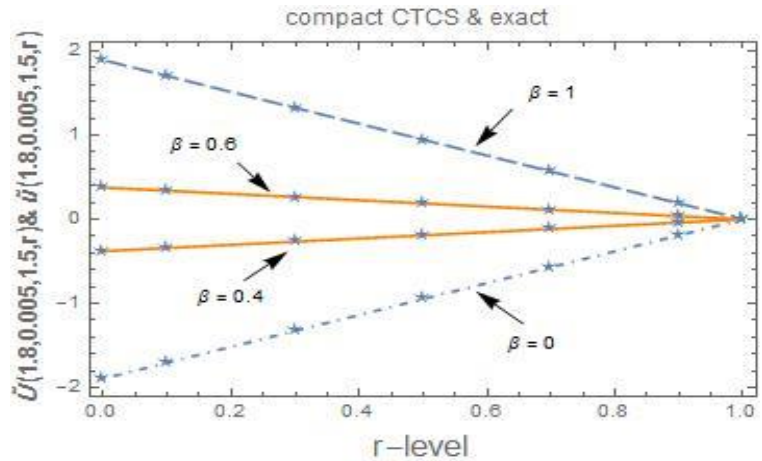

Figure 3. Numerical solution of Eq. (22) by compact CTCS at $t = 0.05$ and $x = 1.8$ for all $r, \beta \in [0,1]$

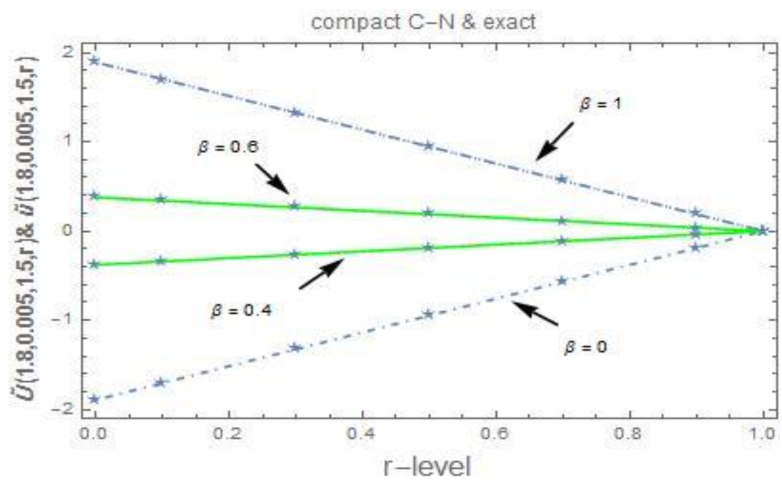

Figure 4. Numerical solution of Eq. (22) by compact C-N at $t = 0.05$ and $x = 1.8$ for all $r, \beta \in [0,1]$

Table 3: Numerical solution of Eq. (22) by CTCS and Compact CTCS at $t = 0.05$ and $x = 1.8$ for all $r, \beta \in [0,1]$

		CTCS		Compact CTCS	
β	r	$\tilde{u}(1.8,0.05;r,\beta)$	$\tilde{E}(1.8,0.05;r,\beta)$	$\tilde{u}(1.8,0.05;r,\beta)$	$\tilde{E}(1.8,0.05;r,\beta)$
	0	-1.892094	1.46803×10^{-3}	-1.8914107694	7.84682×10^{-4}
	0.1	-1.702885	1.32123×3	-1.7022696925	7.06214×10^{-4}
	0.3	-1.324466	1.02762×10^{-3}	-1.32398753845	5.49278×10^{-4}
$\beta = 0$	0.5	-0.946047	7.34015×10^{-4}	-0.9457053847	3.92341×10^{-4}

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Lower solution	0.7	-0.567628	4.40409×10^{-4}	-0.5674232308	2.35405×10^{-4}
	0.9	-0.1892094	1.46803×10^{-4}	-0.1891410769	7.84682×10^{-5}
	1	0	0	0	0
$\beta = 1$ Upper solution	0	1.892094	1.46803×10^{-4}	1.8914107694	7.84682×10^{-4}
	0.1	1.702885	1.32123×10^{-4}	1.7022696925	7.06214×10^{-4}
	0.3	1.324466	1.02762×10^{-4}	1.32398753845	5.49278×10^{-4}
	0.5	0.946047	7.34015×10^{-5}	0.9457053847	3.92341×10^{-4}
	0.7	0.567628	4.40409×10^{-5}	0.5674232308	2.35405×10^{-4}
	0.9	0.1892094	1.46803×10^{-5}	0.1891410769	7.84682×10^{-5}
	1	0	0	0	0

Table 1-2 and Fig.3, Fig.4 shows that the proposed methods have a good agreement with the exact solution at $t = 0.05$, $\alpha = 1.5$ and for some values of $r, \beta \in [0,1]$. In addition, they satisfy the properties of double parametric form of fuzzy number by attaining the triangular fuzzy number shape. In addition, the compact CTCS is slightly more accurate than compact C-N. Also, form table 3 the fuzzy compact CTCS scheme provided slightly more accurate result than fuzzy classical CTCS scheme. Furthermore, As can be seen in Fig.3 and Fig.4, the numerical result for proposed methods are more accurate solution at points that are close to the inflection point ($\beta = 0.5$). It should be noted that the accuracy of the finite difference methods depends upon the value of α .

8. Conclusion

In this paper, two CFDM methods have been developed and applied in order to obtain a numerical solution for FTFWE in double parametric form. The Caputo definition utilized for the time fractional derivative. Both the CCTCS and compact Crank-Nicholson methods produce results that match the properties of fuzzy numbers, specifically taking the shape of triangular fuzzy numbers. The CCTCS scheme has been found to produce more precise solutions compared to the compact Crank-Nicholson method, as demonstrated by a comparative analysis of exact and numerical solutions. In addition, the fuzzy compact CCTCS scheme provided slightly more accurate result than fuzzy classical CTCS scheme.

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