# Finding The HK Completions of Some Sequence Spaces

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**Abstract**

In this paper, we prove the following result: If the space Φ (the finite sequences) is equipped with the norm which is naturally induced by a positive definite Hermitian and diagonally blockwise constructed matrix, then an HK completion exists. Also, we investigate our result through many illustrated examples.

**Keywords:** HK completion; vector space; normed space; sequence

## 1. Introduction and Preliminaries:

An HK space is a Hilbert space of sequences on which coordinate projections are all continuous.
An FK space is a linear topological space of sequences which is a locally convex Frechet space with continuous coordinate projections.
A BK space is the special case of the foregoing in which the Frechet space is a Banach space.
An AD space is an FK space in which Φ is dense.
Throughout this paper, ω will stand for the collection of all complex sequences, C for the complex numbers, and $\ell^2$ will denote the space of all members of ω which are square summable:

$$\text{i.e. } \ell^2 = \{x \,\varepsilon\, \omega / \sum_j |x_i|^2 = \|x_i\|_e^2 < \omega)$$

Let, for $n = 1,2,\ldots, e^n$ be the sequence defined as.
$e_k^n = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$, and finally, let M be the set of all infinite matrices $A = (a_{ij})$ which are Hermitian and positive definite
For $x = \sum_{k=1}^n x_k e^k \varepsilon \phi$ and A ε M, define the norm $\|x\|_A$ induced by A as:

$$\|x\|_A^2 = \sum_{i=1}^n \sum_{j=1}^n x_i \ a_{ij} \ \bar{x}_j$$

This, of course, makes $(\phi, \|.\|_A)$ a normed sequence space: In this paper, we present sufficient conditions for this space to have an HK com**pletion.**
**The** continuity of coordinate projections:
*For n* = 1,2,..., we define the $n^{th}$ coordinate projection $p_n$ as:
$p_n(x) = x_n \qquad (x \,\varepsilon\, \omega)$.
The HK definition requires that $p_n$ be continuous on $(\phi, \|.\|_A)$ for each n. One needs the following.
**Continuity Criterion ([1])**: If X is a linear topological space with topology determined by a family P of seminorms, then the linear functional f is continuous on X iff there exists an s > o and $q_1, q_2, \ldots, q_n$ selected from P such that

$$|f(x)| \leq s \sum_{k=1}^n q_k(x) \text{ for all } x \,\varepsilon\, X$$

In the present setting, $p = \{\|.\|_A\}$; and the following example shows a case where $P_l$ is not continuous on $(\phi, \|.\|_A)$.
**Example:**
For each n, let $E^n = \{x \,\varepsilon\, \omega : x_k = 0 \text{ for all } k > n)$. Consider the matrix $A = (a_{ij})$ where :



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$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ 5 & \text{if } i = j > 1 \\ -2 & \text{if } |i-j| = 1 \\ 0 & \text{elsewhere.} \end{cases}$$

A is clearly Hermitian. As for the positive definiteness:

Let $x \neq 0$ be an arbitrary element in $E^n$. We may assume that $x_n \neq 0$.

$\|x\|_A = x * A\,x$

$= \{\bar{x}_1 - 2\bar{x}_2, -2\bar{x}_1 + 5\bar{x}_2 - 2\bar{x}_3, -2\bar{x}_2 + 5\bar{x}_3 - 2\bar{x}_4, \dots, -2\bar{x}_{n-1} + 5\bar{x}_n\}x.$

$= |x_1|^2 - 2x_1\bar{x}_2 - 2\bar{x}_1 x_2 + 5|x_2|^2 - 2x_2\bar{x}_3 - 2\bar{x}_2 x_3 + 5|x_3|^2 - \dots + 5|x_{n-1}|^2 - 2x_{n-1}\bar{x}_n - 2\bar{x}_{n-1}x_n + 5|x_n|^2.$

$= |x_1|^2 - 2x_1\bar{x}_2 - \overline{2\bar{x}_1 x_2} + 5|x_2|^2 - 2x_2\bar{x}_3 - \overline{2\bar{x}_2 x_3} + 5|x_3|^2 - \dots + 5|x_{n-1}|^2 - 2x_{n-1}\bar{x}_n - \overline{2\bar{x}_{n-1}x_n} + 5|x_n|^2$

$= (|x_1|^2 - 4\mathrm{Re}(x_1\bar{x}_2) + 4|x_2|^2) + (|x_2|^2 - 4\mathrm{Re}(x_2\bar{x}_3) + 4|x_3|^2) + \cdots (|x_{n-1}|^2 - 4\mathrm{Re}(x_{n-1}\bar{x}_n) + 4|x_n|^2) + |xn|^2.$

$= |x_1 - 2x_2|^2 + |x_2 - 2x_3|^2 + \dots + |x_{n-1} - 2x_n|^2 + |x_n|^2 > 0$

Therefore, $A \,\varepsilon\, M$

For $k \geq 2$, choose $x_k = \frac{x_{k-1}}{2}$ to get $x_n = \frac{x_1}{2^{n-1}}$, and no one has:

$\|x\|_A^2 = |x_n|^2 = \frac{|x_1|^2}{2^{2n-2}}$, or $|x_1|^2 = 2^{2n-2}\|x\|_A^2$.

This implies that $p_I$ is not continuous on $(\phi, \|.\|_A)$

We proceed to establish sufficient conditions for coordinate projections to be continuous. We first make the

## 2. Main Discussion:

**Lemma:**

For the nxn positive definite matrix A, let $\lambda = \min\{\lambda_k : \lambda_k$ is an eigen value of A). It is then known that, for $x \,\varepsilon\, E^n$,

$x * A\,x \geq \lambda\|x\|$ where $\|x\|^2 = \sum_{k=1}^{n}|x_k|^2$

Proof:

Let U be a unitary matrix which diagonalizes A. Let $x = U_y$ for some $y \,\varepsilon\, E^n$

Now, $x * A\,x = y * AUy$

$= y * Dy$

$= \sum_{k=1}^{n}|y_k|^2$ where $D = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$

**Theorem:**

If $A \,\varepsilon\, M$ has the form $A = \mathrm{diag}\,(A_1, A_2, \dots)$ where, for each k, $A_k$ is an $S_k \times S_k$ matrix, then for each n = 1,2, ... $P_n$. is continuous on $(\phi, \|.\|_A)$.

Proof :

It is clear that, for each k, $A_k$ is Hermetian and positive definite.

For each k, define the matrix $A'_k = \mathrm{diag}\,(A_k, 0, 0, \dots)$, and the matrix $B_k = \mathrm{diag}\,(A_1, A_2, \dots, A_k, 0, 0, \dots)$

Fix n, let $\lambda = \min\{\lambda_i : \lambda_i$ is an eigen value of $B_i\}$, and for $x \,\varepsilon\, \phi$

let $\|x\|$ be as in (2.3).

Now, $|x_n|^2 \leq \|x\|^2$

$\leq \frac{1}{\lambda}\|x\|^2 B_n$ by (2.3)

$= \frac{1}{\lambda}\sum_{k=1}^{n}\|x\|_{A'_k}^2$

$\leq \frac{1}{\lambda}\sum_{k=1}^{\infty}\|x\|_{A'_k}^2$

$= \frac{1}{\lambda}\|x\|_A^2$

Thus, $P_n$. is continuous on $(\phi, \|.\|_A)$. It is now our objective to show that this matrix diagonal blockwise construction also solves the existence problem. The existing completions are to be shown unique and of desired form.

As for the uniqueness question, the next well-known lemma takes care of it.



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**Lemma**:
Let X and Y be two BK spaces. If S is a dense subspace of X and of Y such that $\|x\|_x = \|x\|_y$ for all x ε S, then X = Y.
Proof:
Let x ε X be arbitrary. Choose a sequence $\{x^n\} \subseteq S$ with $x^n \dots > x$ in X. The sequence $\{x^n\}$ is then a Cauchy sequence in X
and so, for all $\varepsilon > 0$ there exists a positive integer N for which $\|x^m - x^n\|_x < \varepsilon$ whenever $m, n \geq N$. But this says that $\|x^m - x^n\|_y < \varepsilon$ for all $m, n \geq N$. Therefore, $\{x^n\}$ is Cachy in Y, hence converges to some y ε Y.

Now, $x^n \dots > x \, \varepsilon \, X$; so, for each $k, x_k^n \dots > x_k$(in C). Also, $x^n \dots > y \, \varepsilon \, Y$; so, for each $k, x_k^n \dots > y_k$ (in C), implying that x = y, thus X ⊆ Y. //

Luckily enough, inner product spaces transfer their inner products to their completions. But the problem is that a completion of a specific type may not exist, and this is actually the existence problem being considered here.

**Proposition**:
Let (X, <., .>) be an inner product space, and K a completion of X, then K is an inner product space.
Proof:
First of all, one has to note that a completion always exists. Take for example the closure of X in its second dual under the canonical embedding. For x, y ε K, define :
$< x, y \geq \lim_{n\to\infty} < x^n, y^n >_x$ where, $(x^n) \subseteq X, \{y^n\}$ are two sequences converging respectively, to x, and y. This inner product is well defined, indeed;

$$|< x^m, y^m > - < x^n, y^n >| = < x^m, y^m > + < x^m, y^n > - < x^m, y^n > - < x^n, y^n$$
$$> | \leq |< x^m, y^m > - < x^m, y^n >|$$
$$+ |< x^n, y^n > - < x^m, y^n >|$$
$$\leq \|x^m\| \, \|y^m - y^n\| + \|y^n\| \|x^m - x^n\| \dots > 0$$

So the sequence $\{< x^n, y^n >\}$ is a Cachy sequence in C, hence converges to a unique limit. //

As it was remarked earlier, turning ϕ into an inner product space does not necessarily force an FK completion to exist. The following example shows this claim.

**Example:** Suppose that A ε M makes coordinate projections continuous on $(\phi, \|.\|_A)$. For x ε ϕ, let.

$$\|xx\| = (\|x\|_A^2 + |f(x)|^2)^{1/2}$$

where f is a noncontinuous linear functional on $(\phi, \|.\|_A)$. [f exists since ϕ is infinite dimensional; see example 3.3.14 of [2]].
We claim that the space $(\phi, \|.\|)$is an inner product space which is continuously embedded in ω, but has no FK completion.
To prove, we need the following

**Lemma:**
Let X be a linear metric space of sequences on which $P_n$. is continuous for all n. Then X has an FK completion iff for every Cauchy sequence $\{x^n\} \subset X$which converges to zero pointwise, we have $x^n \dots > 0$ in X.
We can now prove our claim.
Note first that the norm $\|.\|$ is given by the inner product
$< x, y > = < x, y >_A + f(x)\overline{f(y)}$.
Define the matrix $B = (b_{nk})$ as:
So, the norm $\|.\| = \|.\|_B$.
Now, $(\phi, \|.\|_B)$ is continuously embedded in ω {Fix n, and let x ε ϕ be arbitrary.
$|x_n| \leq s\|x\|_A$ for some s>0 by (2.1)
$\leq s\|x\|_B$ by the construction of $\|.\|_B$
But $(\phi, \|.\|_B)$ has no FK completion [The set $\{x \, \varepsilon \, \phi : f(x) = 1\}$ is dense in $(\phi, \|.\|_A$. Therefore, there exists a sequence $\{x^n\} \subset \phi$ with $f(x^n) = 1$ for all n, and $\|x^n\|_A \ldotp\ldotp > 0$
Finally, $\{x^n\}$ is Cauchy in $(\phi, \|.\|_B)$ since, $\|x^m - x^n\|_B = (\|x^m - x^n\|_A^2 + |f(x^m - x^n)|^2)^{1/2} = \|x^m - x^n\|_A, x^n \dots > 0$ in ω, but $\|x^n\| \dots > 1 - 0$. Our claim now follows from lemma.

Turning to the existence problem; it is essential to recall the following.

**Remarks:**

(a) ([4] ) :

(b) A finite dimensional inner product space is a Hilbert space.



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(c) ([5]) :

(d) If $(H_n)$ is a sequence of Hilbert spaces, then the direct sum $\oplus_n H_n$ is the Hilbert space H of all sequences $\{x^n : x^n \varepsilon H_n\}$ such that the sequence $\{\|x^n\|_{Hn}\} \varepsilon \ell^2$

Addition, scalar multiplanation and inner product are defined on H as follows:

For $x = \{x^n\}, y = \{y^n\} \varepsilon$ H, and for $\alpha \varepsilon$ C,

Define $x + y = \{x^n + y^n\}$,

$\alpha x = \{\alpha x^n\}$ and

$< x, y >_H = \sum_n < x^n, y^n >_{Hn}$

Consider the matrix A of (2.4)

For each n, let:

$$r_n = \sum_{k=1}^{n} S_k$$

$\phi_n = \{x \varepsilon \phi : x_k = 0$ for all k except, possibly, for $r_{n-1} < k < r_n$,

$A_n = \text{diag}(0.0, \ldots, 0, A_n, 0, 0, \quad)$ where the zero-block appears $(r_{n-1})$-times before the block $A_n$, and let $H_n = (\phi_n, \|.\|_{An})$.

By remark, H is a Hilbert space of sequences, and by its *very* construction, H has AD.

With all this at hand, theorem (2.4), now, easily and clearly implies the

**Theorem (main result):**

Suppose that A is an infinite matrix which is Hermitian and positive definite. If A has the form

$A = \text{diag}\,(A_1, A_2, \ldots)$ where, for each n, $A_n$ is an $S_n \times S_n$ matrix, then the normed sequence space $(\phi, \|.\|_A)$ has an HK completion which has the AD property.

Proof: Done already.

## 3. Conclusion

In this paper, we we proved the following result:

If the space Φ (the finite sequences) is equipped with the norm which is naturally induced by a positive definite Hermitian and diagonally blockwise constructed matrix, then an HK completion exists. Also, we investigated our result through many illustrated examples.

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