On Some Results About P - Lindelöf Spaces

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Abstract

In this work, we study p-Lindelöf spaces**,** a concept which is "stronger" than Lindelöf spaces. Every p-compact space is proven to be p-Lindelöf space, so the concept of p-Lindelöf is a generalization of the concept of p-compact. Most of the theorems which are valid for p-compact spaces will be also valid for p-Lindelöf spaces, (with a slight modification). We also study local p-Lindelöf spaces with many related theorems and examples.

Keywords: Topology; topological space; Lindelof space; countable set

1. Introduction and basic concepts:

In [5], the authors introduce and study p-compact spaces. In this paper, we study p-Lindelöf spaces as a generalization of p-compact space. Most of the theorems which are valid for p-compact will be valid for p-Lindelöf spaces. To make this work as self-contained as possible we are going to give some preliminaries theorems and definitions we need them later.

Definition 1.1 [1]: A topological space (X,τ) is Lindelöf space if every open cover of X has a countable subcover.

Definition1.2 [5]: A subset A of a topological space (X,τ) is called pre-open if $A \subseteq (A)^{O}$. A subset B is a pre-closed if X \ B is pre-open.

Remaks1.3 [4]:1.Every open (closed) subset of (X,τ)is pre-open (pre closed) set, but the converse is not true, for example the set of rational numbers is pre-open subset of (**R**,τu), where τu is an usual topology on **R**, but it is not an open set in **R**.
If G is a subset of X (open) and V is a pre-open set in X then G ∩ V
is open set.

Definition 1.4 [6]: Let (X,τ) be a topological space, A ⊆ X and let F= {Vα: α ∈Ω } be a cover of A is called a pre-open cover of A if Vα is a pre-open subset in X for each α∈Ω . Note that every open cover of A is pre-open cover of A.

Definition 1.5 [2]: A topological space (X,τ) is called p-compact if for every preopen cover of X has a finite subcover.

Remark 1.6: It is known that every p-compact is compact, but the converse is not true, for example, let X = **R**, τ= {φ, X, {0}}, then X is compact but it is not a pre-compact space.

Definition 1.7 [3]: A subset A of (X,τ) is said to be Gδ -open if for each x in A there is a countable family of open sets $\{U i\}_{i=1}^{\infty}$ such that $x \in \bigcap_{i=1}^{\infty} i \subseteq A$. On the other hand, A is Gδ -closed if X \ A is Gδ -open set.

Note that each open set is Gδ -open and each closed set is Gδ -closed, but the converse are not true.

Definition 1.8 [3] : A function f : (X,τx) → (Y,τY) is called Gδ -closed function if image of each closed subset of X is Gδ -closed subset of Y.

Definition 1.9 [6]: A space (X,τ) is said to be p-T2-space (p-Hausdorff space) if for each two distinct elements x, y of X there exist two pre-open subsets of X, U and V such that x ∈ U, y∈V and U ∩ V = φ.

Definition 1.10: A subset A of (X,τ) is said to be pre-Gδ -open if for each x in A there is a countable family of pre-open sets $\{U i\}_{i=1}^{\infty}$ such that $x \in \bigcap_{i=1}^{\infty} Ui \subseteq A$. On the other hand, A is pre-Gδ -closed if X\A is a pre-Gδ -open. Note that Gδ -closed (Gδ -open) set is a pre-Gδ -closed (pre-Gδ -open) set, but the converse is not true.

Doi: https://doi.org/10.54216/PMTCS.020205
Received: January 22, 2023 Revised: April 11, 2023 Accepted: September 26, 2023

Definition 1.11 [6] : A function f : (X,τx) → (Y,τY) is called p** continuous if $f^{-1}(W)$ is a pre-open set in X for each W is a pre-open set in Y.

Definition 1.12*:* A topological space (X,τ) is N-space if for every countable intersection of open sets is also open set.

Remark and Examples 1.13:

1. The discrete space (X, D) is N-space.
2. (**R**, τu) is not N-space.
3. Let (X,τ) be a N-space then every Gδ-open set in X is an open set and Gδ-closed set in X is closed set.

1. **Lindelöf spaces and p- Lindelöf spaces**

Lindelöf spaces have always played a highly expressive role in topology. They were introduced by Alexanddroff and Urysohn back in 1929 and their name is due to Lindelöf's proof in 1903 that from any collection of open sets covering a Euclidean space one can extract a countable subcollection covering the space. The authors in [7] study special kind of Lindelöf space which is called LC-space, but in this section we introduce and study another kind of Lindelöf space which is called p-Lindelöf space. p-Lindelöf space is a generalization of the concept of p-compact space.

Definition 2.1: A topological space (X,τ) is said to be p-Lindelöf space iff every preopen cover of X has countable subcover.

Remark 2.2: If X is a countable indiscrete space then it is a p-Lindelöf space and if X is an uncountable discrete space it is not p-Lindelöf space.

Proposition 2.3: Every p-Lindelöf space X is a Lindelöf space.

Proof: Let {Wα:α∈Ω} be any open cover for X. Since an open set Wα in X is a preopen set in X then {Wα:α∈Ω} is a pre-open cover for X, so there is α1,α2,… ∈Ω such that $X = \bigcup_{i=1}^{\infty} W\alpha i$ because X is a p-Lindelöf space. Hence {Wαi} is a countable open cover for X, therefore X is a Lindelöf space.

Remark 2.4: The following diagram shows the relations between compact, p-compact, Lindelöf and p-Lindelöf:

$$\begin{array}{ccc} \text{Compact} & \rightarrow & \text{Lindelöf} \\ \uparrow & & \uparrow \\ \text{p-compact} & \rightarrow & \text{P-Lindelöf} \end{array}$$

A Lindelöf space need not be p-Lindelöf since an uncountable Lindelöf space is a Lindelöf space, but not p-Lindelöf space.

Proposition 2.5: Every pre-closed subset A of p- Lindelöf space X is a p- Lindelöf space.

Proof: Let F= {Wα:α∈Ω} be any pre-open cover for A. So F* = F∪{X\ A} is a preopen cover of X , but X is a p- Lindelöf space , hence there is α1,α2,… ∈Ω such that $X = \bigcup_{i=1}^{\infty} W\alpha i \cup \{X\backslash A\}$ which implies that $A_{i=1}^{\infty} \subseteq \cup W\alpha i$ i.e. A is a p- Lindelöf space.□

Remark 2.6: It is known that a continuous function from a compact (p-compact) space into a Hausdroff space is a closed function [2]. However, the continuous function from a Lindelöf (p- Lindelöf) space, need not be closed an example may be found in [4].

Proposition 2.7: The continuous function f: X → Y from a Lindelöf X into a T2 – space Y is a Gδ-closed function.

Proof: Let A be any closed subset of X. In order to prove that f (A) is a Gδ-closed subset of Y, i.e. Y\f (A) is a Gδ -open subset of Y, let x ∈Y\f (A) implies that x ≠ y, ∀y∈ f (A). Since Y is T2- space there is U y, V y open subsets of Y such that x∈ U y, y∈ V y and U y ∩ V y = φ . Therefore f (A) ⊆ ∪ {V y: y∈ f (A)} so {V y: y∈ f (A)} is an open cover for f(A) in Y implies

$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\cup \{V y: y \in f(A)\}) = \cup \{f^{-1}(V y): y \in f(A)\}$

But $f^{-1}(V y)$ is an open subset of X, since f is a continuous function. Thus $\{f^{-1}(V y): y \in f(A)\}$ is an open cover for A. By knowing A is a closed subset of Lindelöf space X, then A itself is Lindelöf space which implies there is y1,y2, … such that $A_{i=1}^{\infty} \subseteq \cup f^{-1}(V yi)$, hence $f(A) \subseteq f(\cup_{i=1}^{\infty} f^{-1}(V yi)) = \cup_{i=1}^{\infty} f(f^{-1}(V yi)) \subseteq \cup_{i=1}^{\infty} Vyi$. But for each V yi there is U yi such that U yi ∩ V yi = φ , and x ∈ U yi for each i=1,2,… implies that $x \in \cap_{i=1}^{\infty} U yi \subseteq Y\backslash f(A)$, hence Y\ f (A) is a Gδ -open subset of Y. Therefore, f is a Gδ-closed function. □

Corollary 2.8: Every continuous function from a p- Lindelöf space into a T2 –space is a Gδ-closed function.

Proof: Since p- Lindelöf space is a Lindelöf space then use proposition 2.7 we obtain the result.□

Proposition 2.9: Every Lindelöf subset A of a T2 –space X is a Gδ closed set.

Proof: Let x ∈ X\A and for each y ∈A there is Uy, Vy open subsets of X such that x∈Uy, y∈Vy and Uy ∩ Vy = φ because X is a T2 –space. Therefore {Vy: y ∈A} is an open cover of A, but A is a Lindelöf subset there is y1, y2… in A such that $A \subseteq \cup_{i=1}^{\infty} V yi$ and $x \in \cap_{i=1}^{\infty} U yi \subseteq X\backslash A$. Hence X\A is a Gδ -open subset of X i.e. A is a Gδ-closed set.

Doi: https://doi.org/10.54216/PMTCS.020205
Received: January 22, 2023 Revised: April 11, 2023 Accepted: September 26, 2023

Proposition 2.10: Every p-Lindelöf subset A of a T2 –space X is a pre-Gδ-closed set.

Proof: Let x ∈X\A and for each y ∈A there is Uy, Vy pre-open subsets of X such that x ∈U y, y ∈V y and U y ∩ V y = φ because X is a T2 –space. Therefore {Vy: y ∈A} is a pre-open cover of A, but A is a p- Lindelöf subset there is y1, y2… in A such that $A \subseteq \cup_{i=1}^{\infty} V yi$ and $x \in \cap_{i=1}^{\infty} U yi \subseteq X \backslash A$. Hence X\A is a pre-Gδ -open subset of X i.e. A is a pre-Gδ-closed set.

Proposition 2.11: Every p**-continuous image of a p-Lindelöf space is a p-Lindelöf space.

Proof: Let f: X → Y be a p**-continuous function of a p- Lindelöf space X onto a space Y and let W = {Wα:α∈Ω} be any pre-open cover of Y. Then $W^* = \{f^{-1}(W\alpha): \alpha \in \Omega\}$ is a pre-open cover of X, because f is a p**-continuous function. Since X is a p- Lindelöf space W* has a countable subcover such that $X = \cup_{i=1}^{\infty} (W\alpha i)$. Since f is onto then $f\left(f^{-1}(W\alpha i)\right) = W\alpha i$ for each αi , i = 1,2,…, hence Y = f(X) $= \cup_{i=1}^{\infty} W\alpha i$. This implies that Y is a p- Lindelöf space.□

3.Relative p-Lindelöf and locally p-Lindelöf spaces

We begin with a definition of relative p- Lindelöf and some characterization.

Definition 3.1: A subset A of space X is said to be p- Lindelöf relative to X iff for every cover of A by pre-open sets of X has a countable subcover.

Proposition 3.2: If G is an open subset of space X then G is a p- Lindelöf (as the subspace of X) iff G is a p-Lindelöf relative to X.

Proof: Let G be a p- Lindelöf set in X and let {Wα :α∈Ω} be any pre-open cover of G such that Wα is a pre-open set of X. By using remark 1.3 (2) we have G ∩ Wα is a preopen set in G and in X implies that {G ∩ Wα :α∈Ω} is a pre-open cover of G. But G is a p- Lindelöf set, thus $G \subseteq \cup_{i=1}^{\infty} (G \cap W\alpha i)$i.e. G is a p- Lindelöf relative to X. Conversely, since G is an open subset of X then it is a p- Lindelöf (see remark 1.3 (1)). □

Proposition 3.3: Let A and B be two open subsets of X such that A ⊆B then A is ap- Lindelöf subspace of B if A is a p- Lindelöf subspace of X.

Proof: Let A be a p- Lindelöf relative to B and {Wα :α∈Ω} be any pre open cover of A, where Wα is a pre-open set in X. Therefore, B ∩ Wα is a pre-open set in A. But A is a p- Lindelöf relative to B, hence A ⊆ ∪ {B ∩ Wαi} implies A ⊆ ∪{Wαi} i.e. A is a pLindelöf relative to X. Conversely, since any pre-open set in B, where B is an open set in X, is a pre-open set in X (remark 1.3 (2)) then A is a p- Lindelöf relative to B.□

Definition 3.4 [2]: A space X is said to be locally p- Lindelöf if each element has a neighborhood which is a p- Lindelöf subspace of X.

Remark 3.5: Every p- Lindelöf space is a locally p- Lindelöf space, however the converse may not be true, since an uncountable space is a locally p- Lindelöf but not p-Lindelöf.

Proposition 3.6: A space X is a locally p- Lindelöf if every element of X, there exists an open set which is p-Lindelöf of X.

Proof: Let X be a locally p- Lindelöf, then for each element x in X has a neighborhood which is a p- Lindelöf subspace of X. Thus the element x has a neighborhood which is a p- Lindelöf relative to X (Proposition 3.2). Conversely, let x in X and Nx be an open set to x then Nx is a p- Lindelöf set (Proposition 3.2). Thus Nx is a neighborhood which is a p- Lindelöf subspace of X i.e. X is a locally p- Lindelöf.

Proposition 3.7: A space X is a locally p- Lindelöf if for every x in X, there exists an open locally p-Lindelöf subspace A containing x.

Proof: Let X be a locally p- Lindelöf space, then there is a neighborhood Nx such that x ∈Nx and Nx is a p-Lindelöf implies that Nx is a locally p- Lindelöf containing x (Proposition 3.6). Conversely, since A is an open locally p- Lindelöf subspace of X containing x, there exists a neighborhood V of x in A which is a p- Lindelöf subspace of A.

Since A is an open set in X, so is V .By using proposition 3.3 V is a p- Lindelöf subspace of X. This shows that X is a locally p- Lindelöf space.

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Doi: https://doi.org/10.54216/PMTCS.020205
Received: January 22, 2023 Revised: April 11, 2023 Accepted: September 26, 2023

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Doi: https://doi.org/10.54216/PMTCS.020205
Received: January 22, 2023 Revised: April 11, 2023 Accepted: September 26, 2023