Investigating Inclusion, Neighborhood, and Partial Sums Properties for a General Subclass of Analytic Functions

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Abstract

The study of geometric properties within the subclass of analytic functions has garnered significant attention in recent years due to its complex and intricate interplay between geometric function theory and complex analysis. This area of study provides deep insights into both mathematical theory and its practical applications. The exploration of these properties is not only of theoretical interest but also offers valuable implications for various applications in mathematical and engineering disciplines. In particular, this paper focuses on a detailed examination of the inclusion, neighborhood, and partial sums properties within a broad and general subclass of analytic functions. This class of functions is defined through a generalized multiplier transformation operator, which adds a layer of complexity to their analysis. By investigating these specific properties, this study aims to validate and build upon many existing findings documented in the literature, offering new perspectives and contributing to a deeper understanding of the field.

Keywords: Analytic functions; inclusion; neighborhood; partial sums

Mathematics Subject Classification: 30C45

1 Introduction

Let $\mathcal{A}$ represent the set of analytic functions f defined on the open unit disk $\mathbb{U}$, which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Consequently, every function $f \in \mathcal{A}$ can be expressed as a Taylor-Maclaurin series of the form:

$$f(\upsilon) = \upsilon + \sum_{\kappa=2}^{\infty} a_\kappa \upsilon^\kappa, \quad \upsilon \in \mathbb{U}. \tag{1}$$

In 1975, Silverman[1] introduced and analyzed a specific subset of $\mathcal{A}$ comprised of functions where the coefficients, starting from the second term, are negative. In other words, the analytic functions f within this subset can be represented as

$$f(\upsilon) = \upsilon - \sum_{\kappa=2}^{\infty} a_\kappa \upsilon^\kappa, \quad \upsilon \in \mathbb{U}. \quad a_n \geqslant 0 \tag{2}$$

DOI: https://doi.org/10.54216/IJNS.250341
Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

This subclass is known as the class of analytic functions with negative coefficients and is denoted as $\mathcal{A}^*$. Following Silverman's contribution, there has been considerable interest in studying functions with negative coefficients. Building on Silverman's work, many additional subclasses of $\mathcal{A}$ have been investigated in academic literature. For functions $f \in \mathcal{A}$, Cho and Srivastava[2] introduced the generalized multiplier transformation operator as follows:

$$I_\psi^m f(\upsilon) = \upsilon + \sum_{\kappa=2}^{\infty} \left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \upsilon^\kappa, \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ and } \psi \geqslant 0). \tag{3}$$

It is worth noting that for $\psi = 1$, the multiplier transformation I_ψ^m was introduced and examined by Uralegaddi and Somanatha,[3] whereas for $\psi = 0$, the multiplier transformation I_ψ^m was introduced and analyzed by Salagean.[4]

Yousef *et al.* in[5] initially introduced the class $\mathcal{B}_\Sigma^\mu(\varphi, \delta, \alpha)$. In subsequent work,[6] Illafe *et al.* incorporated the operator $I_\psi^m f(\upsilon)$ to define a new subclass, $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha)$, and established the necessary and sufficient conditions for a function $f(\upsilon)$ to belong to this subclass $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha)$, as outlined in the following definition and theorem. For more work related to the class $\mathcal{B}_\Sigma^\mu(\varphi, \delta, \alpha)$, we refer the reader to[7]-.[23]

Definition 1.1. *For $\varphi \geqslant 1, \delta \geqslant 0$ and $0 \leqslant \alpha < 1$, an analytic function f represented by (1) is in $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha)$ if for all $\upsilon \in \mathbb{U}$.*

$$\operatorname{Re}\left\{(1-\varphi)\frac{I_\psi^m f(\upsilon)}{\upsilon} + \varphi(I_\psi^m f(\upsilon))' + \delta\upsilon(I_\psi^m f(\upsilon))''\right\} > \alpha. \tag{4}$$

Theorem 1.2. *A function $f \in \mathcal{A}^*$ defined by (2) belong to the class $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha)$ if and only if*

$$\sum_{\kappa=2}^{\infty} [(1-\varphi) + \kappa\varphi + \kappa(\kappa-1)\delta] \left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \leqslant 1-\alpha. \tag{5}$$

2 Inclusion Relations

This section investigates the inclusion properties of the subclass $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha)$. By examining various inclusion relations, we aim to understand how different subclasses relate to each other within this framework. The results presented offer insights into how certain classes are contained within others, contributing to a broader understanding of the hierarchical structure within the class of analytic functions under consideration.

Theorem 2.1. *let $0 \leqslant \alpha_1 \leqslant \alpha_2 < 1$. Then*

$$\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha_1) \supseteq \mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha_2). \tag{6}$$

Proof. Consider the function $f(\upsilon)$ defined by equation (2) to belong to the class $\mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha_2)$. Then, by Theorem 1.2, we have

$$\begin{aligned}\sum_{\kappa=2}^{\infty} [(1-\varphi) + \kappa\varphi + \kappa(\kappa-1)\delta] \left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa &\leqslant 1-\alpha_2 \\ &\leqslant 1-\alpha_1.\end{aligned}$$

Thus, $f \in \mathcal{B}_{m,\psi}^*(\varphi, \delta, \alpha_1)$.

Therefore, the inclusion relation shown by (6) holds. □

Theorem 2.2. *let $1 \leqslant \varphi_1 \leqslant \varphi_2$. Then*

$$\mathcal{B}_{m,\psi}^*(\varphi_1, \delta, \alpha) \supseteq \mathcal{B}_{m,\psi}^*(\varphi_2, \delta, \alpha). \tag{7}$$

DOI: https://doi.org/10.54216/IJNS.250341

Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

Proof. Consider the function $f(\upsilon)$ defined by equation (2) to belong to the class $\mathcal{B}^*_{m,\psi}(\varphi_2,\delta,\alpha)$. Then, by Theorem 1.2, we have

$$\sum_{\kappa=2}^{\infty}[(1-\varphi_1)+\kappa\varphi_1+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \leqslant \sum_{\kappa=2}^{\infty}[(1-\varphi_2)+\kappa\varphi_2+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \leqslant 1-\alpha.$$

Thus, $f\in\mathcal{B}^*_{m,\psi}(\varphi_1,\delta,\alpha)$.

Therefore, the inclusion relation shown by (7) holds. □

Theorem 2.3. *let* $0\leqslant\delta_1\leqslant\delta_2$. *Then*

$$\mathcal{B}^*_{m,\psi}(\varphi,\delta_1,\alpha)\supseteq\mathcal{B}^*_{m,\psi}(\varphi,\delta_2,\alpha). \tag{8}$$

Proof. Consider the function $f(\upsilon)$ defined by equation (2) to belong to the class $\mathcal{B}^*_{m,\delta}(\varphi,\delta_2,\alpha)$. Then, by Theorem 1.2, we have

$$\sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta_1]\left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \leqslant \sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta_2]\left(\frac{\kappa+\psi}{1+\psi}\right)^m a_\kappa \leqslant 1-\alpha.$$

Thus, $f\in\mathcal{B}^*_{m,\psi}(\varphi,\delta_1,\alpha)$.

Therefore, the inclusion relation shown by (8) holds. □

Theorem 2.4. *let* $0\leqslant m_1\leqslant m_2$. *Then*

$$\mathcal{B}^*_{m_1,\psi}(\varphi,\delta,\alpha)\supseteq\mathcal{B}^*_{m_2,\psi}(\varphi,\delta,\alpha). \tag{9}$$

Proof. Consider the function $f(\upsilon)$ defined by equation (2) to belong to the class $\mathcal{B}^*_{m_2,\delta}(\varphi,\delta,\alpha)$. Then, by Theorem 1.2, we have

$$\sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi}{1+\psi}\right)^{m_1} a_\kappa \leqslant \sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi}{1+\psi}\right)^{m_2} a_\kappa \leqslant 1-\alpha.$$

Thus, $f\in\mathcal{B}^*_{m_1,\delta}(\varphi,\delta,\alpha)$.

Therefore, the inclusion relation shown by (9) holds. □

Theorem 2.5. *let* $0\leqslant\psi_1\leqslant\psi_2$. *Then*

$$\mathcal{B}^*_{m,\psi_1}(\varphi,\delta,\alpha)\supseteq\mathcal{B}^*_{m,\psi_2}(\varphi,\delta,\alpha). \tag{10}$$

Proof. Consider the function $f(\upsilon)$ defined by equation (2) to belong to the class $\mathcal{B}^*_{m,\psi_2}(\varphi,\delta,\alpha)$. Then, by Theorem 1.2, we have

$$\sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi_1}{1+\psi_1}\right)^m a_\kappa \leqslant \sum_{\kappa=2}^{\infty}[(1-\varphi)+\kappa\varphi+\kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi_2}{1+\psi_2}\right)^m a_\kappa \leqslant 1-\alpha.$$

Thus, $f\in\mathcal{B}^*_{m,\psi_1}(\varphi,\delta,\alpha)$.

Therefore, the inclusion relation shown by (10) holds. □

DOI: https://doi.org/10.54216/IJNS.250341
Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

3 Neighborhoods

In the study of analytic functions, neighborhoods play a crucial role in understanding the behavior of functions within certain bounds. Specifically, this section explores neighborhoods defined around a given function f in terms of the coefficients of its series expansion.

Consider the neighborhood $\mathcal{N}_\varepsilon(f)$ defined by

$$\mathcal{N}_\varepsilon(f) = \left\{ g \in \mathcal{A}^* : g(\upsilon) = \upsilon - \sum_{\kappa=2}^{\infty} b_\kappa \upsilon^\kappa \quad \text{and} \quad \sum_{\kappa=2}^{\infty} \kappa \left| a_\kappa - b_\kappa \right| \leqslant \varepsilon \right\}.$$

Here, $\mathcal{A}^*$ denote the class of analytic functions with negative coefficients, and a_κ and b_κ are the positive coefficients of the series expansion of f and g, respectively.

For the identity function $e(\upsilon) = \upsilon$, the neighborhood is given by

$$\mathcal{N}_\varepsilon(e) = \left\{ g \in \mathcal{A}^* : g(\upsilon) = \upsilon - \sum_{\kappa=2}^{\infty} b_\kappa \upsilon^\kappa \quad \text{and} \quad \sum_{\kappa=2}^{\infty} \kappa b_\kappa \leqslant \varepsilon \right\}.$$

This section will investigate the neighborhood property, focusing on $\mathcal{N}_\varepsilon(e)$, and will prove several important results concerning the relationship between different function classes and their neighborhoods.

Theorem 3.1. *If* $\varphi \geqslant 2\delta + 1$ *and* $\varepsilon = \frac{2(1-\alpha)}{(1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m}$*, then* $\mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha) \subset \mathcal{N}_\varepsilon(e)$.

Proof. Let $f(\upsilon) \in \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$. Then, by Theorem 1.2, we have

$$(1 + \varphi + 2\delta) \left(\frac{2 + \psi}{1 + \psi} \right)^m \sum_{\kappa=2}^{\infty} a_\kappa \leqslant 1 - \alpha,$$

which implies

$$\sum_{\kappa=2}^{\infty} a_\kappa \leqslant \frac{1 - \alpha}{(1 + \varphi + 2\delta) \left(\frac{2+\psi}{1+\psi} \right)^m} \tag{11}$$

Now, from (5) and (11) we have

$$\begin{aligned} \varphi \left(\frac{2 + \psi}{1 + \psi} \right)^m \sum_{\kappa=2}^{\infty} n a_\kappa &\leqslant 1 - \alpha - (1 - \varphi + 2\delta) \left(\frac{2 + \psi}{1 + \psi} \right)^m \sum_{\kappa=2}^{\infty} a_\kappa \\ &\leqslant 1 - \alpha - \frac{(1 - \alpha)(1 - \varphi + 2\delta)}{1 + \varphi + 2\delta}. \end{aligned} \tag{12}$$

Hence,

$$\sum_{\kappa=2}^{\infty} \kappa a_\kappa \leqslant \frac{2(1 - \alpha)}{(1 + \varphi + 2\delta) \left(\frac{2+\psi}{1+\psi} \right)^m} = \varepsilon,$$

and $f(\upsilon) \in \mathcal{N}_\varepsilon(e)$. □

A function $f \in \mathcal{A}^*$ is said to be in the subclass $\mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$ if there exists a function $g \in \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$ such that

$$\left| \frac{f(\upsilon)}{g(\upsilon)} - 1 \right| \leqslant 1 - \beta, \quad (0 \leqslant \beta < 1, \upsilon \in \mathbb{U}).$$

DOI: https://doi.org/10.54216/IJNS.250341
Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

Theorem 3.2. *If* $g \in \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$ *and*

$$\beta = 1 - \frac{\varepsilon(1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m}{2\left((1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m + \alpha - 1\right)},$$

then $\mathcal{N}_\varepsilon(g) \subset \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$.

Proof. Let $f(\upsilon) \in \mathcal{N}_\varepsilon(g)$. Then

$$\sum_{\kappa=2}^{\infty} \kappa\,|a_\kappa - b_\kappa| \leqslant \varepsilon \Rightarrow \sum_{\kappa=2}^{\infty} |a_\kappa - b_\kappa| \leqslant \frac{\varepsilon}{2}.$$

Since $g \in \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$, then

$$\sum_{\kappa=2}^{\infty} b_\kappa \leqslant \frac{1-\alpha}{(1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m}.$$

Letting $|\upsilon| \to 1$, we have

$$\begin{aligned} \left|\frac{f(\upsilon)}{g(\upsilon)} - 1\right| &\leqslant \frac{\sum_{\kappa=2}^{\infty}|a_\kappa - b_\kappa|}{1 - \sum_{\kappa=2}^{\infty} b_\kappa} \qquad (13)\\ &\leqslant \frac{\varepsilon}{2}\left(\frac{(1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m}{(1+\varphi+2\delta)\left(\frac{2+\psi}{1+\psi}\right)^m + \alpha - 1}\right)\\ &\leqslant 1 - \beta, \end{aligned}$$

Thus, $f \in \mathcal{B}^*_{m,\psi}(\varphi, \delta, \alpha)$. □

4 Partial Sums

Silverman[24] sharp bounds on the real part of the ratios between normalized convex or starlike functions and their corresponding partial sum sequences. In this section, we build upon Silverman's work[24] and the studies mentioned in[25] concerning partial sums of the analytic functions to examine the proportion of real parts of a function of the form (2) and its corresponding partial sum sequences, which is defined by:

$$f_N(\upsilon) = \upsilon - \sum_{\kappa=2}^{N} a_\kappa \upsilon^\kappa, \quad N \in \mathbb{N}\backslash\{1\} \text{ and } a_\kappa \geqslant 0. \qquad (14)$$

Let

$$\Xi^m_\kappa = \Xi^m_\kappa(\varphi, \delta, \psi) := [(1-\varphi) + \kappa\varphi + \kappa(\kappa-1)\delta]\left(\frac{\kappa+\psi}{1+\psi}\right)^m,$$

where $m \in \mathbb{N}_0, \varphi \geqslant 1, \delta \geqslant 0$ and $\psi \geqslant 0$.

Theorem 4.1. *If* $f \in \mathcal{A}^*$ *in the form of equation (2) satisfies the condition (5), then*

$$Re\left\{\frac{f(\upsilon)}{f_N(\upsilon)}\right\} \geqslant \frac{\Xi^m_{N+1} + \alpha - 1}{\Xi^m_{N+1}}, \qquad (15)$$

and

$$Re\left\{\frac{f_N(\upsilon)}{f(\upsilon)}\right\} \geqslant \frac{\Xi^m_{N+1}}{\Xi^m_{N+1} - \alpha + 1}. \qquad (16)$$

The results are sharp for every $N \in \mathbb{N}\backslash\{1\}$, *with the extremal functions given by*

$$f(\upsilon) = \upsilon - \frac{1-\alpha}{\Xi^m_{N+1}}\upsilon^{N+1}. \qquad (17)$$

DOI: https://doi.org/10.54216/IJNS.250341

Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

Proof. To prove (15), it is sufficient to prove that

$$\frac{\Xi_{N+1}^m}{1-\alpha}\left(\frac{f(\upsilon)}{f_N(\upsilon)}-\frac{\Xi_{N+1}^m+\alpha-1}{\Xi_{N+1}^m}\right)\prec\frac{1+\upsilon}{1-\upsilon},\quad(\upsilon\in\mathbb{U}).$$

Define the function $\omega(\upsilon)$ by

$$\frac{\Xi_{N+1}^m}{1-\alpha}\left(\frac{1-\sum_{\kappa=2}^{\infty}a_\kappa\upsilon^{\kappa-1}}{1-\sum_{\kappa=2}^{N}a_\kappa\upsilon^{\kappa-1}}-\frac{\Xi_{N+1}^m+\alpha-1}{\Xi_{N+1}^m}\right)=\frac{1+\omega(\upsilon)}{1-\omega(\upsilon)},\quad(\upsilon\in\mathbb{U}),$$

which implies

$$\omega(\upsilon)=\frac{\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)\sum_{\kappa=N+1}^{\infty}a_\kappa\upsilon^{\kappa-1}}{-2+2\sum_{\kappa=2}^{N}a_\kappa\upsilon^{\kappa-1}+\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)\sum_{\kappa=N+1}^{\infty}a_\kappa\upsilon^{\kappa-1}}.$$

Obviously, $\omega(0)=0$, and

$$|\omega(\upsilon)|\leqslant\frac{\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)\sum_{\kappa=N+1}^{\infty}a_\kappa}{2-2\sum_{\kappa=2}^{N}a_\kappa-\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)\sum_{\kappa=N+1}^{\infty}a_\kappa}.$$

Now, $|\omega(\upsilon)|\leqslant 1$ if and only if

$$2\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)\sum_{\kappa=N+1}^{\infty}a_\kappa\leqslant 2-2\sum_{\kappa=2}^{N}a_\kappa,$$

or, equivalently,

$$\sum_{\kappa=2}^{N}a_\kappa+\sum_{\kappa=N+1}^{\infty}\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)a_\kappa\leqslant 1,$$

In view of (5), it is sufficient to show that

$$\sum_{\kappa=2}^{N}a_\kappa+\sum_{\kappa=N+1}^{\infty}\left(\frac{\Xi_{N+1}^m}{1-\alpha}\right)a_\kappa\leqslant\sum_{\kappa=2}^{\infty}\left(\frac{\Xi_{\kappa}^m}{1-\alpha}\right)a_\kappa,$$

which is equivalent to to showing that

$$\sum_{\kappa=2}^{N}\left(\frac{\Xi_{\kappa}^m+\alpha-1}{1-\alpha}\right)a_\kappa+\sum_{\kappa=N+1}^{\infty}\left(\frac{\Xi_{\kappa}^m-\Xi_{N+1}^m}{1-\alpha}\right)a_\kappa\geqslant 0.\tag{18}$$

We observe that the first term of the first series in (18) is positive and, since Ξ_κ^m is a non-decreasing sequence, all the subsequent terms in the first series are also positive. Additionally, the first term of the second series in (18) is zero, and all the remaining terms in this series are positive as well. Consequently, the inequality (18) holds true. Therefore, the proof of (15) is complete.

The proof of (16) follows by showing that

$$\frac{\Xi_{N+1}^m-\alpha+1}{1-\alpha}\left(\frac{f_N(\upsilon)}{f(\upsilon)}-\frac{\Xi_{N+1}^m}{\Xi_{N+1}^m-\alpha+1}\right)\prec\frac{1+\upsilon}{1-\upsilon},\quad(\upsilon\in\mathbb{U}),$$

using similar arguments to those in (15), and is hence omitted.

Finally, it can be verified that the function given by (17) gives the sharp result in (15) and (16), when, $\upsilon=re^{2\pi i/N}$ and $r\to 1^-$. □

DOI: https://doi.org/10.54216/IJNS.250341

Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

Theorem 4.2. *If $f \in \mathcal{A}^*$ in the form of equation (2) satisfies the condition (5), then*

$$Re\left\{\frac{f'(\upsilon)}{f'_N(\upsilon)}\right\} \geqslant \frac{\Xi^m_{N+1} - (N+1)(1-\alpha)}{\Xi^m_{N+1}}, \tag{19}$$

and

$$Re\left\{\frac{f'_N(\upsilon)}{f'(\upsilon)}\right\} \geqslant \frac{\Xi^m_{N+1}}{\Xi^m_{N+1} + (N+1)(1-\alpha)}. \tag{20}$$

The results are sharp for every $N \in \mathbb{N}\backslash\{1\}$, with the extremal functions given by (17).

Proof. To prove (19), it is sufficient to prove that

$$\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\left(\frac{f'(\upsilon)}{f'_N(\upsilon)} - \frac{\Xi^m_{N+1} - (N+1)(1-\alpha)}{\Xi^m_{N+1}}\right) \prec \frac{1+\upsilon}{1-\upsilon}, \quad (\upsilon \in \mathbb{U}).$$

Define the function $\omega(\upsilon)$ by

$$\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\left(\frac{1-\sum_{\kappa=2}^{\infty}\kappa a_\kappa \upsilon^{\kappa-1}}{1-\sum_{\kappa=2}^{N}\kappa a_\kappa \upsilon^{\kappa-1}} - \frac{\Xi^m_{N+1} - (N+1)(1-\alpha)}{\Xi^m_{N+1}}\right) = \frac{1+\omega(\upsilon)}{1-\omega(\upsilon)}, \quad (\upsilon \in \mathbb{U}),$$

which implies

$$\omega(\upsilon) = \frac{\left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\sum_{\kappa=N+1}^{\infty}\kappa a_\kappa \upsilon^{\kappa-1}}{-2+2\sum_{\kappa=2}^{N}\kappa a_\kappa \upsilon^{\kappa-1} + \left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\sum_{\kappa=N+1}^{\infty}\kappa a_\kappa \upsilon^{\kappa-1}}.$$

Obviously, $\omega(0) = 0$, and

$$|\omega(\upsilon)| \leqslant \frac{\left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\sum_{\kappa=N+1}^{\infty}\kappa a_\kappa}{2-2\sum_{\kappa=2}^{N}\kappa a_\kappa - \left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\sum_{\kappa=N+1}^{\infty}\kappa a_\kappa}.$$

Now, $|\omega(\upsilon)| \leqslant 1$ if and only if

$$2\left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\sum_{\kappa=N+1}^{\infty}\kappa a_\kappa \leqslant 2 - 2\sum_{\kappa=2}^{N}\kappa a_\kappa,$$

or, equivalently,

$$\sum_{\kappa=2}^{N}\kappa a_\kappa + \sum_{\kappa=N+1}^{\infty}\left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\kappa a_\kappa \leqslant 1,$$

In view of (5), it is sufficient to show that

$$\sum_{\kappa=2}^{N}\kappa a_\kappa + \sum_{\kappa=N+1}^{\infty}\left(\frac{\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)\kappa a_\kappa \leqslant \sum_{\kappa=2}^{\infty}\frac{\Xi^m_\kappa}{(1-\alpha)}a_\kappa,$$

which is equivalent to to showing that

$$\sum_{\kappa=2}^{N}\left(\frac{\Xi^m_\kappa - \kappa(1-\alpha)}{(1-\alpha)}\right)a_\kappa + \sum_{\kappa=N+1}^{\infty}\left(\frac{(N+1)\Xi^m_\kappa - n\Xi^m_{N+1}}{(N+1)(1-\alpha)}\right)a_\kappa \geqslant 0. \tag{21}$$

We observe that the first term of the first series in (21) is positive and, since Ξ^m_κ is a non-decreasing sequence, all the subsequent terms in the first series are also positive. Additionally, the first term of the second series in

Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

(21) is zero, and all the remaining terms in this series are positive as well. Consequently, the inequality (21) holds true. Therefore, the proof of (19) is complete.

The proof of (20) follows by showing that

$$\frac{\Xi_{N+1}^{m}+(N+1)(1-\alpha)}{(N+1)(1-\alpha)}\left(\frac{f_N'(\upsilon)}{f'(\upsilon)}-\frac{\Xi_{N+1}^{m}}{\Xi_{N+1}^{m}+(N+1)(1-\alpha)}\right)\prec\frac{1+\upsilon}{1-\upsilon},\quad(\upsilon\in\mathbb{U}),$$

using similar arguments to those in (19), and is hence omitted.

Finally, it can be verified that the function given by (17) gives the sharp result in (19) and (20), when, $\upsilon = re^{2\pi i/N}$ and $r\to 1^{-}$. □

5 Concluding Remarks

In this work, we thoroughly investigate the geometric properties of a general subclass of analytic functions with negative coefficients, denoted as $\mathcal{B}^{*}m,\psi(\varphi,\delta,\alpha)$, which is defined by the application of a generalized multiplier transformation operator $I^{m}\psi$. This class of functions plays a significant role in complex analysis, particularly in understanding the behavior of various function families in terms of their starlikeness and convexity.

Our analysis focuses on several important aspects, including the inclusion relations between this subclass and other well-known classes of analytic functions. We also examine the neighborhood properties of functions within this subclass, which provides insight into how these functions relate to others in terms of proximity and structural similarity in the complex plane. Furthermore, we delve into the partial sums of these functions, which is a crucial area of study in determining the approximation properties and convergence behaviors.

The results we have obtained not only generalize certain known theorems but also provide new insights into the structure of these analytic functions. Our findings have the potential to verify and extend numerous existing results in the literature, thereby contributing to a deeper understanding of the interplay between geometric function theory and operator theory. Through this work, we aim to offer a comprehensive exploration of the subject, paving the way for future research in this area.

Author contributions

The authors contributed equally to this work.

Acknowledgments

This research has been done during the year 2024 when the third author was on Sabbatical leave in the USA. The authors would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

Disclosure Statement

The authors report no financial or non-financial interests that could be construed as a potential conflict of interest in relation to this research. The research was conducted with full transparency and impartiality, and no funding or support was received from any external entities that could influence the outcomes or interpretation of the results.

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Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024

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DOI: https://doi.org/10.54216/IJNS.250341
Received: March 29, 2024 Revised: June 28, 2024 Accepted: November 10, 2024