Neutrosophic $\mathcal{N}$-structures over Hilbert algebras

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Abstract

The notions of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$-structures to be neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras are provided. The Cartesian product of neutrosophic $\mathcal{N}$-structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals.

Keywords: Hilbert algebra; neutrosophic $\mathcal{N}$-subalgebra; neutrosophic $\mathcal{N}$-ideal; homomorphic pre-image.

1 Introduction

Zadeh[19] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov[1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophic" [French neuter, Latin neuter, neutral, and Greek sophia, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy/intuitionistic" logic/set and "neutrosophic" logic/set, that is, the included middle component, that is, the neutral/indeterminate/unknown part (besides the truth/membership and falsehood/non-membership components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). The concept of the neutrosophic set developed by Smarandache[16,17] is a more general platform that extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm). Diego[5] proved that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag[2,3] and Jun[9] and some of their filters forming deductive systems were recognized. Dudek[6] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras.

The negative structure of sets is constantly being defined and studied. Jun et al.[10] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures in 2009. Jun et al.[11,18] considered neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras and neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras in 2017. Jun et al.[12] studied neutrosophic positive implicative $\mathcal{N}$-ideals in BCK-algebras in 2018. Rangsuk et al.[15] introduced the notions of (special) neutrosophic $\mathcal{N}$-UP-subalgebras, (special) neutrosophic $\mathcal{N}$-near UP-filters, (special) neutrosophic $\mathcal{N}$-UP-filters, (special) neutrosophic $\mathcal{N}$-UP-ideals, and (special) neutrosophic $\mathcal{N}$-strong UP-ideals of UP-algebras in 2019.

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In this paper, the notions of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras are introduced, and several properties are investigated. Conditions for neutrosophic $\mathcal{N}$-structures to be neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras are provided. The Cartesian product of neutrosophic $\mathcal{N}$-structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals.

2 Preliminaries

Before we begin our study, we will give the definition of a Hilbert algebra.

Definition 2.1. [5] A *Hilbert algebra* is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,

(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,

(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In,[6] the following conclusion was established.

Lemma 2.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,

(2) $(\forall x \in X)(1 \cdot x = x)$,

(3) $(\forall x \in X)(x \cdot 1 = 1)$,

(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,

(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.3. [20] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.4. [4,7] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

(1) $1 \in D$,

(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,

(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

A *fuzzy set*[19] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5. [14] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

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Definition 2.6. [8] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,

(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,

(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7. [1] A neutrosophic set in a nonempty set H is defined to be a structure

$$A := \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in H\}, \tag{1}$$

where $T_A : H \to [0, 1]$ is a truth membership function, $I_A : H \to [0, 1]$ is an indeterminate membership function, and $F_A : H \to [0, 1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (X, T_A, I_A, F_A)$.

Definition 2.8. [13] We denote the family of all functions from a nonempty set X to the closed interval $[-1, 0]$ of the real line by $\mathcal{F}(X, [-1, 0])$. An element of $\mathcal{F}(X, [-1, 0])$ is called a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). An ordered pair of a nonempty set X and an $\mathcal{N}$-function on X is called an *$\mathcal{N}$-fuzzy structure*. A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over a nonempty universe of discourse X is defined to be the structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, where $T_{\mathcal{N}}, I_{\mathcal{N}}$, and $F_{\mathcal{N}}$ are $\mathcal{N}$-functions on X which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function on X, respectively.

For the sake of simplicity, we will use the notation $X_{\mathcal{N}}$ instead of the neutrosophic $\mathcal{N}$-structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$.[10]

Definition 2.9. [15] Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure over a nonempty set X. The neutrosophic $\mathcal{N}$-structure $\overline{X_{\mathcal{N}}} = (X, \overline{T_{\mathcal{N}}}, \overline{I_{\mathcal{N}}}, \overline{F_{\mathcal{N}}})$ defined by

$$(\forall x \in X) \begin{pmatrix} \overline{T_{\mathcal{N}}}(x) = -1 - T_{\mathcal{N}}(x) \\ \overline{I_{\mathcal{N}}}(x) = -1 - I_{\mathcal{N}}(x) \\ \overline{F_{\mathcal{N}}}(x) = -1 - F_{\mathcal{N}}(x) \end{pmatrix} \tag{2}$$

is called the *complement* of $X_{\mathcal{N}}$ in X.

3 Neutrosophic $\mathcal{N}$-fuzzy subalgebras and ideals of Hilbert algebras

In what follows, let X denote a Hilbert algebra $(X, \cdot, 1)$ unless otherwise specified.

Definition 3.1. A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is called a neutrosophic $\mathcal{N}$-fuzzy subalgebra of X if

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \\ I_{\mathcal{N}}(x \cdot y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \\ F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \end{pmatrix}. \tag{3}$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define a neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X as follows:

X	1	x	y	z	0
$T_{\mathcal{N}}$	-1	-0.8	-0.8	-0.7	-0.4
$I_{\mathcal{N}}$	-0.3	-0.5	-0.7	-0.3	-0.6
$F_{\mathcal{N}}$	-1	-0.8	-0.8	-0.7	-0.4

Then $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X.

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Proposition 3.3. *Every neutrosophic $\mathcal{N}$-subalgebra of X satisfies*

$$(\forall x \in X) \left(\begin{array}{c} T_{\mathcal{N}}(1) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}(1) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}(1) \leq F_{\mathcal{N}}(x) \end{array} \right). \tag{4}$$

Proof. For any $x \in X$, we have

$$T_{\mathcal{N}}(1) = T_{\mathcal{N}}(x \cdot x) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x),$$
$$I_{\mathcal{N}}(1) = I_{\mathcal{N}}(x \cdot x) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(x),$$
$$F_{\mathcal{N}}(1) = F_{\mathcal{N}}(x \cdot x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(x)\} = F_{\mathcal{N}}(x).$$

□

Definition 3.4. A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if it satisfies (4) and

$$(\forall x, y \in X) \left(\begin{array}{c} T_{\mathcal{N}}(x \cdot y) \leq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x \cdot y) \geq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x \cdot y) \leq F_{\mathcal{N}}(y) \end{array} \right), \tag{5}$$

$$(\forall x, y_1, y_2 \in X) \left(\begin{array}{c} T_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{T_{\mathcal{N}}(y_1), T_{\mathcal{N}}(y_2)\} \\ I_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{I_{\mathcal{N}}(y_1), I_{\mathcal{N}}(y_2)\} \\ F_{\mathcal{N}}((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{F_{\mathcal{N}}(y_1), F_{\mathcal{N}}(y_2)\} \end{array} \right). \tag{6}$$

Example 3.5. From Example 3.2, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.

Proposition 3.6. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then*

$$(\forall x, y \in X) \left(\begin{array}{c} T_{\mathcal{N}}((y \cdot x) \cdot x) \leq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}((y \cdot x) \cdot x) \geq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}((y \cdot x) \cdot x) \leq F_{\mathcal{N}}(y) \end{array} \right). \tag{7}$$

Proof. Let $x, y \in X$. By (6), we have

$$T_{\mathcal{N}}((y \cdot x) \cdot x) = T_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \leq \max\{T_{\mathcal{N}}(1), T_{\mathcal{N}}(y)\} = T_{\mathcal{N}}(y),$$
$$I_{\mathcal{N}}((y \cdot x) \cdot x) = I_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \geq \min\{I_{\mathcal{N}}(1), I_{\mathcal{N}}(y)\} = I_{\mathcal{N}}(y),$$
$$F_{\mathcal{N}}((y \cdot x) \cdot x) = F_{\mathcal{N}}((1 \cdot (y \cdot x)) \cdot x) \leq \max\{F_{\mathcal{N}}(1), F_{\mathcal{N}}(y)\} = F_{\mathcal{N}}(y).$$

□

Lemma 3.7. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \left\{ \begin{array}{l} T_{\mathcal{N}}(x) \geq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x) \leq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x) \geq F_{\mathcal{N}}(y) \end{array} \right. \right). \tag{8}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{array}{rcl} T_{\mathcal{N}}(y) & = & T_{\mathcal{N}}(1 \cdot y) \\ & = & T_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \leq & \max\{T_{\mathcal{N}}(x \cdot y), T_{\mathcal{N}}(x)\} \\ & \leq & \max\{T_{\mathcal{N}}(1), T_{\mathcal{N}}(x)\} \\ & = & T_{\mathcal{N}}(x), \end{array}$$

$$\begin{array}{rcl} I_{\mathcal{N}}(y) & = & I_{\mathcal{N}}(1 \cdot y) \\ & = & I_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \geq & \min\{I_{\mathcal{N}}(x \cdot y), I_{\mathcal{N}}(x)\} \\ & \geq & \min\{I_{\mathcal{N}}(1), I_{\mathcal{N}}(x)\} \\ & = & I_{\mathcal{N}}(x), \end{array}$$

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$$\begin{aligned} F_{\mathcal{N}}(y) &= F_{\mathcal{N}}(1 \cdot y) \\ &= F_{\mathcal{N}}(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\leq \max\{F_{\mathcal{N}}(x \cdot y), F_{\mathcal{N}}(x)\} \\ &\leq \max\{F_{\mathcal{N}}(1), F_{\mathcal{N}}(x)\} \\ &= F_{\mathcal{N}}(x). \end{aligned}$$

□

Theorem 3.8. *Every neutrosophic $\mathcal{N}$-ideal of X is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. By (5), we have

$$T_{\mathcal{N}}(x \cdot y) \leq T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\},$$

$$I_{\mathcal{N}}(x \cdot y) \geq I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\},$$

$$F_{\mathcal{N}}(x \cdot y) \leq F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}.$$

Hence, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

Proposition 3.9. *If $\{X^i_{\mathcal{N}} \mid i \in \Delta\}$ is a family of neutrosophic $\mathcal{N}$-subalgebras of X, then $\bigwedge_{i\in\Delta} X^i_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Let $\{X^i_{\mathcal{N}} \mid i \in \Delta\}$ be a family of neutrosophic $\mathcal{N}$-subalgebras of X. Let $x, y \in X$. Then

$$\begin{aligned} (\bigwedge_{i\in\Delta} T_{\mathcal{N}i})(x \cdot y) &= \sup_{i\in\Delta}\{T_{\mathcal{N}i}(x \cdot y)\} \\ &\leq \sup_{i\in\Delta}\{\max\{T_{\mathcal{N}i}(x), T_{\mathcal{N}i}(y)\}\} \\ &\leq \max\{\sup_{i\in\Delta}\{T_{\mathcal{N}i}(x)\}, \sup_{i\in\Delta}\{T_{\mathcal{N}i}(y)\}\} \\ &= \max\{(\bigwedge_{i\in\Delta} T_{\mathcal{N}i})(x), (\bigwedge_{i\in\Delta} T_{\mathcal{N}i})(y)\}, \end{aligned}$$

$$\begin{aligned} (\bigwedge_{i\in\Delta} I_{\mathcal{N}i})(x \cdot y) &= \inf_{i\in\Delta}\{I_{\mathcal{N}i}(x \cdot y)\} \\ &\geq \inf_{i\in\Delta}\{\min\{I_{\mathcal{N}i}(x), I_{\mathcal{N}i}(y)\}\} \\ &\geq \min\{\inf_{i\in\Delta}\{I_{\mathcal{N}i}(x)\}, \inf_{i\in\Delta}\{I_{\mathcal{N}i}(y)\}\} \\ &= \min\{(\bigwedge_{i\in\Delta} I_{\mathcal{N}i})(x), (\bigwedge_{i\in\Delta} I_{\mathcal{N}i})(y)\}, \end{aligned}$$

$$\begin{aligned} (\bigwedge_{i\in\Delta} F_{\mathcal{N}i})(x \cdot y) &= \sup_{i\in\Delta}\{F_{\mathcal{N}i}(x \cdot y)\} \\ &\leq \sup_{i\in\Delta}\{\max\{F_{\mathcal{N}i}(x), F_{\mathcal{N}i}(y)\}\} \\ &\leq \max\{\sup_{i\in\Delta}\{F_{\mathcal{N}i}(x)\}, \sup_{i\in\Delta}\{F_{\mathcal{N}i}(y)\}\} \\ &= \max\{(\bigwedge_{i\in\Delta} F_{\mathcal{N}i})(x), (\bigwedge_{i\in\Delta} F_{\mathcal{N}i})(y)\}. \end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} X^i_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

The following proposition can be proved similarly to Proposition 3.9.

Proposition 3.10. *If $\{X^i_{\mathcal{N}} \mid i \in \Delta\}$ is a family of neutrosophic $\mathcal{N}$-ideals of X, then $\bigwedge_{i\in\Delta} X^i_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Definition 3.11. Let $X_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure over a nonempty set X. The neutrosophic $\mathcal{N}$-structures $\oplus X_{\mathcal{N}}, \otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are defined as $\oplus X_{\mathcal{N}} = (X, T_{\mathcal{N}}, \overline{T_{\mathcal{N}}}, F_{\mathcal{N}})$, $\otimes X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, and $\odot X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, \overline{I_{\mathcal{N}}})$.

Theorem 3.12. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, then $\oplus X_{\mathcal{N}}, \otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are neutrosophic $\mathcal{N}$-subalgebras of X.*

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Proof. Let $x, y \in X$. Then

$$\begin{array}{rcl} \overline{T_{\mathcal{N}}}(x \cdot y) & = & -1 - T_{\mathcal{N}}(x \cdot y) \\ & \geq & -1 - \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \\ & = & \min\{-1 - T_{\mathcal{N}}(x), -1 - T_{\mathcal{N}}(y)\} \\ & = & \min\{\overline{T_{\mathcal{N}}}(x), \overline{T_{\mathcal{N}}}(y)\}, \end{array}$$

$$\begin{array}{rcl} \overline{I_{\mathcal{N}}}(x \cdot y) & = & -1 - I_{\mathcal{N}}(x \cdot y) \\ & \leq & -1 - \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \\ & = & \max\{-1 - I_{\mathcal{N}}(x), -1 - I_{\mathcal{N}}(y)\} \\ & = & \max\{\overline{I_{\mathcal{N}}}(x), \overline{I_{\mathcal{N}}}(y)\}. \end{array}$$

Hence, $\oplus X_{\mathcal{N}}, \otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are neutrosophic $\mathcal{N}$-subalgebras of X. □

The following theorem can be proved similarly to Theorem 3.12.

Theorem 3.13. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $\oplus X_{\mathcal{N}}, \otimes X_{\mathcal{N}}$, and $\odot$ are neutrosophic $\mathcal{N}$-ideals of X.*

Theorem 3.14. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, then the sets $X_{T_{\mathcal{N}}} = \{x \in X \mid T_{\mathcal{N}}(x) = T_{\mathcal{N}}(1)\}$, $X_{I_{\mathcal{N}}} = \{x \in X \mid I_{\mathcal{N}}(x) = I_{\mathcal{N}}(1)\}$, and $X_{F_{\mathcal{N}}} = \{x \in X \mid F_{\mathcal{N}}(x) = F_{\mathcal{N}}(1)\}$ are subalgebras of X.*

Proof. Let $x, y \in X_{T_{\mathcal{N}}}$. Then $T_{\mathcal{N}}(x) = T_{\mathcal{N}}(1) = T_{\mathcal{N}}(y)$ and $T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} = T_{\mathcal{N}}(1)$. By (4), we have $T_{\mathcal{N}}(x \cdot y) = T_{\mathcal{N}}(1)$; hence $x \cdot y \in X_{T_{\mathcal{N}}}$. Let $x, y \in X_{I_{\mathcal{N}}}$. Then $I_{\mathcal{N}}(x) = I_{\mathcal{N}}(1) = I_{\mathcal{N}}(y)$ and $I_{\mathcal{N}}(x \cdot y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} = I_{\mathcal{N}}(1)$. By (4), we have $I_{\mathcal{N}}(x \cdot y) = I_{\mathcal{N}}(1)$; hence $x \cdot y \in X_{I_{\mathcal{N}}}$. Let $x, y \in X_{F_{\mathcal{N}}}$. Then $F_{\mathcal{N}}(x) = F_{\mathcal{N}}(1) = T_{\mathcal{N}}(y)$ and $F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} = F_{\mathcal{N}}(1)$. By (4), we have $F_{\mathcal{N}}(x \cdot y) = F_{\mathcal{N}}(1)$; hence $x \cdot y \in X_{F_{\mathcal{N}}}$. Hence, the sets $X_{T_{\mathcal{N}}}, X_{I_{\mathcal{N}}}$, and $X_{F_{\mathcal{N}}}$ are subalgebras of X. □

The following proposition can be proved similarly to Theorem 3.14.

Theorem 3.15. *If $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then the sets $X_{T_{\mathcal{N}}}, X_{I_{\mathcal{N}}}$, and $X_{F_{\mathcal{N}}}$ are ideals of X.*

For any numbers $a^+, a^-, b^+, b^-, c^+, c^- \in [-1, 0]$ such that $a^+ > a^-, b^+ > b^-, c^+ > c^-$ and a nonempty subset G of X, define a neutrosophic $\mathcal{N}$-structure

$$X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix} = \left(X, {T_{\mathcal{N}}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}, {I_{\mathcal{N}}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}, {F_{\mathcal{N}}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix} \right)$$

over X, where

$$\begin{array}{l} {T_{\mathcal{N}}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x) = \begin{cases} a^- & \text{if } x \in G \\ a^+ & \text{otherwise,} \end{cases} \\ {I_{\mathcal{N}}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}(x) = \begin{cases} b^+ & \text{if } x \in G \\ b^- & \text{otherwise,} \end{cases} \\ {F_{\mathcal{N}}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x) = \begin{cases} c^- & \text{if } x \in G \\ c^+ & \text{otherwise.} \end{cases} \end{array}$$

Lemma 3.16. *If the constant 1 of X is in a nonempty subset G of X, then the neutrosophic $\mathcal{N}$-structure $X^G \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ over X satisfies* (4).

Proof. If $1 \in G$, then ${T_{\mathcal{N}}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1) = a^-$, ${I_{\mathcal{N}}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}(1) = b^+$, and ${F_{\mathcal{N}}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix}(1) = c^-$. Thus,

$$(\forall x \in X) \left(\begin{array}{l} {T_{\mathcal{N}}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1) = a^- \leq {T_{\mathcal{N}}}^G \begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x) \\ {I_{\mathcal{N}}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}(1) = b^+ \geq {I_{\mathcal{N}}}^G \begin{bmatrix} b^+ \\ b^- \end{bmatrix}(x) \\ {F_{\mathcal{N}}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix}(1) = c^- \leq {F_{\mathcal{N}}}^G \begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x) \end{array} \right). \tag{9}$$

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Hence, $X^G\begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ satisfies (4). □

Lemma 3.17. *If the neutrosophic $\mathcal{N}$-structure $X^G\begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ over X satisfies* (4)*, then the constant* 1 *of X is in a nonempty subset G of X.*

Proof. Assume that the neutrosophic $\mathcal{N}$-structure $X^G\begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ in X satisfies (4). Then $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1) \leq T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus, $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(g) = a^-$ and so $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1) \geq a^- = T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(g) \geq T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1)$, that is, $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(1) = a^-$. Hence, $1 \in G$. □

Theorem 3.18. *The neutrosophic $\mathcal{N}$-structure $X^G\begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ in X is a neutrosophic $\mathcal{N}$-subalgebra of X if and only if a nonempty subset G of X is a subalgebra of X.*

Proof. Assume that $X^G\begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. Let $x, y \in G$. Then $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x) = a^- = T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(y)$. Thus,

$$\begin{aligned} T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x \cdot y) &\leq \max\left\{T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x), T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(y)\right\} \\ &= \max\{a^-, a^-\} \\ &= a^- \\ &\leq T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x \cdot y) \end{aligned}$$

and so $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x \cdot y) = a^-$. Thus, $x \cdot y \in G$. Hence, G is a subalgebra of X.

Conversely, assume that G is a subalgebra of X. Let $x, y \in X$.

Case 1: Let $x, y \in G$. Then

$$T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x) = a^- = T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(y),$$

$$I_{\mathcal{N}}{}^G\begin{bmatrix} b^+ \\ b^- \end{bmatrix}(x) = b^+ = I_{\mathcal{N}}{}^G\begin{bmatrix} b^+ \\ b^- \end{bmatrix}(y),$$

$$F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x) = c^- = F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(y).$$

Since G is a subalgebra of X, we have $x \cdot y \in G$ and so $T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x \cdot y) = a^-$, $I_{\mathcal{N}}{}^G\begin{bmatrix} b^+ \\ b^- \end{bmatrix}(x \cdot y) = b^+$, and $F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x \cdot y) = c^-$. Thus,

$$T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x \cdot y) = a^- \leq a^- = \max\{a^-, a^-\} = \max\left\{T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(x), T_{\mathcal{N}}{}^G\begin{bmatrix} a^- \\ a^+ \end{bmatrix}(y)\right\},$$

$$I_{\mathcal{N}}{}^G\begin{bmatrix} b^- \\ b^+ \end{bmatrix}(x \cdot y) = b^+ \geq b^+ = \min\{b^+, b^+\} = \min\left\{I_{\mathcal{N}}{}^G\begin{bmatrix} b^- \\ b^+ \end{bmatrix}(x), I_{\mathcal{N}}{}^G\begin{bmatrix} b^- \\ b^+ \end{bmatrix}(y)\right\},$$

$$F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x \cdot y) = c^- \leq c^- = \max\left\{c^-, c^-\right\} = \max\left\{F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(x), F_{\mathcal{N}}{}^G\begin{bmatrix} c^- \\ c^+ \end{bmatrix}(y)\right\}.$$

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Case 2: Let $x \notin G$ or $y \notin G$. Then

$$T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(x) = a^+ \text{ or } T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(y) = a^+,$$

$$I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(x) = b^- \text{ or } I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(y) = b^-,$$

$$F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(x) = c^+ \text{ or } F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(y) = c^+.$$

Thus,

$$\max\left\{T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(x), T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(y)\right\} = a^+,$$

$$\min\left\{I_{\mathcal{N}}{}^{G}\begin{bmatrix}a^+\\a^-\end{bmatrix}(x), I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(y)\right\} = b^-,$$

$$\max\left\{F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(x), F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(y)\right\} = c^+.$$

Therefore,

$$T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(x\cdot y) \leq a^+ = \max\left\{T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(x), T_{\mathcal{N}}{}^{G}\begin{bmatrix}a^-\\a^+\end{bmatrix}(y)\right\}$$

$$I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(x\cdot y) \geq b^- = \min\left\{I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(x), I_{\mathcal{N}}{}^{G}\begin{bmatrix}b^+\\b^-\end{bmatrix}(y)\right\}$$

$$F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(x\cdot y) \leq c^+ = \max\left\{F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(x), F_{\mathcal{N}}{}^{G}\begin{bmatrix}c^-\\c^+\end{bmatrix}(y)\right\}.$$

Hence, $X^{G}\begin{bmatrix}a^-, & b^+, & c^-\\a^+, & b^-, & c^+\end{bmatrix}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

The following theorem can be proved similarly to Theorem 3.18.

Theorem 3.19. *The neutrosophic $\mathcal{N}$-structure $X^{G}\begin{bmatrix}a^-, & b^+, & c^-\\a^+, & b^-, & c^+\end{bmatrix}$ over X is a neutrosophic $\mathcal{N}$-ideal of X if and only if a nonempty subset G of X is an ideal of X.*

Definition 3.20. Let f be an $\mathcal{N}$-function on a nonempty set X. For any $t \in [-1, 0]$, the sets $U(f : t) = \{x \in X \mid f(x) \geq t\}$ is called an upper t-level subset of f, $L(f : t) = \{x \in X \mid f(x) \leq t\}$ is called a lower t-level subset of f, and $E(f : t) = \{x \in X \mid f(x) = t\}$ is called an equal t-level subset of f.

Theorem 3.21. *A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$-subalgebra of X if and only if for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or subalgebras of X.*

Proof. Assume that $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. Let $a, b, c \in [-1, 0]$ be such that $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are nonempty. Let $x, y \in L(T_{\mathcal{N}} : a)$. Then $T_{\mathcal{N}}(x) \leq a$ and $T_{\mathcal{N}}(y) \leq a$, so a is an upper bound of $\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. By (3), we have $T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \leq a$. Thus, $x \cdot y \in L(T_{\mathcal{N}} : a)$. Let $x, y \in U(I_{\mathcal{N}} : b)$. Then $I_{\mathcal{N}}(x) \geq b$ and $I_{\mathcal{N}}(y) \geq b$, so b is a lower bound of $\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. By (3), we have $I_{\mathcal{N}}(x \cdot y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \geq b$. Thus, $x \cdot y \in U(I_{\mathcal{N}} : b)$. Let $x, y \in L(F_{\mathcal{N}} : c)$. Then $F_{\mathcal{N}}(x) \leq c$ and $F_{\mathcal{N}}(y) \leq c$, so c is an upper bound of $\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. By (3), we have $F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \leq c$. Thus, $x \cdot y \in L(F_{\mathcal{N}} : c)$. Hence, $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are subalgebras of X.

Conversely, assume that for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or subalgebras of X. Let $x, y \in X$. Then $T_{\mathcal{N}}(x) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$ and $T_{\mathcal{N}}(x) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. Thus, $x, y \in L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\})$ is a subalgebra of X. Then $x \cdot y \in L(T_{\mathcal{N}} : \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\})$. Thus, $T_{\mathcal{N}}(x \cdot y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\}$. Let $x, y \in X$. Then $I_{\mathcal{N}}(x) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$ and $I_{\mathcal{N}}(x) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. Thus, $x, y \in U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\})$ is a subalgebra of X. Then $x \cdot y \in U(I_{\mathcal{N}} : \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. Thus,

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$I_{\mathcal{N}}(x \cdot y) \geq \max\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\}$. Let $x, y \in X$. Then $F_{\mathcal{N}}(x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$ and $F_{\mathcal{N}}(x) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. Thus, $x, y \in L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}) \neq \emptyset$. By the assumption, we have $L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\})$ is a subalgebra of X. Then $x \cdot y \in L(F_{\mathcal{N}} : \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\})$. Thus, $F_{\mathcal{N}}(x \cdot y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\}$. Hence, $X_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

The following theorem can be proved similarly to Theorem 3.21.

Theorem 3.22. *A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$-ideal of X if and only if for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or ideals of X.*

The following two corollaries are a straightforward result of Theorems 3.21 and 3.22.

Corollary 3.23. *A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$-subalgebra of X if and only if for all $a, b, c \in [-1, 0]$, the set $L(T_{\mathcal{N}} : a) \cap U(I_{\mathcal{N}} : b) \cap L(F_{\mathcal{N}} : c)$ is either empty or a subalgebra of X.*

Corollary 3.24. *A neutrosophic $\mathcal{N}$-structure $X_{\mathcal{N}}$ over X is a neutrosophic $\mathcal{N}$-ideal of X if and only if for all $a, b, c \in [-1, 0]$, the set $L(T_{\mathcal{N}} : a) \cap U(I_{\mathcal{N}} : b) \cap L(F_{\mathcal{N}} : c)$ is either empty or an ideal of X.*

Definition 3.25. Let $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ be neutrosophic $\mathcal{N}$-structures of X and Y, respectively. The Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}} = (X \times Y, \Delta, \Theta, \Lambda)$ defined by

$$(\forall (x, y) \in X \times Y) \left(\begin{array}{l} \Delta(x, y) = \max\{T_{\mathcal{N}X}(x), T_{\mathcal{N}Y}(y)\} \\ \Theta(x, y) = \min\{I_{\mathcal{N}X}(x), I_{\mathcal{N}Y}(y)\} \\ \Lambda(x, y) = \max\{F_{\mathcal{N}X}(x), F_{\mathcal{N}Y}(y)\} \end{array} \right), \tag{10}$$

where Δ, Θ, and Λ are $\mathcal{N}$-functions on $X \times Y$.

Remark 3.26. Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Then $(X \times Y, \diamond, (1_X, 1_Y))$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.27. *If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$-subalgebras of Hilbert algebras X and Y, respectively, then the Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X \times Y$.*

Proof. Assume that $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$-subalgebras of Hilbert algebras X and Y, respectively. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{array}{rcl} \Delta((x_1, y_1) \diamond (x_2, y_2)) & = & \Delta(x_1 \cdot x_2, y_1 \star y_2) \\ & = & \max\{T_{\mathcal{N}X}(x_1 \cdot x_2), T_{\mathcal{N}Y}(y_1 \star y_2)\} \\ & \leq & \max\{\max\{T_{\mathcal{N}X}(x_1), T_{\mathcal{N}X}(x_2)\}, \max\{T_{\mathcal{N}Y}(y_1), T_{\mathcal{N}Y}(y_2)\}\} \\ & = & \max\{\max\{T_{\mathcal{N}X}(x_1), T_{\mathcal{N}Y}(y_1)\}, \max\{T_{\mathcal{N}X}(x_2), T_{\mathcal{N}Y}(y_2)\}\} \\ & = & \max\{\Delta(x_1, y_1), \Delta(x_2, y_2)\}, \end{array}$$

$$\begin{array}{rcl} \Theta((x_1, y_1) \diamond (x_2, y_2)) & = & \Theta(x_1 \cdot x_2, y_1 \star y_2) \\ & = & \min\{I_{\mathcal{N}X}(x_1 \cdot x_2), I_{\mathcal{N}Y}(y_1 \star y_2)\} \\ & \geq & \min\{\min\{I_{\mathcal{N}X}(x_1), I_{\mathcal{N}X}(x_2)\}, \min\{I_{\mathcal{N}Y}(y_1), I_{\mathcal{N}Y}(y_2)\}\} \\ & = & \min\{\min\{I_{\mathcal{N}X}(x_1), I_{\mathcal{N}Y}(y_1)\}, \min\{I_{\mathcal{N}X}(x_2), I_{\mathcal{N}Y}(y_2)\}\} \\ & = & \min\{\Theta(x_1, y_1), \Theta(x_2, y_2)\}, \end{array}$$

$$\begin{array}{rcl} \Lambda((x_1, y_1) \diamond (x_2, y_2)) & = & \Lambda(x_1 \cdot x_2, y_1 \star y_2) \\ & = & \max\{F_{\mathcal{N}X}(x_1 \cdot x_2), F_{\mathcal{N}Y}(y_1 \star y_2)\} \\ & \leq & \max\{\max\{F_{\mathcal{N}X}(x_1), F_{\mathcal{N}X}(x_2)\}, \max\{F_{\mathcal{N}Y}(y_1), F_{\mathcal{N}Y}(y_2)\}\} \\ & = & \max\{\max\{F_{\mathcal{N}X}(x_1), F_{\mathcal{N}Y}(y_1)\}, \max\{F_{\mathcal{N}X}(x_2), F_{\mathcal{N}Y}(y_2)\}\} \\ & = & \max\{\Lambda(x_1, y_1), \Lambda(x_2, y_2)\}. \end{array}$$

Hence, $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X \times Y$. □

The following proposition can be proved similarly to Proposition 3.27.

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Proposition 3.28. *If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$-ideals of Hilbert algebras X and Y, respectively, then the Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X \times Y$.*

The following two theorems are a straightforward result of Propositions 3.27 and 3.28, and Theorems 3.12 and 3.13.

Theorem 3.29. *If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$-subalgebras of Hilbert algebras X and Y, respectively, then $\oplus(X_{\mathcal{N}} \times Y_{\mathcal{N}}), \otimes(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, and $\odot(X_{\mathcal{N}} \times Y_{\mathcal{N}})$ are neutrosophic $\mathcal{N}$-subalgebras of X.*

Theorem 3.30. *If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic $\mathcal{N}$-ideals of Hilbert algebras X and Y, respectively, then $\oplus(X_{\mathcal{N}} \times Y_{\mathcal{N}}), \otimes(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, and $\odot(X_{\mathcal{N}} \times Y_{\mathcal{N}})$ are neutrosophic $\mathcal{N}$-ideals of X.*

Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. A mapping $f : X \to Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1_X) = 1_Y$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any neutrosophic $\mathcal{N}$-structure $Y_{\mathcal{N}}$ over Y, we define a new neutrosophic $\mathcal{N}$-structure $f^{-1}(Y_{\mathcal{N}}) = (X, T_{f^{-1}(Y_{\mathcal{N}})}, I_{f^{-1}(Y_{\mathcal{N}})}, F_{f^{-1}(Y_{\mathcal{N}})})$ over X by

$$(\forall x \in X) \left(\begin{array}{l} T_{f^{-1}(Y_{\mathcal{N}})}(x) = T_{\mathcal{N}Y}(f(x)) \\ I_{f^{-1}(Y_{\mathcal{N}})}(x) = I_{\mathcal{N}Y}(f(x)) \\ F_{f^{-1}(Y_{\mathcal{N}})}(x) = F_{\mathcal{N}Y}(f(x)) \end{array} \right).$$

Theorem 3.31. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Let $f : X \to Y$ be a homomorphism and $Y_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure over Y. If $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of Y, then $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Assume that $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of Y. Let $x, y \in X$. Then

$$\begin{array}{rcl} T_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) & = & T_{\mathcal{N}Y}(f(x \cdot y)) \\ & = & T_{\mathcal{N}Y}(f(x) \cdot f(y)) \\ & \leq & \max\{T_{\mathcal{N}Y}(f(x)), T_{\mathcal{N}Y}(f(y))\} \\ & = & \max\{T_{f^{-1}(Y_{\mathcal{N}})}(x), T_{f^{-1}(Y_{\mathcal{N}})}(y)\}, \end{array}$$

$$\begin{array}{rcl} I_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) & = & I_{\mathcal{N}Y}(f(x \cdot y)) \\ & = & I_{\mathcal{N}Y}(f(x) \cdot f(y)) \\ & \geq & \min\{I_{\mathcal{N}Y}(f(x)), I_{\mathcal{N}Y}(f(y))\} \\ & = & \min\{I_{f^{-1}(Y_{\mathcal{N}})}(x), I_{f^{-1}(Y_{\mathcal{N}})}(y)\}, \end{array}$$

$$\begin{array}{rcl} F_{f^{-1}(Y_{\mathcal{N}})}(x \cdot y) & = & F_{\mathcal{N}Y}(f(x \cdot y)) \\ & = & F_{\mathcal{N}Y}(f(x) \cdot f(y)) \\ & \leq & \max\{F_{\mathcal{N}Y}(f(x)), F_{\mathcal{N}Y}(f(y))\} \\ & = & \max\{F_{f^{-1}(Y_{\mathcal{N}})}(x), F_{f^{-1}(Y_{\mathcal{N}})}(y)\}. \end{array}$$

Hence, $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

The following theorem can be proved similarly to Theorem 3.31.

Theorem 3.32. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Let $f : X \to Y$ be a homomorphism and $Y_{\mathcal{N}}$ be a neutrosophic $\mathcal{N}$-structure over Y. If $Y_{\mathcal{N}}$ is a neutrosophic $\mathcal{N}$-ideal of Y, then $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic $\mathcal{N}$-ideal of X.*

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4 Conclusion

In this paper, we have introduced the notions of neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras. Conditions for neutrosophic $\mathcal{N}$-structures to be neutrosophic $\mathcal{N}$-subalgebras and neutrosophic $\mathcal{N}$-ideals of Hilbert algebras are provided. From our study, we found that the Cartesian product of neutrosophic $\mathcal{N}$-subalgebras (ideals) of a Hilbert algebra is a neutrosophic $\mathcal{N}$-subalgebra (ideal). Finally, we found that the homomorphic pre-image of a neutrosophic $\mathcal{N}$-subalgebra (ideal) of a Hilbert algebra is a neutrosophic $\mathcal{N}$-subalgebra (ideal).

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