



## A New Proof of Feuerbach's Theorem

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### Abstract

Feuerbach's theorem on the tangent of the circle of the nine points and the inscribed and exinscribed circle is considered one of the most beautiful theorems in geometry. In this paper, we offer a basic proof of this theorem starting from one of Gh. Buicliu's ideas [1].

**Keywords:** Feuerbach's theorem; circle of the nine points; inscribed circles; exinscribed circles

**The Theorem:** The circle of the nine points of a triangle is tangent to the inscribed circle and to the exinscribed circles of the triangle (K. Feuerbach – 1822).

**Proof:** May  $O_9$  be the center of the circle of the 9 points of the  $ABC$  triangle and  $I$  the center of its inscribed circle. We denote by  $D$  the contact of the inscribed circle with  $BC$  and with  $D'$  its diameter within the inscribed circle. May  $AH_a$  be the height from  $A$  (see Figure 1); we will denote by  $K$  and  $M$  the intersections of the parallel going through  $I$  to  $AD'$  with  $AH_a$ , respectively with  $BC$  and  $L$  the intersection of the parallel at  $AD'$ , drawn through  $D$ , with  $AH_a$ .

We also note with  $P$  the intersection of the  $IL$  and  $BC$  lines and with  $Q$  the intersection of the half-line  $(D'K$  with the inscribed circle).

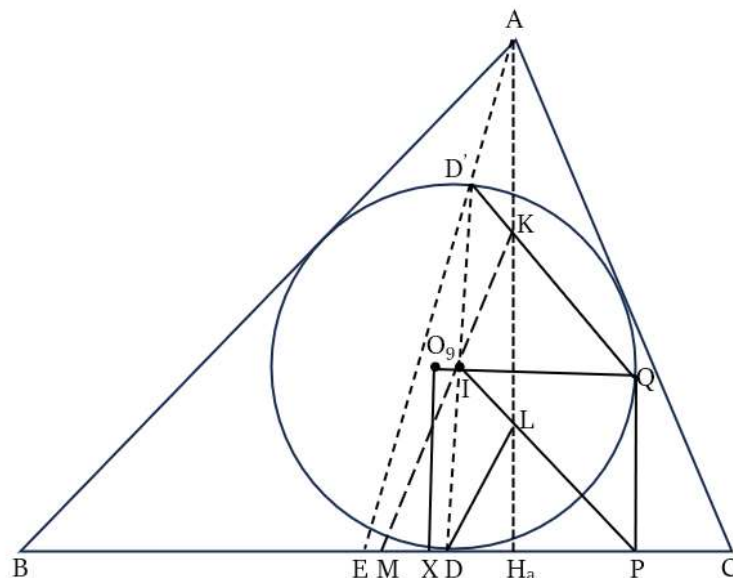


Figure 1

We will prove that  $Q$  is the tangency point of the inscribed circle with the circle of the 9 points. ( $Q$  is one of Feuerbach's points.) From the fact that the inscribed circle ( $I$ ) and the exinscribed circle  $A$  ( $I_a$ ) are homothetical through the homothety of pole  $A$ , we know that the point  $E$  is the exinscribed ( $I_a$ ) circle's contact with  $BC$ , and that

points  $D$  and  $E$  are isotomical (symmetrical to  $M$ , the middle of  $BC$ ). We note that  $\alpha = m(\widehat{ID'Q})$ , because  $ID' = IQ$ , thus  $m(\widehat{IQD'}) = \alpha$ . The  $ID'AK$  and  $IDLK$  quadrilaterals are parallelograms.

$ID' \parallel AK$  și  $ID' = KL = r$  (the ray of the inscribed circle), thus the  $ID'KL$  quadrilateral is a parallelogram, so  $m(\widehat{IQD'}) = m(\widehat{LIK}) = \alpha$ . Likewise,  $m(\widehat{DIL}) = m(\widehat{LIQ}) = \alpha$ , which, along with  $ID = IQ = r$ , shows that the points  $D$  and  $Q$  are symmetrical to  $IP$ . Moreover, because  $PD$  is tangent in  $D$  to the  $(I)$  circle, we have that  $PQ$  is tangent in  $Q$  to the  $(I)$  inscribed circle, so  $PD = PQ$ .

The fact that the  $(MI; DL)$  and  $(DD'; AH_a)$  lines are parallel leads to:  $\frac{PM}{PD} = \frac{PI}{PL} = \frac{PD}{PH_a}$ .

We remember that  $PD^2 = PM \cdot PH_a$ , and seeing as how  $PD = PQ$ , we have that  $PQ^2 = PM \cdot PH_a$ , which shows  $PQ$  is tangent to the circumscribed circle of the  $QMH_a$  triangle.

We prove this circle is the circle of the 9 points of the  $ABC$  triangle.

It is sufficient to prove that  $Q$  belongs to the circle of the 9 points, so we show that  $O_9Q = \frac{R}{2}$  ( $R$  being the ray of the circumscribed circle and  $O_9$  the center of the circle of the 9 points). We denote by  $X$  the orthogonal projection of  $O_9$  on  $BC$ . We have that  $O_9X^2 + PX^2 = PO_9^2$ .

On another note,  $O_9M = \frac{R}{2}$  and  $O_9X^2 = \frac{R^2}{4} - MX^2$ . We know that  $X$  este is the middle of the  $MH_a$  segment (because  $O_9$  is the middle of  $OH$  – where  $O$  and  $H$  are, respectively, the center of the circumscribed circle and the orthocenter of  $ABC$ ).

We obtain  $PO_9^2 = PX^2 - MX^2 + \frac{R^2}{4}$ .

But  $PX^2 - MX^2 = (PX + MX)(PX - MX) = PM \cdot (PH_a + H_aX - MX)$ .

However,  $MX = MH_a$ , thus  $PX^2 - MX^2 = PM \cdot PH_a = PD^2 = PQ^2$ .

We obtain  $PO_9^2 = PQ^2 + \frac{R^2}{4}$ .

We interpret this relationship as follows: the triangle with the length sides  $PO_9$ ,  $PQ$  and  $\frac{R}{2}$  is rectangular. Because  $PQ$  is the tangent of the circumscribed circle to the  $MH_aQ$  triangle, we obtain the  $\frac{R}{2}$  length segment with an end in  $Q$  and the other on the  $IQ$  line. But the circle circumscribed to the  $MH_aQ$  triangle has its center at the intersection of the bisection of the  $MH_a$  segment with perpendiculars in  $Q$  on  $PQ$ , just as  $O_9$  is the bisection of  $MH_a$  and is at the distance of  $O_9M = \frac{R}{2}$  of  $M$  and  $H_a$ , thus  $O_9Q = \frac{R}{2}$ , so  $O_9$  is the center of the circumscribed circle of the  $MH_aQ$  triangle.

We continue to prove that the circle of the 9 points is tangent to the A-exinscribed circle in a Feuerbach which we note by  $Q_a$ . We build on the  $E$  point the orthogonal projection of  $I_a$  on  $BC$  and  $E'$  its diameter in the  $(I_a)$  circle.

We draw the parallels to  $AE'$  through  $I_a$  and  $E$  and we denote by  $K_a$  respectively  $L_a$  their intersections with the height  $AH_a$  of the  $ABC$  triangle (see *Figure 2*). The intersection of the  $I_aK_a$  lines and  $BC$  is  $M$  – the middle of  $BC$ , but the intersection of the  $I_aL_a$  line with  $BC$  we denote by  $P_a$ . We affirm that the  $Q_a$  Feuerbach point is the intersection of the exinscribed A-circle with the  $E'K_a$  segment.

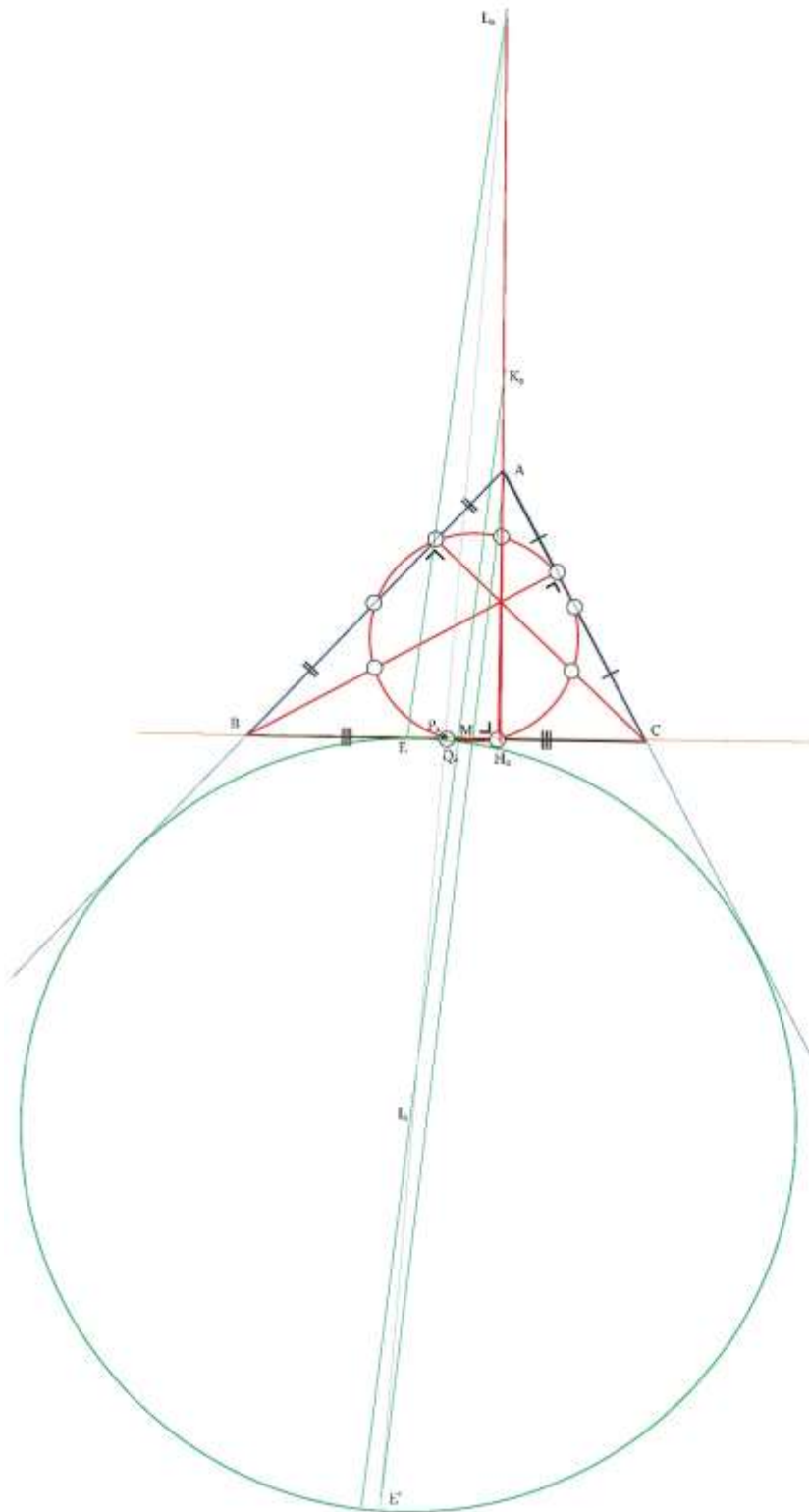


Figure 2

We prove that the  $Q_a$  point is symmetrical to  $P_a$  on the  $I_aK_a$  line. From what we have so far, the  $E'I_aL_aK_a$  quadrilateral is a parallelogram, so  $\sphericalangle EI_aP_a = \sphericalangle I_aE'Q_a$  (alt. int.). because the  $I_aF'Q_a$  triangle is isosceles we have that:  $\sphericalangle I_aE'Q_a = \sphericalangle I_aQ_aE'$ .

The  $\sphericalangle EI_aQ_a$  angle is exterior to the  $E'I_aQ_a$  triangle, so  $\sphericalangle EI_aQ_a = 2 \cdot \sphericalangle I_aQ_aE'$  and seeing as  $\sphericalangle EI_aP_a = \sphericalangle I_aQ_aE'$ , we have that  $\sphericalangle EI_aP_a = \sphericalangle I_aQ_aE'$ , relationship which, together with  $I_aE = I_aQ_a$  implies  $\Delta P_aI_aE \cong \Delta Q_aI_aP_a$ , and

from here we obtain that  $P_aQ_a = P_aE$  and  $P_aQ_a \perp I_aK_a$ . From the parallelism of the  $EL_a$  line with the  $I_aK_a$  line and  $EE'$  with  $AH_a$  we have that:  $\frac{P_aM}{P_aE} = \frac{I_aP_a}{P_aL_a} = \frac{P_aE}{P_aH_a}$ .

We remember that  $P_aE^2 = P_aM \cdot P_aH_a$ , thus  $P_aQ_a^2 = P_aM \cdot P_aH_a$ .

This relationship shows that the A-exinscribed circle and the circumscribed circle of the  $MH_aQ_a$  triangle are exterior tangents in the  $Q_a$  point.

We continue to prove that the circumscribed circle of the  $MH_aQ_a$  triangle is the circle of the 9 points ( $O_9$ ).

We denote by  $T$  the orthogonal projection of  $O_9$  on  $BC$ , we have  $P_aO_9^2 = P_aT^2 + O_9T^2$ .

But  $O_9T^2 = O_9M^2 - MT^2 = \frac{R}{4} - MT^2$ . We obtain  $P_aT^2 - MT^2 + \frac{R}{4} = P_aO_9^2$ . But  $MT = H_aT = \frac{1}{2}MH_a$ , we have  $P_aT^2 - MT^2 = (P_aT + TH_a)(P_aT - MT) = P_aH_a \cdot P_aM$ . We obtained  $P_aO_9^2 = P_aQ_a^2 + \frac{R^2}{4}$ .

We interpret this relationship as follows: we can build a rectangular triangle with the hypotenuses  $P_aO_9$  with a  $P_aQ_a$  cathetus and a  $\frac{R}{2}$  length cathetus. Because  $m(\widehat{P_aQ_aI_a}) = 90^\circ$ , the rectangular triangle with the  $P_aO_9$  hypotenuses will have a  $90^\circ$  angle in  $Q_a$  and the third corner will be on the prolongment of the  $I_aQ_a$  segment. That corner will be on the bisection of the  $MH_a$  segment at the distance of  $OM = OH_a = \frac{R}{2}$ . Because  $O_9M = O_9H_a = \frac{R}{2}$ , we have that  $O_9$  is the circumscribed center of the  $MH_aQ_a$  triangle, meaning this last circle is the circle of the 9 points of the  $ABC$  triangle. The theorem is proved the same way for the B-exinscribed and C-exinscribed circles.

## References

- [1]. Gh. Buicliu: *Probleme de construcții geometrice cu rigla și compasul*. Tehnica (publishing house), Bucharest, 1967.
- [2]. Florentin Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*. Education Publisher, Columbus, Ohio, 2012.