The Runge-Kutta Numerical Method of Rank Seven for the Solutions of Some Refined Neutrosophic Differential Problems

Belal Batiha[1,*]

[1]Department of Mathematics, Faculty of Science, Jadara University, Irbid, Jordan

Email: B.bateha@jadara.edu.jo

Abstract

In this paper, we present a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provide the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provide numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

Keywords: Refined neutrosophic number; Refined neutrosophic Runge-Kutta of rank seven; Numerical error; absolute error

1. Introduction

Neutrosophic logic presented by Smarandache [6] is considered as a new generalization of fuzzy logic that takes into account the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena.

It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even in computer science [10-13].

The Runge-Kutta method is one of the reference methods in numerical analysis that has been dealt with and developed by many researchers around the world. Different ranks of this method have been used in order to find numerical solutions and approximate errors for various problems in applied mathematics [4-9].

The applications of neutrosophic methods in numerical analysis have been studied by many authors see [14-18]. Where, we can see neutrosophic modelled problems with their approximate solutions and absolute errors were presented numerically [19-25].

The main goal of numerical analysis is to study the numerical approximations of solutions for many different problems (algebraic or differential) with a clear computation of errors that emerge from the approximation process [1-3].

This has motivated us to we present a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provide the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provide numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

2. Main Discussion

The Refined neutrosophic Runge-Kutta method of rank 7:

$$\begin{aligned} t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2 &= t_n + j_nI_1 + e_nI_2 + \frac{m + nI_1 + lI_2}{192}(32(s_1 + s_1I_1 + s_1I_2) + 80(s_2 + s_2I_1 + s_2I_2) + 48(s_3 \\ &+ s_3I_1 + s_3I_2) + 24(s_4 + s_4I_1 + s_4I_2) + 4(s_5 + s_5I_1 + s_5I_2) + 2(s_6 + s_6I_1 + s_6I_2) + (s_7 \\ &+ s_7I_1 + s_7I_2) + (s_8 + s_8I_1 + s_8I_2)) \end{aligned}$$

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Where:

$s_1 + s_1I_1 + s_1I_2 = f(h_n + g_nI_1 + r_nI_2, t_n + j_nI_1 + e_nI_2)$

$s_2 + s_2I_1 + s_2I_2 = f(h_n + g_nI_1 + r_nI_2 + \frac{m+nI_1+lI_2}{2}, t_n + j_nI_1 + e_nI_2 + \frac{1}{2}(m + nI_1 + lI_2)(s_1 + s_1I_1 + s_1I_2))$

$s_3 + s_3I_1 + s_3I_2 = f(h_n + g_nI_1 + r_nI_2 + \frac{m+nI_1+lI_2}{2}, t_n + j_nI_1 + e_nI_2 + \frac{1}{2}(m + nI_1 + lI_2)(s_2 + s_2I_1 + s_2I_2))$

$s_4 + s_4I_1 + s_4I_2 = f(h_n + g_nI_1 + r_nI_2 + m + nI_1 + lI_2, t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)(s_3 + s_3I_1 + s_3I_2))$

$s_5 + s_5I_1 + s_5I_2 = f(h_n + g_nI_1 + r_nI_2 + m + nI_1 + lI_2, t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)(s_4 + s_4I_1 + s_4I_2))$

$s_6 + s_6I_1 + s_6I_2 = f(h_n + g_nI_1 + r_nI_2 + m + nI_1 + lI_2, t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)(s_5 + s_5I_1 + s_5I_2))$

$s_7 + s_7I_1 + s_7I_2 = f(h_n + g_nI_1 + r_nI_2 + m + nI_1 + lI_2, t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)(s_6 + s_6I_1 + s_6I_2))$

$s_8 + s_8I_1 + s_8I_2 = f(h_n + g_nI_1 + r_nI_2 + m + nI_1 + lI_2, t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)(s_7 + s_7I_1 + s_7I_2))$

The stability analysis for refined variables:

$t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2 =$

$t_n + j_nI_1 + e_nI_2 + (m + nI_1 + lI_2)\phi(h_n + g_nI_1 + r_nI_2, t_n + j_nI_1 + e_nI_2, +nI_1 + lI_2)$ (1)

The general is :

$$(t_n + j_nI_1 + e_nI_2)(h_n + g_nI_1 + r_nI_2) - (t_{n+1} + j_{n+1}I_1 + e_{n+1}I_2) = O((m + nI_1 + lI_2)^{p+1}) \quad (2)$$

we get the following:

$$(t_n + j_nI_1 + e_nI_2)(h_n + g_nI_1 + r_nI_2) = E(\lambda(m + nI_1 + lI_2))(t_n + j_nI_1 + e_nI_2) \quad (3)$$

we get :

$$(t_n + j_nI_1 + e_nI_2)' = \lambda(t_n + j_nI_1 + e_nI_2), (t_n + j_nI_1 + e_nI_2)(h_0 + g_0I_1 + r_0I_2) = (t + jI_1 + eI_2)_0 \quad (4)$$

Thus:

$E(\overline{(m + nI_1 + lI_2)}) = 1 + \overline{(m + nI_1 + lI_2)} + \frac{1}{2!}\overline{(m + nI_1 + lI_2)}^2 + \dots\dots\dots + \frac{1}{p!}\overline{(m + nI_1 + lI_2)}^p +$
$O(\overline{(m + nI_1 + lI_2)}^{p+1})$ (5)

RK		stability period
7	$1 + \overline{(m + nI_1 + lI_2)} + \frac{\overline{(m + nI_1 + lI_2)}^2}{2!} + \frac{\overline{(m + nI_1 + lI_2)}^3}{3!} + \frac{\overline{(m + nI_1 + lI_2)}^4}{4!} + \frac{\overline{(m + nI_1 + lI_2)}^5}{5!} + \frac{\overline{(m + nI_1 + lI_2)}^6}{6!} + \frac{\overline{(m + nI_1 + lI_2)}^7}{7!}$	$(-2.12265 + 1.1937I_1 + 0.0012I_2)$

Numerical applications:

Example.

Solve the following system of neutrosophic differential equations:

$(t + jI_1 + eI_2)_1' = (t + jI_1 + eI_2)_2, (t + jI_1 + eI_2)_1(0) = 1 + I_1 + I_2$

$(t + jI_1 + eI_2)_2' = -1001 - 1001I_1 - 1001I_2,\ (t + jI_1 + eI_2)_2 - 1000(I_1 + I_2)\ (t + jI_1 + eI_2)_1, (t + jI_1 + eI_2)_2(I) = -1 - I_1 - I_2$

We will take the value of the step length $h = 0.001(1 + I_1 + I_2)$

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Example:

Solve the following system of differential equations:

$$(t + jI_1 + eI_2)'_1 = (600 + 300I)(t + jI_1 + eI_2)_1{}^2\left((t + jI_1 + eI_2)_2 - (t + jI_1 + eI_2)_1{}^3\right), (t + jI_1 + eI_2)_1(I) = 0.1 + I$$

$$(t + jI_1 + eI_2)'_2 = (-200 - 100(I_1+I_2))\left((t + jI_1 + eI_2)_2 - (t + jI_1 + eI_2)_1{}^3\right) + 2(1 - (t + jI_1 + eI_2)_2), (t + jI_1 + eI_2)_2(I_1 + I_2) = -0.1 - (I_1+I_2)$$

Table 1: the results of solving the problem in the first example

Refined method of Seventh rank error values		**Truth values**		**refined method-Kutta of the Seventh values**	
***First variable* error values**	***Second variable* error values**	**First variable Values**	***Second variable* Values**	***First variable* Values**	***Second variable* Values**
(I_1+I_2)	(I_1+I_2)	1+(I_1+I_2)	-1-(I_1+I_2)	1+(I_1+I_2)	-1-(I_1+I_2)
2.16675e-14+I_1+I_2	3.2213e-14+I_1+I_2	0. 0638+0.928 +I_1+I_2	-0. 0638-0.928(I_1+I_2)	0. 0638+0. 0638((I_1+I_2))	-0. 0638-0. 0638(I_1+I_2)
5.23639e-14+I_1+I_2	6.31e-14+I_1+I_2	0.93311+I_1+I_2	-(0.93311+I_1+I_2)	0.9447+I_1+I_2	-(0.90637+I_1+I_2)
9.114e-14+I_1+I_2	9.3452e-14+I_1+I_2	0.90632+I_1+I_2	-(0.9332+I_1+I_2)	0.9332+I_1+I_2	-(0.9332+I_1+I_2)
1.20063e-11+I_1+I_2	1.22173e-11+I_1+I_2	0.9673913+I_1+I_2	-(0.9739143+I_1+I_2)	0.9739143+I_1+I_2	-(0.9739143+I_1+I_2)
1.10342e-11+I_1+I_2	1.10342e-11+I_1+I_2	0.9115+I_1+I_2	-(0.9115+I_1+I_2)	0.973915+I_1+I_2	-(0. 0739115+I_1+I_2)
1.309049113e-11+I_1+I_2	1.27391913e-11+I_1+I_2	0.60904807 +I_1+I_2	-(0.6570904107+I_1+I_2)	0.65739107+I_1+I_2	-(0. 73090407+I_1+I_2)
2.11295e-11+I_1+I_2	2.11295e-11+I_1+I_2	0.9443861+I_1+I_2	0.9443861+I_1+I_2	0.9443861+I_1+I_2	0.9443861+I_1+I_2
2.3090487e-11+I_1+I_2	2. 090487e-11+I_1+I_2	0.909041293 +I_1+I_2	-(0.909041293+I_1+I_2)	0.909041293+I_1+I_2	-(0.909041293+I_1+I_2)
2.20904876e-11+I_1+I_2	2. 09048876e-11+I_1+I_2	0.9088+I_1+I_2	_(0.9088+I_1+I_2)	0.9088+I_1+I_2	-(0.9088+I_1+I_2)
3.0904145e-11+I_1+I_2	3. 090445e-11+I_1+I_2	0.709042+I_1 +I_2	-(0.77602+I_1+I_2)	0.7709042+I_1+I_2	-(0.7709042+I_1+I_2)

Table 2: results of solving the problem in the second example

Refined method of Seventh rank error values		**Truth values**		**refined method-Kutta of the Seventh values**	
***First variable* error values**	***Second variable* error values**	**First variable Values**	***Second variable* Values**	***First variable* Values**	***Second variable* Values**
I_1+I_2	I_1+I_2	0.1+I_1+I_2	-0.1-I_1+I_2	0.1+I_1+I_2	-0.1-I_1+I_2
5.46645e-9+I_1+I_2	1.212e-7+I_1+I_2	0.092237+I_1 +I_2	-0.073498-I_1+I_2	0. 092234+I_1+I_2	-0.073498-I_1+I_2
1.221376e-8+I_1+I_2	4.2437e-7+I_1+I_2	0.0911369+I_1+I_2	-0.0379445-I_1+I_2	0.0911366+I_1+I_2	-0.0379444-I_1+I_2
1.0987e-8+I_1+I_2	5.7787e-7+I_1+I_2	0.094696+I_1 +I_2	-0.046064-I_1+I_2	0.094693+I_1+I_2	-0.049621-I_1+I_2

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2.311209e-8+I_1+I_2	6.52014e-7+I_1+I_2	0.092064+I_1+I_2	-0.022398-I_1+I_2	0.092060+I_1+I_2	-0.0333913-I_1+I_2
2.3329e-9+I_1+I_2	6.66783e-8+I_1+I_2	0.095833+I_1+I_2	-0.231998+I_1+I_2	0.095830+I_1+I_2	-0.011690133-I_1+I_2
2.1092e-9+I_1+I_2	6.11038e-8+I_1+I_2	0.094155+I_1+I_2	-0.0100078-I_1+I_2	0.094151+I_1+I_2	-0.0017110-I_1+I_2
2.51375e-9+I_1+I_2	6.1167e-05+I_1+I_2	0.097964+I_1+I_2	-0.01437-I_1+I_2	0.097961+I	-0.01433-I_1+I_2
2.256248e-9+I_1+I_2	6.2334e-05+I_1+I_2	0.03326+I_1+I_2	-0.01125-I_1+I_2	0.03323+I_1+I_2	-0.01121+I_1+I_2
2.3143e-9+I_1+I_2	5.673108e-05+I_1+I_2	0.08873+I_1+I_2	-0.008514-I_1+I_2	0.08870+I_1+I_2	-0.008510-I_1+I_2
2.100123e-9+I_1+I_2	5.4431e-05+I_1+I_2	0.0897664+I_1+I_2	-0.0023638-I_1+I_2	0.0897661+I_1+I_2	-0.0023634-I_1+I_2

Refined Implicit RK Method of the seventh rank:

$$t_{n+1}+j_{n+1}I_1+e_{n+1}I_2 = t_n+j_nI_1+e_nI_2-\frac{m+nI_1+lI_2}{192}(32(s_1+s_1I_1+s_1I_2)+80(s_2+s_2I_1+s_2I_2)+48(s_3+s_3I_1+s_3I_2)+24(s_4+s_4I_1+s_4I_2)+4(s_5+s_5I_1+s_5I_2)+2(s_6+s_6I_1+s_6I_2)+(s_7+s_7I_1+s_7I_2)+(s_8+s_8I_1+s_8I_2))$$

Where:

$s_1+s_1I_1+s_1I_2 = f(h_n+g_nI_1+r_nI_2, t_n+j_nI_1+e_nI_2)$

$s_2+s_2I_1+s_2I_2 = f(h_n+g_nI_1+r_nI_2-\frac{m+nI_1+lI_2}{2}, t_n+j_nI_1+e_nI_2-\frac{1}{2}(m+nI_1+lI_2)(s_1+s_1I_1+s_1I_2))$

$s_3+s_3I_1+s_3I_2 = f(h_n+g_nI_1+r_nI_2-\frac{m+nI_1+lI_2}{2}, t_n+j_nI_1+e_nI_2-\frac{1}{2}(m+nI_1+lI_2)(s_2+s_2I_1+s_2I_2))$

$s_4+s_4I_1+s_4I_2 = f(h_n+g_nI_1+r_nI_2-(m+nI_1+lI_2), t_n+j_nI_1+e_nI_2-(m+nI_1+lI_2)(s_3+s_3I_1+s_3I_2))$

$s_5+s_5I_1+s_5I_2 = f(h_n+g_nI_1+r_nI_2-(m+nI_1+lI_2), t_n+j_nI_1+e_nI_2-(m+nI_1+lI_2)(s_4+s_4I_1+s_4I_2))$

$s_6+s_6I_1+s_6I_2 = f(h_n+g_nI_1+r_nI_2-(m+nI_1+lI_2), t_n+j_nI_1+e_nI_2-(m+nI_1+lI_2)(s_5+s_5I_1+s_5I_2))$

$s_7+s_7I_1+s_7I_2 = f(h_n+g_nI_1+r_nI_2-(m+nI_1+lI_2), t_n+j_nI_1+e_nI_2-(m+nI_1+lI_2)(s_6+s_6I_1+s_6I_2))$

$s_8+s_8I_1+s_8I_2 = f(h_n+g_nI_1+r_nI_2-(m+nI_1+lI_2), t_n+j_nI_1+e_nI_2-(m+nI_1+lI_2)(s_7+s_7I_1+s_7I_2))$

Stability of Seventh Order refined neutrosophic RK method:

Consider the general formula:

$$t_{n+1}+j_{n+1}I_1+e_{n+1}I_2 = y_n+(m+nI_1+lI_2)\phi(h_{n+1}+g_{n+1}I_1+r_{n+1}I_2, t_{n+1}+j_{n+1}I_1+e_{n+1}I_2, m+nI_1+lI_2) \quad (6)$$

The refined neutrosophic IRK formula (6) of phase R is said to be of rank P if:

$(t+jI_1+eI_2)(h_{n+1}+g_{n+1}I_1+r_{n+1}I_2)-(t_{n+1}+j_{n+1}I_1+e_{n+1}I_2) = O((m+nI_1+lI_2)^{p+1})$

Hence, $(t+jI_1+eI_2)_{n+1} = E(\lambda(m+nI_1+lI_2))(t+jI_1+eI_2)_n$ (7)

So that:

$$(t+jI_1+eI_2)' = \lambda(t+jI_1+eI_2), (t+jI_1+eI_2)((h_n+g_nI_1+r_nI_2)_0) = t+jI_1+eI_2)_0 \quad (8)$$

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We get:

$$E\left(\overline{m+nI_1+lI_2}\right) = 1 + I + \overline{m+nI_1+lI_2} + \frac{1}{2!}\overline{(m+nI_1+lI_2)}^2 + \ldots\ldots\ldots + \frac{1}{(p+1)!}\overline{(m+nI_1+lI_2)}^{p+1} + O(\overline{(m+nI_1+lI_2)}^{p+1}) \qquad (9)$$

Where $\overline{m+nI_1+lI_2} = \lambda(\overline{m+nI_1+lI_2})$ are polynomials of degree R in $\overline{m+nI_1+lI_2}$

RK	r_2	stability period
7	$1 + \overline{m+nI_1+lI_2} + \frac{\overline{m+nI_1+lI_2}^2}{2!} + \frac{\overline{m+nI_1+lI_2}^3}{3!} + \frac{\overline{m+nI_1+lI_2}^4}{4!} + \frac{\overline{m+nI_1+lI_2}^5}{5!} + \frac{\overline{m+nI_1+lI_2}^6}{6!} + \frac{\overline{m+nI_1+lI_2}^7}{7!} + \frac{\overline{m+nI_1+lI_2}^8}{8!}$	(-4.112809+I_1,0.2213 I_1 + 0.55715I_2)

Example:

Solve the system of equations in the first example using the neutrosophic IRK method of the seventh rank.

We will take the step length value h=0.002+0.001I_1 + 0.001I_2.

Example:

Solve the system of equations in the second example using the neutrosophic IRK method of the seventh rank.

We will take the step length value h=0.001+ 0.001I_1 + 0.001I_2.

Table 3: results of neutrosophic IRK of seventh order for the first example

Refined method of Seventh rank error values		**Truth values**		**refined method-Kutta of the Seventh values**	
***First variable* error values**	**Second variable error values**	**First variable Values**	***First variable* error values**	**Second variable error values**	**First variable Values**
+0.00151I_1+ 0.001661I_2	+00151I_1+ 0.001661I_2	+I_1+I_2	-I_1+I_2	1+00152I_1+ 0.001663I_2	-1-00152I_1+ 0.001663I_2
3.108872e-15+I_1+I_2	3.108877e-15+I_1+I_2	0.90887+I_1+ I_2	-(0.90887+I_1+I_2)	0.90887+I_1+I_2	-(0.908870887+I_1+ I_2)
6.7233486e-15+I_1+I_2	6.7233482e-15+I_1+I_2	0.923345+I_1 +I_2	-(0.923345+I_1+ I_2)	0.923345+I_1+I_2	-(0.923345+I_1+I_2)
1.02334551e-15+I_1+I_2	1.02334531e-15+I_1+I_2	0.9348344+ I_1+I_2	-(0.923344+I_1+ I_2)	0.923344+I_1+I_2	-(0.923344+I_1+I_2)
1.32219e-14+I_1+I_2	1.32216e-14+I_1+I_2	0.93482+I_1+ I_2	-(0.93482+I_1+I_2	0.93482+I_1+I_2	-(0.93482+I_1+I_2)
1.649304e-14+I_1+I_2	1.649301e-14+I_1+I_2	0.991+I_1+I_2	-(0.991+I_1+I_2	0.991+I_1+I_2	-(0.991+I_1+I_2)
2.0808879e-14++I_1+I_2	2.0808875e-14+I_1+I_2	0.9808871+ I_1+I_2	-(0.9808871+I_1+ I_2)	0.9808871+I_1+I_2	-(0.9088731+I_1+I_2

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2.32334e-14+I	2.312334e-14+I_1+I_2	0.98571534 +I_1+I_2	-(0.982334+I_1+ I_2)	0.92334+I_1+I_2	-(0.92334+I_1+I_2)
2.6497e-14+I_1+I_2	2.6492e-14+I_1+I_2	0.9833+I_1+ I_2	-(0.93483+I_1+I_2)	0.93483+I_1+I_2	-(0.934833+I_1+I_2)
3. 0887558e-14+I_1+I_2	2. 0571572e-14+I_1+I_2	0. 088721+I_1+ I_2	-(0. 0834821+I_1+I_2)	0. 08348721+I_1+I_2	-(0. 0834821+I_1+I_2)
3.32185e-14+I_1+I_2	3.35715180e-14+I_1+I_2	0. 01611+I_1 +I_2	-(0. 01611+I_1+I_2)	0. 01621+I_1+I_2	-(0.100161+I_1+I_2)

Table 4: results of neutrosophic IRK of seventh order for the second example

Refined method of Seventh rank error values		**Truth values**		**refined method-Kutta of the Seventh values**	
***First variable* error values**	***Second variable* error values**	**First variable Values**	***First variable* error values**	**Second variable error values**	**First variable Values**
+0.00151I_1+ 0.001661I_2	+00151I_1+ 0.001661I_2	+I_1+I_2	-I_1+I_2	1+00152I_1+ 0.001663I_2	-1-00152I_1+ 0.001663I_2
2.2231e-07+I_1+I_2	8.95715476e-06+I_1+I_2	0.0562421+ I_1+I_2	-0.0797-I_1+I_2	0.09211I_1+I_2	-0.079715-I_1+I_2
9.15624634e-07+I_1+I_2	2.556245e-05+I	0.09864+I_1+ I_2	-0.06415-I_1+I_2	0.09851+I_1+I_2	-0.063142-I_1+I_2
1.3562465e-06+I_1+I_2	3.585624395e-05+I_1+I_2	0.0986675+ I_1+I_2	-0.049564-I_1+I_2	0.098101+I_1+I_2	-0.049602-I_1+I_2
1.773654e-06+I_1+I_2	4.95715e-05+I_1+I_2	0.092316+I_1 +I_2	-0.033798-I_1+I_2	0.09793+I_1+I_2	-0.03284-I_1+I_2
1.4895e-06+I_1+I_2	4.10496e-05+I_1+I_2	0.0989886+ I_1+I_2	-0.016901369- I_1+I_2	0.016901234	-0.029302-I_1+I_2
1.5856375e-07+I_1+I_2	368485e-06+I_1+I_2	0.09847364 +I_1+I_2	-0.0211988- I_1+I_2	0.091657150141 +I_1+I_2	-0.0221784- I_1+I_2
1.11690124e-07+I_1+I_2	3.156244e-06+I_1+I_2	0.09562411 +I_1+I_2	-0.017769-I_1+I_2	0.0169016+I_1+I_2	-0.01422313- I_1+I_2
1.01690185e-07+I_1+I_2	2.856246e-06+I_1+I_2	0.097332+I_1 +I_2	-0.010015-I_1+I_2	0.01690165+I_1+I_2	-0.01100173I_1+ I_2
1. 169013e-07+I_1+I_2	2.10038e-06+I_1+I_2	0.0969984+ I_1+I_2	-0.0056315I_1+I_2	0.0965543+I_1+I_2	-0.005567-I_1+I_2
1.0169010285e-07+I_1+I_2	1.93562494e-06+I_1+I_2	0.05715453 +I_1+I_2	-0.0057152486- I_1+I_2	0.057159014+I_1+ I_2	-0.005624101- I_1+I_2

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3. Conclusion

In This paper, we presented a numerical approach to the seventh rank refined neutrosophic Runge-Kutta numerical method, where we provided the theoretical basis of this formula to be applicable on refined neutrosophic differential equations. In addition, we provided numerical tables to compare the validity of this new method with other methods, as well as a clear computation of absolute errors in terms of refined neutrosophic numbers.

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