On Some Results about Complex Order Bessel's Equation

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Abstract

In this paper, we derive the Bessel's equation of complex order (n + i) from the classical well-known Bessel's equation. In addition, we generalize that recurrence relation from Bessel's equation of order (n) to Bessel equation of complex order (n + i). On the other hand, we present an algorithm to solve the novel complex order equation with many illustrated examples that explain the validity of our approach.

Keywords: Bessel's equation; complex order; differential equation; partial derivation

1. Introduction

The linear differential equation of the second order and the following variable coefficients:

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad , \quad n \geq 0} \quad \ldots(1.1)$$

(1.1) is known as the Bessel's equation of order n and this equation appears a lot in Applied Mathematics, Physics and engineering problems. Any solution that achieves this equation is called the Bessel functions of rank n. When solving this equation, if the value of n is an integer, the general solution is as follows:

$$y(x) = c_1 J_n(x) + c_2 y_n(x) \quad \ldots(1.2)$$

Since $y_n(x), J_n(x)$ are Bessel functions of the first and second types, respectively, and c_2, c_1 are optional constants, and:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} (\frac{x}{2})^{n+2r} \quad \ldots(1.3)$$

$$y_n(x) = \frac{1}{sin\ n\pi}[J_n(x)\ cos\ n\pi - J_{-n}(x)] \quad \ldots(1.4)$$

If n is not an integer, the general solution is as follows:

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x)$$

Since $J_{-n}(x), J_n(x)$ are Bessel functions of the first kind of rank *-n* , *n*, respectively, and c_2, c_1 are optional constants, and if:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} (\frac{x}{2})^{n+2r} \quad \ldots(1.5)$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(-n+r+1)} (\frac{x}{2})^{-n+2r} \quad \ldots(1.6)$$

Bernoulli was the first to give the concept of the Bessel function in 1732, using the zero-order function as a solution to the problem of oscillation of a one-sided suspended chain, and later the French scientist Willem Bessel (1784-1846) introduced a generalization of this equation used by other mathematicians such as Euler and Lord Rayleigh....... Etc.

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Bessel's equation was discussed in many cases, for example, of zero rank, $\frac{1}{2}$ rank , 1 rank, integral rank, and solutions were obtained for these cases ([3], [4], [7], [8], [9], [11])

Our study will be summarized in this research on the case where n is a non integer, as the study is carried out on the Bessel equation of nodal rank (n+ i) and the study of some of its properties.

Definition (1.1): Regular singular point [2].

If we have the following differential equation:

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$$

If both $p_1(x)$ and $p_2(x)$ are functions that are not analytic at point x_0, then this point is called the Singular point of the differential equation. If the functions $(x - x_0)p_1(x)$ and $(x - x_0)^2p_2(x)$ are analytic functions at point x_0, then this point is called a regular singular point.

For a differential equation, if one or both of them is not an analytical function at the point x_0, this point is called an irregular singular point.

Definition (1.2): Gamma Function [1]:

defined as:

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

Which approaches when $n > 0$. There is also another Formula:

$$\Gamma(n + 1) = n\Gamma(n)$$

It is also known as the factorial function, thus:

$$\Gamma(n + 1) = n!$$

Definition (1.3): Wronskian determinant. [8]

If $y_2(x), y_1(x)$ are real-valued or nodal functions, the wronskian determinant is equal to:

$$w(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = y_1y`_2 - y_2y`_1$$

If $w(y_1(x), y_2(x)) = 0$,then the two solutions are linearly dependent, and if $w(y_1(x), y_2(x)) \neq 0$, then the two solutions are linearly independent.

2. Solve the Bessel equation of nodal rank by the method of power sequences

Let be a linear differential equation of the second order and the following variable coefficients:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - (n + i)^2)y = 0 \quad , \quad n \geq 0 \qquad \ldots(2.1)$$

Bessel equation of rank (n +). Point x = 0 regular singular point we begin solving equation (2.1) by the Frobenus method by assuming that:

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots) \quad , \quad m > 0 \qquad \ldots(2.2)$$

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} \quad , \quad a_r \neq 0$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m + r)x^{m+r-1} \qquad \ldots(2.3)$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m + r)(m + r - 1)x^{m+r-2} \qquad \ldots(2.4)$$

We substitute equation (2.2) and equation (2.4) into equation (2.1) and get:

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$$x^2\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}+x\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}+(x^2-(n+i)^2)\sum_{r=0}^{\infty}a_rx^{m+r}=0$$

$$\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r}+\sum_{r=0}^{\infty}a_r(m+r)x^{m+r}+\sum_{r=0}^{\infty}a_rx^{m+r-2}-(n+i)^2\sum_{r=0}^{\infty}a_rx^{m+r}=0$$

$$\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r}+\sum_{r=0}^{\infty}a_r(m+r)x^{m+r}+\sum_{r=2}^{\infty}a_{r-2}x^{m+r}-(n+i)^2\sum_{r=0}^{\infty}a_rx^{m+r}=0$$

$$m(m-1)a_0x^m+m(m+1)a_1x^{m+1}+ma_0x^m+(m+1)a_1x^{m+1}-(n+i)^2a_0x^m-(n+i)^2a_1x^{m+1}$$
$$+\sum_{r=2}^{\infty}[[(m+r)(m+r-1)+(m+r)-(n+i)^2]a_r+a_{r-2}]x^{m+r}=0$$

Equating the coefficient x^m,we get:

$$[m(m-1)+m-(n+i)^2]a_0=0 \quad , \quad a_0\neq 0$$
$$m^2-(n+i)^2=0\Rightarrow m^2=(n+i)^2$$

Either $m=n+i$ or $m=-(n+i)$.

Equating the coefficient x^{m+1}, we get:

$$[m(m+1)+m+1-(n+i)^2]a_1=0$$

The first case: when $m=n+i$ we get:

$$[2m+1]a_1=0$$

Since $2m+1\neq 0$ because $m>0$.

So, $a_1=0$

Equating the coefficient x^{m+r}, we get:

$$[(m+r)(m+r-1)+(m+r)-(n+i)^2]a_r+a_{r-2}=0 \quad , \quad a_0\neq 0$$
$$[m^2+2mr+r^2-(n+i)^2]a_r=-a_{r-2}$$

Since $m=n+i$, we get:

$$[(n+i)^2+2(n+i)r+r^2-(n+i)^2]a_r=-a_{r-2}$$
$$r[2(n+i)+r]a_r=-a_{r-2}$$

$a_r=\frac{-a_{r-2}}{r[2(n+i)+r]} \quad , \quad r\geq 2 \quad \ldots(2.5)$

$$a_2=\frac{-a_{r-2}}{2[2(n+i)+2]}\Rightarrow a_2=\frac{-a_0}{2^2[(n+i)+1]}$$
$$a_3=\frac{-a_1}{3[2(n+i)+3]}\because a_1=0\Rightarrow a_3=0$$
$$a_4=\frac{-a_2}{4[2(n+i)+4]}\Rightarrow a_4=\frac{-1}{4[2(n+i)+4]}\cdot\frac{-a_0}{2^2[(n+i)+1]}$$
$$a_4=\frac{a_0}{4[2(n+i)+4]\,2^2[(n+i)+1]}$$
$$a_5=\frac{-a_3}{5[2(n+i)+5]}\because a_3=0\Rightarrow a_5=0$$

Then:

$$a_1=a_3=a_5=a_7=\ldots=0$$

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And therefore:

$$a_{2n-1} = 0 \quad , \quad n = 1,2,3,\ldots\ldots$$

This means that the coefficients of equation (2.1) are only even coefficients. So, we substitute r in the iterative formula (2.5) by 2k, we get:

$$a_{2k} = \frac{-a_{2k-2}}{2k[2(n+i)+2k]} \quad , \quad k = 1,2,3,4,\ldots$$

$$a_{2k} = \frac{-a_{2k-2}}{4k[(n+i)+k]}$$

$$k = 1 \Rightarrow a_2 = \frac{-a_0}{4[(n+i)+1]}$$

$$k = 2 \Rightarrow a_4 = \frac{-a_2}{4.2[(n+i)+2]} \Rightarrow a_4 = \frac{-1}{4.2[(n+i)+2]} \cdot \frac{-a_0}{4[(n+i)+1]}$$

$$\Rightarrow a_4 = \frac{a_0}{4^2.2.1[(n+i)+2][(n+i)+1]}$$

$$k = 3 \Rightarrow a_6 = \frac{-a_4}{4.3[(n+i)+3]}$$

$$\Rightarrow a_6 = \frac{-1}{4.3[(n+i)+3]} \cdot \frac{a_0}{4^2.2.1[(n+i)+2][(n+i)+1]}$$

$$\Rightarrow a_6 = \frac{-a_0}{4^2..3.2.1[(n+i)+3][(n+i)+2][(n+i)+1]}$$

Therefore:

$$a_{2k} = \frac{(-1)^k}{4^k k!\,[(n+i)+k]\ldots\ldots\ldots\ldots\ldots\ldots[(n+i)+1]}$$

We substitute the coefficients in equation (2.2) with *m*= *n*+ *i*, and get:

$$y = x^{n+i}(a_0 - \frac{a_0}{4[(n+i)+1]}x^2 + \frac{a_0}{4^2.2.1[(n+i)+2][(n+i)+1]}x^4 - \frac{-a_0}{4^3.3.2.1[(n+i)+3][(n+i)+2][(n+i)+1]}x^6 + \ldots)$$

$$y = a_0 x^{n+i}(1 - \frac{1}{4[(n+i)+1]}x^2 + \frac{1}{4^2.2.1[(n+i)+2][(n+i)+1]}x^4 - \frac{1}{4^3.3.2.1[(n+i)+3][(n+i)+2][(n+i)+1]}x^6 + \ldots)$$

….(2.6)

If we choose the value of the constant $\boldsymbol{a_0}$ in the form [3]:

$$a_0 = \frac{1}{2^{n+i}(n+i)!}$$

We substitute equation (2.7) into equation (2.6), and get:

$$y = \frac{1}{2^{n+i}(n+i)!}x^{n+i}(1 - \frac{1}{4[(n+i)+1]}x^2 + \frac{1}{4^2.2.1[(n+i)+2][(n+i)+1]}x^4 - \frac{1}{4^3.3.2.1[(n+i)+3][(n+i)+2][(n+i)+1]}x^6 + \ldots)$$

$$y = (\frac{x}{2})^{n+i}[\frac{1}{(n+i)!} - \frac{1}{4[(n+i)+1]}x^2 - \frac{1}{4^2.2.1[(n+i)+2]!}x^4 - \frac{1}{4^3.3.2.1[(n+i)+3]!}x^6 + \ldots]$$

…(2.8)

Since $\Gamma(n+i+1) = (n+i)!$ [12], thus:

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$$y = (\frac{x}{2})^{n+i}[\frac{1}{\lceil(n+i+1)} - \frac{1}{4\lceil(n+i+2)}x^2 + \frac{1}{4^2.2.1\lceil(n+i+3)}x^4 - \frac{1}{4^3.3.2.1\lceil n+i+4}x^6 + \ldots]$$

$$y = (\frac{x}{2})^{n+i}[\frac{1}{\lceil(n+i+1)} - \frac{1}{4\lceil(n+i+2)}(\frac{x}{2})^2 + \frac{1}{2.1\lceil(n+i+3)}(\frac{x}{2})^4 - \frac{1}{3.2.1\lceil n+i+4}(\frac{x}{2})^6 + \ldots]$$

$y_{n+i} = J_{n+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil(n+i+r+1)}(\frac{x}{2})^{n+i+2r}$ …(2.9)

$J_{n+i}(x)$ is called a Bessel function of the first type and rank $n + i$. to find the other solution of Bessel's equation we take $m = -(n + i)$ the iterative formula (2.5) becomes:

$$a_r = \frac{-a_{r-2}}{r[r-2(n+i)]} \quad , \quad r \geq 2 \quad , \quad r \neq 2(n+i)$$

Using the same steps that we performed to obtain $J_{n+i}(x)$, we get the other solution to Bessel's equation, which takes the following form:

$J_{-(n+i)}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil(-(n+i)+r+1)}(\frac{x}{2})^{-(n+i)+2r}$ …(2.10)

This function is called a Bessel function of the first type and rank $-(n + i)$ and therefore the general solution of the Bessel equation of nodal rank $n + i$ has the following form:

$$y(x) = c_1 J_{n+i}(x) + c_2 J_{-(n+i)}(x)$$

Example (1): Find the general solution of the following Bessel equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - (3+i)^2)y = 0$$

Solution:

Bessel equation of Rank 3 + i and the general solution of this equation is:

$$y(x) = c_1 J_{3+i}(x) + c_2 J_{-3-i}(x)$$

Since c_1 and c_2 are optional constants, thus:

$$J_{3+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil(3+i+r+1)}(\frac{x}{2})^{3+i+2r}$$

$$J_{3+i}(x) = (\frac{x}{2})^{3+i}[\frac{1}{0!\lceil(3+i+1)} - \frac{1}{1!\lceil(3+i+2)}(\frac{x}{2})^2 + \frac{1}{2!\lceil(3+i+3)}(\frac{x}{2})^4 - \frac{1}{3!\lceil(3+i+4)}(\frac{x}{2})^6 + \ldots..]$$

$$J_{3+i}(x) = (\frac{x}{2})^{3+i}[\frac{1}{\lceil(4+i)} - \frac{1}{4\lceil(5+i)}x^2 + \frac{1}{4^2.2.1\lceil(6+i)}x^4 - \frac{1}{4^3.3.2.1\lceil(7+i)}x^6 + \ldots..]$$

$$J_{3+i}(x) = (\frac{x}{2})^{3+i}[\frac{1}{(3+i)!} - \frac{1}{4(4+i)!}x^2 + \frac{1}{4^2.2.1(5+i)!}x^4 - \frac{1}{4^3.3.2.1(6+i)!}x^6 + \ldots..]$$

$$J_{3+i}(x) = (\frac{x}{2})^{3+i}[\frac{1}{1.55+4.98!} - \frac{1}{4(1.22+21.47i)}x^2 + \frac{1}{32(-15.37+108.57i)}x^4 - \frac{1}{384(-20079+636.05i)}x^6 + \ldots..]$$

$$J_{3+i}(x) = (\frac{x}{2})^{3+i}[0.0569 - 0.18306i - (0.00065 - 0.0116i)x^2 - (0.00003 + 0.0002i)x^4 + (0.000001 + 0.000003i)x^6 + \ldots]$$

$$J_{-3-i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\lceil(-3-i+r+1)}(\frac{x}{2})^{-3-i+2r}$$

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$$J_{-3-i}(x) = (\frac{x}{2})^{-3-i}[\frac{1}{0!\,\Gamma(-3-i+1)} - \frac{1}{1!\,\Gamma(-3-i+2)}(\frac{x}{2})^2 + \frac{1}{2!\,\Gamma(-3-i+3)}(\frac{x}{2})^4 - \frac{1}{3!\,\Gamma(-3-i+4)}(\frac{x}{2})^6 + \ldots..]$$

$$J_{-3-i}(x) = (\frac{x}{2})^{-3-i}[\frac{1}{\Gamma(-2-1)} - \frac{1}{4\Gamma(-1-i)}x^2 + \frac{1}{4^2.2.1\Gamma(-i)}x^4 - \frac{1}{4^3.3.2.1\Gamma(1-i)}x^6 + \ldots..]$$

$$J_{-3-i}(x) = (\frac{x}{2})^{-3-i}[\frac{1}{(-3-i)!} - \frac{1}{4(-2-i)!}x^2 + \frac{1}{4^2.2.1(-1-i)!}x^4 - \frac{1}{4^3.3.2.1(-i)!}x^6 + \ldots..]$$

$$J_{-3-i}(x) = (\frac{x}{2})^{-3-i}[\frac{10}{1.339+0.963i} - \frac{1}{2(-0.343-0.653i)}x^2 + \frac{1}{32(0.155+0.498i)}x^4 - \frac{1}{384(0.498+0.155i)}x^6 + \ldots..]$$

$$J_{-3-i}(x) = (\frac{x}{2})^{-3-i}[4.9224 - 3.5401i + (0.3152 - 0.6001i)x^2 - (0.0178 + 0.0572i)x^4 - (0.0047 - 0.0014i)x^6 + \ldots..]$$

Case (2.1):

If $J_{-(n+i)}(x), J_{n+i}(x)$are Bessel functions of the first and second types, respectively, then these functions are linearly independent of equation (2.1) when $x \neq 0$. And that the general solution of equation (2.1) has the form:

$$y(x) = c_1 J_{n+i}(x) + \boldsymbol{c_2} J_{-(n+i)}(x)$$

Since c_1 and c_2 are optional constants.

Proof:

To prove that Bessel functions $J_{-(n+i)}(x), J_{n+i}(x)$ are linearly independent functions it suffices to show that the Wronskian is not equal to zero.

$$\mathrm{W}(J_{n+i}(x), J_{-(n+i)}(x)) = \begin{vmatrix} J_{n+i}(x) & J_{-(n+i)}(x) \\ J`_{n+i}(x) & J`_{-(n+i)}(x) \end{vmatrix}$$

Since $J_{-(n+i)}(x), J_{n+i}(x)$ satisfies equation (2.1)

$$x^2 J``_{n+i}(x) + \boldsymbol{x} J`_{n+i}(x) + (x^2 - (n+i)^2) J_{n+i}(x) = 0$$

$$x^2 J``_{-(n+i)}(x) + \boldsymbol{x} J`_{-(n+i)}(x) + (x^2 - (n+i)^2) J_{-(n+i)}(x) = 0$$

We multiply these two equations by $J_{n+i}(x), J_{-(n+i)}(x)$ respectively and subtract them from each other and divide the result by x, we get:

$$x(J_{n+i}(x) J``_{-(n+i)}(x) - J_{-(n+i)}(x) J``_{n+i}(x) + J_{n+i}(x) J`_{-(n+i)}(x) - J_{-(n+i)}(x) J`_{n+i}(x) = 0$$

This is equivalent to:

$$\frac{d}{dx}(x(J_{n+i}(x) J`_{-(n+i)}(x) - J_{-(n+i)}(x) J`_{n+i}(x) = \frac{d}{dx}(x\, W(J_{n+i}(x), J_{-(n+i)}(x)) = 0$$

$$W(J_{n+i}(x), J_{-(n+i)}(x)) = \frac{c}{x} \quad , \quad x \neq 0$$

$$W(J_{n+i}(x), J_{-(n+i)}(x)) \neq 0$$

So, the Bessel functions $J_{-(n+i)}(x), J_{n+i}(x)$ are linearly independent.

Theorem (2.2):

If

$$J_{n+i}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+i+r+1)} (\frac{x}{2})^{n+i+2r} \quad , \quad n > -1$$

Then:

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$$\lim_{x\to 0}\frac{J_{n+i}(x)}{x^{n+i}}=\frac{1}{2^{n+i}\lceil(n+i+1)}$$

Proof: from the equation (2.8):

$$J_{n+i}(x)=(\frac{x}{2})^{n+i}[\frac{1}{\lceil(n+i+1)}-\frac{1}{4\lceil(n+i+2)}x^2+\frac{1}{4^2.2.1\lceil(n+i+3)}x^4-\frac{1}{4^3.3.2.1\lceil(n+i+4)}x^6+\ldots]$$

$$\frac{J_{n+i}(x)}{x^{n+i}}=\frac{1}{2^{n+i}\lceil(n+i+1)}[1-\frac{1}{4(n+i+1)}x^2+\frac{1}{4^2.2.1(n+i+2)(n+i+1)}x^4-\frac{1}{4^3.3.2.1(n+i+3)(n+i+2)(n+i+1)}x^6+\ldots]$$

$$\lim_{x\to 0}\frac{J_{n+i}(x)}{x^{n+i}}=\lim_{x\to 0}\frac{1}{x^{n+i}\lceil(n+i+1)}[1-\frac{1}{4(n+i+1)}x^2+\frac{1}{4^2.2.1(n+i+2)(n+i+1)}x^4-\frac{1}{4^3.3.2.1(n+i+3)(n+i+2)(n+i+1)}x^6+\ldots]$$

$$\lim_{x\to 0}\frac{J_{n+i}(x)}{x^{n+i}}=\frac{1}{2^{n+i}\lceil(n+i+1)}$$

case (2.2): the following relationship connects Bessel functions of the first type.

$$J_{-(n+i)}(x)=(-1)^{n+i}J_{n+i}(x)$$

Proof:

$$J_{-(n+i)}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\lceil-(n+i)+r+1)}(\frac{x}{2})^{-(n+i)+2r}=\sum_{r=0}^{(n+i)-1}\frac{(-1)^r(\frac{x}{2})^{-(n+i)+2r}}{r!\lceil(-(n+i)+r+1)}+\sum_{r=n+i}^{\infty}\frac{(-1)^r(\frac{x}{2})^{-(n+i)+2r}}{r!\lceil(-(n+i)+r+1)}=0+\sum_{r=n+i}^{\infty}\frac{(-1)^r(\frac{x}{2})^{-(n+i)+2r}}{r!\lceil(-(n+i)+r+1)}$$

Now, we put instead of $r=(n+i)+k$, we get:

$$J_{-(n+i)}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{(n+i)+k}(\frac{x}{2})^{(n+i)+2k}}{((n+i)+k)!\lceil(k+1)}=(-1)^{n+i}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^{n+i+2k}}{k!\lceil(n+i+k+1)}$$

$$J_{-(n+i)}(x)=(-1)^{n+i}J_{n+i}(x)$$

3. Resatioler neceerrcu

Resatioler neceerrcu are mentioned in [3]. In the case of n is an integer. We will consider these relations in the nodal case (n+i)

Case (3.1):

If:

$$J_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\lceil(n+i+r+1)}(\frac{x}{2})^{n+i+2r}$$

Then:

1. $xJ`_{n+i}=(n+i)J_{n+i}-xJ_{n+1+i}$
2. $xJ`_{n+i}=-(n+i)J_{n+i}+xJ_{n-1+i}$

Proof of relation 1:

$$xJ`_{n+i}=(n+i)J_{n+i}-xJ_{n+1+i}$$

It is known that:

$$J_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\lceil(n+i+r+1)}(\frac{x}{2})^{n+i+2r}$$

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$$J`_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+i+2r)}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$= (n+i) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r 2r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$xJ`_{n+i} = x(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r} \frac{1}{2}$$

$$+x \sum_{r=1}^{\infty} \frac{(-1)^r\, 2}{(r-1)!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$= x(n+i) \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$+x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1}$$

$$= (n+i)J_{n+i} + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1}$$

We assume that $s = r - 1$.

$$xJ`_{n+i} = (n+i)J_{n+i} + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\lceil(n+i+s+2)} (\frac{x}{2})^{n+i+2s+1}$$

$$xJ`_{n+i} = (n+i)J_{n+i} - x \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\lceil(n+i+s+2)} (\frac{x}{2})^{n+i+2s+1}$$

$$xJ`_{n+i} = (n+i)J_{n+i} - xJ_{n+1+i}$$

Proof of relation 2:

$$xJ`_{n+i} = (n+i)J_{n+i} + xJ_{n-1+i}$$

It is known that:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

$$J`_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+i+2r)}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [2(n+i)+2r-(n+i)]}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r-1} \frac{1}{2}$$

$$xJ`_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r [2(n+i)+2r-(n+i)]}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r [2(n+i)+2r]}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

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$$-(n+i)\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+i+r+1)}(\frac{x}{2})^{n+i+2r}$$

$$xJ`_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r 2[(n+i)+r]}{r!\,\Gamma(n+i+r+1)}(\frac{x}{2})^{n+i+2r}-(n+i)J_{n+i}$$

$$xJ`_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r 2[(n+i)+r]}{r!\,(n+i+r)\Gamma(n+i+r)}(\frac{x}{2})^{n+i+2r}-(n+i)J_{n+i}$$

$$xJ`_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r 2}{r!\,\Gamma(n+i+r)}(\frac{x}{2})^{n+i+2r}-(n+i)J_{n+i}$$

$$xJ`_{n+i}=x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+i+r)}(\frac{x}{2})^{n+i+2r-1}-(n+i)J_{n+i}$$

$$xJ`_{n+i}=x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+i+r)}(\frac{x}{2})^{n-1+i+2r}-(n+i)J_{n+i}$$

$$xJ`_{n+i}=xJ_{n-1+i}-(n+i)J_{n+i}$$

Result (3.2):

If

$$J_{n+i}=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+i+r+1)}(\frac{x}{2})^{n+i+2r}$$

Then:

3. $2\,J`_{n+i}=J_{n-1+i}-J_{n+1+i}$
4. $2(n+i)\,J_{n+i}=x(J_{n-1+i}+J_{n+1+i})$

Proof of relation 3:

$$2\,J`_{n+i}=J_{n-1+i}-J_{n+1+i}$$

From case (3.1), we get:

$x\,J`_{n+i}=(n+i)\,J_{n+i}-xJ_{n+1+i}$ ..(3.1)

$x\,J`_{n+i}=-(n+i)\,J_{n+i}+xJ_{n-1+i}$ ..(3.2)

Adding (3.1) and (3.2), we get:

$$2x\,J`_{n+i}=-x\,J_{n+1+i}+xJ_{n-1+i}$$

By dividing on x, $x\neq 0$

$$2J`_{n+i}=-J_{n+1+i}+J_{n-1+i}$$

$$2J`_{n+i}=J_{n+1+i}-J_{n+1+i}$$

Proof of relation 4:

$$2(n+i)\,J_{n+i}=x(J_{n-1+i}+J_{n+1+i})$$

From case (3.1), we get:

$x\,J`_{n+i}=(n+i)\,J_{n+i}-xJ_{n+1+i}$ ..(3.3)

$x\,J`_{n+i}=-(n+i)\,J_{n+i}+xJ_{n-1+i}$ ..(3.4)

Substituting equation (3.4) into equation (3.3), we obtain:

$$-(n+i)\,J_{n+i}+xJ_{n-1+i}=(n+i)J_{n+i}-xJ_{n+1+i}$$

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$$2(n+i)\,J_{n+i} - xJ_{n+1+i} - xJ_{n-1+i} = 0$$
$$2(n+i)\,J_{n+i} = xJ_{n+1+i} + xJ_{n-1+i}$$
$$2(n+i)\,J_{n+i} = x(J_{n+1+i} + xJ_{n-1+i})$$

Case (3.3):

If:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

Then:

5. $\frac{d}{dx}(x^{-(n+i)}J_{n+i}) = -x^{-(n+i)}J_{n+1+i}$
6. $\frac{d}{dx}(x^{(n+i)}J_{n+i}) = x^{(n+i)}J_{n-1+i}$

Proof of relation 5:

$$\frac{d}{dx}(x^{-(n+i)}J_{n+i}) = -x^{-(n+i)}J_{n+1+i}$$

It is known that:

$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

$$x^{-(n+i)}J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!\,2^{n+i+2r}\lceil(n+i+r+1)}$$

$$\frac{d}{dx}(x^{-(n+i)}J_{n+i}) = \sum_{r=0}^{\infty} \frac{(-1)^r\ 2r\ x^{2r-1}}{r!\,2^{n+i+2r}\lceil(n+i+r+1)}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r\ \ 2\ \ x^{2r-1}}{(r-1)!\,2^{n+i+2r}\lceil(n+i+r+1)}$$

We assume that $r = k + 1$.

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ \ 2\ \ x^{2k+1}}{k!\,2^{n+i+2k+2}\lceil(n+i+k+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ \ 2\ \ x^{n+i+2k+1}\ \ x^{-(n+i)}}{k!\,2^{n+i+2k+1}\ 2\lceil(n+i+k+2)}$$

$$= x^{-(n+i)} = \sum_{k=0}^{\infty} \frac{(-1)^k\ \ (-1)\ \ \ x^{n+i+2k+1}}{k!\,2^{n+i+2k+1}\ \lceil(n+i+k+2)}$$

$$= -x^{-(n+i)} = \sum_{k=0}^{\infty} \frac{(-1)^k\ \ \ x^{n+i+2k+1}}{k!\,2^{n+i+2k+1}\ \lceil(n+i+k+2)}$$

$$= -x^{-(n+i)}J_{n+1+i}$$

Proof of relation 6:

$$\frac{d}{dx}(x^{(n+i)}J_{n+i}) = x^{(n+i)}J_{n-1+i}$$

It is known that:

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$$J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\lceil(n+i+r+1)} (\frac{x}{2})^{n+i+2r}$$

$$x^{(n+i)} J_{n+i} = \sum_{r=0}^{\infty} \frac{(-1)^r \; x^{2(n+i)+2r}}{r!\, 2^{n+i+2r} \lceil(n+i+r+1)}$$

$$\frac{d}{dx}[x^{n+i} \; J_{n+i}] = \sum_{r=0}^{\infty} \frac{(-1)^r \;\; (2(n+i)+2r) \;\; x^{2(n+i)+2r-1}}{r! \;\; 2^{n+i+2r} \;\; \lceil(n+i+r+1)}$$

$$\frac{d}{dx}[x^{n+i} \; J_{n+i}] = x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; 2(n+i) \;\; x^{n+i+2r-1}}{r! \;\; 2^{n+i+2r} \;\; \lceil(n+i+r+1)}$$

$$+x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; 2r \;\; x^{n+i+2r-1}}{r! \;\; 2^{n+i+2r} \;\; \lceil(n+i+r+1)}$$

$$\frac{d}{dx}[x^{n+i} \; J_{n+i}] = x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; (n+i) \;\; x^{n+i+2r-1}}{r! \;\; 2^{n+i+2r-1} \;\; \lceil(n+i+r+1)}$$

$$+x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; r \;\; x^{n+i+2r-1}}{r! \;\; 2^{n+i+2r-1} \;\; \lceil(n+i+r+1)}$$

$$= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; (\frac{x}{2})^{n+i+2r-1} \;\; (n+i+r)}{r! \;\; \lceil(n+i+r+1)}$$

$$= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; (\frac{x}{2})^{n+i+2r-1} \;\; (n+i+r)}{r! \;\; (n+i+r) \;\; \lceil(n+i+r+1)}$$

$$= x^{n+i} \sum_{r=0}^{\infty} \frac{(-1)^r \;\; (\frac{x}{2})^{n+i+2r-1}}{r! \;\; \lceil(n+i+r)}$$

$$\frac{d}{dx}(x^{(n+i)} J_{n+i}) = x^{(n+i)} J_{n-1+i}$$

4. Illustrative examples as an application of previous relations

Example (1): Write the Bessel function J_{5+i} in the function of both Bessel functions J_{1+i}, J_{2+i}

Solution: the fourth relation of the result (3.2)

$2(n+i)J_{n+i} = x(J_{n-1+i} + J_{n+1+i})$ ..(3.5)

When n = 4

$$2(4+i)J_{4+i} = x(J_{3+i} + J_{5+i})$$

$$(\frac{8+2i}{x})J_{4+i} = (J_{3+i} + J_{5+i})$$

$J_{5+i} = (\frac{8+2i}{x})J_{4+i} - J_{3+i}$ ….(3.6)

When n = 3 in equation (3.5), we get:

$$2(3+i)J_{3+i} = x(J_{2+i} + J_{4+i})$$

$$(\frac{6+2i}{x})J_{3+i} = (J_{2+i} + J_{4+i})$$

$J_{4+i} = (\frac{6+2i}{x})J_{3+i} - J_{2+i}$ ….(3.7)

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When n = 2 in equation (3.5), we get:

$$2(2+i)J_{2+i} = x(J_{1+i} + J_{3+i})$$

$$(\frac{4+2i}{x})J_{2+i} = (J_{1+i} + J_{3+i})$$

$$J_{3+i} = (\frac{4+2i}{x})J_{2+i} - J_{1+i} \quad \text{....(3.8)}$$

We substitute equation (3.7) and equation (3.8) into equation (3.6) and get:

$$J_{5+i} = (\frac{8+2i}{x})((\frac{6+2i}{x})[(\frac{4+2i}{x})J_{2+i} - J_{1+i}] - J_{2+i}) - [(\frac{4+2i}{x})J_{2+i} - J_{1+i}]$$

$$J_{5+i} = \frac{44+28i}{x}(\frac{4+2i}{x}J_{2+i} - J_{1+i}) - \frac{8+2i}{x}J_{2+i} - \frac{4+2i}{x}J_{2+i} - J_{1+i}$$

$$J_{5+i} = \frac{120+200i}{x^3}J_{2+i} - \frac{44+28i}{x^2}J_{1+i} - \frac{8+2i}{x}J_{2+i} - \frac{4+2i}{x}J_{2+i} + J_{1+i}$$

$$J_{5+i} = \frac{120+200i}{x^3}J_{2+i} - \frac{44+28i}{x^2}J_{1+i} - \frac{12+4i}{x}J_{2+i} + J_{1+i}$$

$$J_{5+i} = [\mathbf{1} - \frac{44+28i}{x^2}]J_{1+i} + [\frac{120+200i}{x^3} - \frac{12+4i}{x}]J_{2+i}$$

Example (2): prove that:

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = 2[\frac{n+i}{x}J_{n+i}^2(x) - \frac{(n+1+i)}{x}J_{n+1+i}^2(x)]$$

Proof: from case (3.1), we get:

$$xJ`_{n+i}(x) = (n+i)J_{n+i}(x) - xJ_{n+1+i}(x) \quad \text{...(3.9)}$$

$$xJ`_{n+i}(x) = -(n+i)J_{n+i}(x) + xJ_{n-1+i}(x) \quad \text{...(3.10)}$$

We put the formula (3.10) instead of every n a (n+1), we get:

$$xJ`_{n+1+i}(x) = -(n+1+i)J_{n+1+i}(x) + xJ_{n+i}(x) \quad \text{...(3.11)}$$

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = 2J_{n+i}(x)\,J`_{n+i}(x) + 2J_{n+1+i}(x)\,J`_{n+1+i}(x) \quad \text{...(3.12)}$$

The values of $J`_{n+i}(x)$ and $J`_{n+1+i}(x)$ in the formula (3.11), (3.9), respectively, we substitute them in (3.12) and we get:

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = 2J_{n+i}(x)\left[\frac{n+i}{x}J_{n+i}(x) - J_{n+1+i}(x)\right]$$

$$+2J_{n+1+i}(x)\left[\frac{-(n+1+i)}{x}J_{n+1+i}(x) + J_{n+i}(x)\right]$$

$$\frac{d}{dx}[J_{n+i}^2(x) + J_{n+1+i}^2(x)] = \frac{2}{x}J_{n+i}(x)[(n+i)J_{n+i}(x) - xJ_{n+1+i}(x)]$$

$$+\frac{2}{x}J_{n+1+i}(x)[-(n+1+i)J_{n+1+i}(x) + xJ_{n+i}(x)]$$

$$= 2\frac{n+i}{x}J_{n+i}^2(x) - \mathbf{2}J_{n+i}(x)\,J_{n+1+i}(x) - 2\frac{(n+1+i)}{x}J_{n+1+i}^2(x) + 2J_{n+1+i}(x)J_{n+i}(x)$$

$$2\left[\frac{n+i}{x}J_{n+i}^2(x) - \frac{(n+1+i)}{x}J_{n+1+i}^2(x)\right]$$

Example (3): if *n* > -1, prove that:

$$\int_0^x x^{-(n+i)}J_{n+1+i}dx = \frac{1}{2^{n+i}\Gamma(n+i+1)} - x^{-(n+i)}J_{n+i}(x)$$

Proof: the fifth relation of case (3.3), we get:

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$$\frac{d}{dx}\left(x^{-(n+i)}J_{n+i}\right) = -x^{-(n+i)}J_{n+1+i}$$

$$\int_0^x x^{-(n+i)}J_{n+1+i}(x)\,dx = -\left[x^{-(n+i)}J_{n+i}(x)\right]_0^x$$

$$= -x^{-(n+i)}J_{n+i} + \lim_{x\to 0}\frac{J_{n+i}(x)}{x^{n+i}}$$

By theorem (2.2) we obtain:

$$= -x^{-(n+i)}J_{n+i} + \frac{1}{2^{n+i}\Gamma(n+i+1)}$$

$$= \frac{1}{2^{n+i}\Gamma(n+i+1)} - x^{-(n+i)}J_{n+i}(x)$$

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