Application of Integral Operator Generated by Touchard Polynomials to Certain Subclasses of Harmonic Functions

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Abstract

Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the unite disk $\mathbb{U} = \{z : |z| < 1\}$ where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = \sum\limits_{k=1}^{\infty} b_k z^k$ $(|b_1| < 1)$. In this paper we establish connections between various subclasses of harmonic univalent functions by applying certain integral operator involving the Touchard Polynomials.

Keywords: Harmonic univalent; Touchard Polynomials; Integral Operator

1 Introduction and preliminary results

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see[6]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \qquad |b_1| < 1. \tag{1}$$

Note that $\mathcal{S}_{\mathcal{H}}$ reduce to class of $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of its member is zero. Let the subclass $\mathcal{S}^0_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$ defined by

$$\mathcal{S}^0_{\mathcal{H}} = \{f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}} \,:\, g'(0) = b_1 = 0\}.$$

Analogous to well-known subclasses of the family $\mathcal{S}$, one can define various subclasses of the family $\mathcal{S}_{\mathcal{H}}$. A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ is in the class $\mathcal{S}^*_{\mathcal{H}}$ if the range $f(\mathbb{U})$ is starlike with respect to the

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origin. A function $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ is called a harmonic starlike of order α, $(0 \leq \alpha < 1)$ mapping in $\mathbb{U}$. Likewise a function f defined in $\mathbb{U}$ belongs to the class $\mathcal{C}_{\mathcal{H}}$ if $f \in \mathcal{S}_{\mathcal{H}}$ and if $f(\mathbb{U})$ is a convex domain. A function $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$ is called harmonic convex of order α, $(0 \leq \alpha < 1)\mathbb{U}$. Analytically, we have

$$f \in \mathcal{S}^*_{\mathcal{H}}(\alpha) \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

$$f \in \mathcal{C}_{\mathcal{H}}(\alpha) \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{zh''(z) + h'(z) - \overline{zg''(z) + zg'(z)}}{h'(z) - \overline{g'(z)}}\right\} > \alpha, \qquad z \in \mathbb{U}.$$

These classes have been extensively studied by Jahangiri[7] and.[2]
For $\alpha = 0$, these classes $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ and $\mathcal{C}_{\mathcal{H}}(\alpha)$ were denoted by $\mathcal{S}^*_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}$ respectively were studied by Avci and Zlotkiewicz,[8] Silverman[4] and Silvia.[5] Further, we let $\mathcal{S}^{*,0}_{\mathcal{H}}$, $\mathcal{C}^0_{\mathcal{H}}$ and $\mathcal{K}^0_{\mathcal{H}}$ denote the subclasses of $\mathcal{S}^0_{\mathcal{H}}$ of harmonic function which are, respectively, starlike, convex and close-to-convex.

Recently, the author[3] introduce a series with Touchard polynomials coefficients after the second force as following:

$$F_n(z,m) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n}{(k-1)!} e^{-m} z^k. \tag{2}$$

It can be easily by ratio test showed that the above series is convergent ant the radius of convergence is infinity.

For harmonic functions $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k + \overline{\sum\limits_{k=1}^{\infty} b_k z^k}$and $\Omega(z) = z + \sum\limits_{k=2}^{\infty} \psi_k z^k + \overline{\sum\limits_{k=1}^{\infty} \varphi_k z^k}$ the convolution of f and Ω is given by

$$(f * \Omega)(z) = f(z) * \Omega(z) = z + \sum_{k=2}^{\infty} a_k \psi_k z^k + \overline{\sum_{k=1}^{\infty} b_k \varphi_k z^k}. \tag{3}$$

Now, we introduce the integral operator $I : \mathcal{S}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$ as following:

$$I_n(f) \equiv I_n(m_1, m_2)f(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = h(z) * \int_0^z \frac{F_n(t, m_1)}{t} dt, \;\; G(z) = g(z) * \int_0^z \frac{F_n(t, m_2)}{t} dt,$$

or equivalently

$$H(z) = z + \sum_{k=2}^{\infty} \frac{m_1^{k-1}(k-1)^n}{k!} e^{-m_1} a_k z^k, \;\; G(z) = b_1 z + \sum_{k=2}^{\infty} \frac{m_1^{k-1}(k-1)^n}{k!} e^{-m_1} b_k z^k. \tag{4}$$

In this paper we will apply the integral operator I for the various subclasses of harmonic univalent function.

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2 Preliminary Lemmas

To prove our results, we need the following Lemmas:

Lemma 2.1. *([1]) If $f = h + \overline{g} \in \mathcal{C}^0_{\mathcal{H}}$ where h and g are given by (1.1) with $b_1 = 0$, then*

$$|a_k| \le \frac{k+1}{2}, \ \ |b_k| \le \frac{k-1}{2},$$

Lemma 2.2. *([2]) If $f = h + \overline{g}$ where h and g are given by (1.1). If for some $\alpha(0 \le \alpha < 1)$ and the inequality*

$$\sum_{k=2}^{\infty} (k-\alpha)a_k + \sum_{k=1}^{\infty} (k+\alpha)b_k \le 1-\alpha, \tag{5}$$

is satisfied, then f is harmonic, sense-preserving, univalent function in $\mathbb{U}$ and $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Define $\mathcal{TS}^*_{\mathcal{H}}(\alpha) = \mathcal{S}^*_{\mathcal{H}}(\alpha) \cap \mathcal{T}$, where $\mathcal{T}$ consists of the function $f = h + \overline{g}$ in $\mathcal{S}_{\mathcal{H}}$ so that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k| z^k \qquad |b_1| < 1. \tag{6}$$

Lemma 2.3. *In ([2]), it is also show that $f = h + \overline{g}$ where h and g are given by (2.2) is in the class $\mathcal{TS}_{\mathcal{H}}(\alpha)$, if and only if the coefficient conation (2.3) holds. Moreover, if $f \in \mathcal{TS}^*_{\mathcal{H}}(\alpha)$, then*

$$|a_n| \le \frac{1-\alpha}{k-\alpha}, \ k \ge 2 \ \ |b_k| \le \frac{1-\alpha}{k+\alpha}, \ k \ge 1.$$

Lemma 2.4. *([2]) If $f = h + \overline{g}$ where h and g are given by (1.1). If for some $\alpha(0 \le \alpha < 1)$ and the inequality*

$$\sum_{k=2}^{\infty} k(k-\alpha)a_k + \sum_{k=1}^{\infty} k(k+\alpha)b_k \le 1-\alpha, \tag{7}$$

is satisfied, then f is harmonic, sense-preserving, univalent function in $\mathbb{U}$ and $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$.

Lemma 2.5. *In ([2]), it is also show that $f = h + \overline{g}$ where h and g are given by (2.2) is in the class $\mathcal{TC}_{\mathcal{H}}(\alpha)$, if and only if the coefficient conation (2.3) holds. Moreover, if $f \in \mathcal{TC}_{\mathcal{H}}(\alpha)$, then*

$$|a_k| \le \frac{1-\alpha}{k(k-\alpha)}, \ k \ge 2 \ \ |b_k| \le \frac{1-\alpha}{k(k+\alpha)}, \ k \ge 1.$$

Lemma 2.6. *([1]) Let $f = h + \overline{g} \in \mathcal{S}^{*,0}_{\mathcal{H}}$ or $\mathcal{C}^0_{\mathcal{H}}$, where h and g are given by (1.1), then*

$$|a_k| \le \frac{(2k+1)(k+1)}{6}, \ \ |b_k| \le \frac{(2k-1)(k-1)}{6}, \ k \ge 2.$$

3 Main Results

In our first result, we determine conditions which guarantee that the integral operator I is a harmonic starlike in $\mathbb{U}$.

Theorem 3.1. *If $0 \le \alpha < 1$, $m_1, m_2 > 0$. Also, suppose $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ is given by (1.1). If the inequalities*

$$(i) \sum_{k=2}^{\infty} |a_k| + \sum_{k=1}^{\infty} |a_k| \le 1, \ \ |b_1| < 1,$$

$$(ii) \sum_{i=0}^{n} \binom{n}{k} k^{n-i} (2 - e^{-m_1} - e^{-m_2}) \le 1 - |b_1|$$

are satisfied, then the integral operator I is sense-preserving, harmonic univalent function and maps $\mathcal{S}_{\mathcal{H}}$ in to $\mathcal{S}^*_{\mathcal{H}}$.

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Proof. Note that

$$I(m_1, m_2)f(z) = H(z) + \overline{G(z)},$$

where $H(z)$ and $G(z)$ are given by (1.4). In order to show that $I(f)$ is locally univalent and sense-preserving it suffices to show that $|H'(z)| - |G'(z)| > 0$ in $\mathbb{U}$. Using the condition (i), we have

$$\begin{aligned}
& |H'(z)| - |G'(z)| \\
> \ & 1 - \sum_{k=2}^{\infty} k\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!} - \sum_{k=2}^{\infty} k\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!} - |b_1| \\
= \ & 1 - \sum_{k=0}^{\infty} \frac{e^{-m_1}m_1^{k+1}(k+1)^n}{(k+1)!} - \sum_{k=2}^{\infty} \frac{e^{-m_2}m_2^{k+1}(k+1)^n}{(k+1)!} - |b_1| \\
= \ & 1 - \sum_{i=0}^{n} \binom{n}{k} k^{n-i} \sum_{k=0}^{\infty} e^{-m_1}\frac{m_1^{k+1}}{(k+1)!} - |b_1| - \sum_{i=0}^{n} \binom{n}{k} k^{n-i} \sum_{k=0}^{\infty} e^{-m_2}\frac{m_2^{k+1}}{(k+1)!} \\
= \ & 1 - \sum_{i=0}^{n} \binom{n}{k} k^{n-i}(1-e^{-m_1}) - |b_1| - \sum_{i=0}^{n} \binom{n}{k} k^{n-i}(1-e^{-m_2}) \\
= \ & 1 - |b_1| - \sum_{i=0}^{n} \binom{n}{k} k^{n-i}(2 - e^{-m_1} - e^{-m_2}) \\
\geq \ & 0, from(ii).
\end{aligned}$$

To show that $I(f)$ is univalent in $\mathbb{U}$, we follow the method of Theorem 1 in.[2] That is, for $z_1 \neq z_2$ in $\mathbb{U}$, it suffices to prove that

$$\Re\frac{f(z_2) - f(z_1)}{z_2 - z_1} > \int_0^1 (\Re H'(z(t)) - |G'(z(t))|)dt. \tag{8}$$

Since from the given condition (i), we have

$$\Re H'(z(t)) - |G'(z(t))|) > 1 - \sum_{k=2}^{\infty} k\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!} - \sum_{k=2}^{\infty} k\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!} - |b_1|,$$

it follows from the given hypothesis that the last inequality is positive.

Therefore, from the inequality (3.1) we have

$$\Re\frac{f(z_2) - f(z1)}{z_2 - z_1} > 0.$$

This proves the univalence of $I(f)$.

In order to prove that $I(f) \in \mathcal{S}^*_{\mathcal{H}} \equiv \mathcal{S}^*_{\mathcal{H}}(0)$, by using Lemma 2.2 it suffices to show that

$$\sum_{k=2}^{\infty} k\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}|a_k| + |b_1| + \sum_{k=2}^{\infty} k\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}|b_k| \leq 1.$$

Since $|a_k| \leq 1,\ |b_k| \leq 1,\ \forall\, k \geq 2$, because of given condition (i), we obtain that

$$\begin{aligned}
& \sum_{k=2}^{\infty} k\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}|a_k| + |b_1| + \sum_{k=2}^{\infty} k\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}|b_k| \\
\leq \ & \sum_{k=2}^{\infty} \frac{e^{-m_1}m_1^{k-1}(k-1)^n}{(k-1)!} + |b_1| + \sum_{k=2}^{\infty} \frac{e^{-m_2}m_2^{k-1}(k-1)^n}{(k-1)!} \\
= \ & \sum_{i=0}^{n} \binom{n}{k} k^{n-i}(1-e^{-m_1}) + |b_1| + \sum_{i=0}^{n} \binom{n}{k} k^{n-i}(1-e^{-m_2}) \\
\leq \ & 1, from(ii).
\end{aligned}$$

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This completes the proof of Theorem 3.1.

We next find a sufficient condition for which the integral operator I maps $\mathcal{C}^0_{\mathcal{H}}$ into $\mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Theorem 3.2. *If* $0 \leq \alpha < 1,\ m_1, m_2 > 0$ *and the inequality*

$$\sum_{i=0}^{n} \binom{n}{k} k^{n-i}\Big[m_1 + m_2 + (2-\alpha)(1-e^{-m_1}) + \alpha(1-e^{-m_2}) - \frac{\alpha}{m_1}(1-e^{-m_1} - m_1 e^{-m_1}) - \frac{\alpha}{m_2}(1-e^{-m_2} - m_2 e^{-m_2})\Big] \leq 2(1-\alpha)$$

is satisfied, then $I(\mathcal{C}^0_{\mathcal{H}}) \subset \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Proof. Let $f = h + \overline{g} \in \mathcal{C}^0_{\mathcal{H}}$ where h and g are given (1.1) with $b_1 = 0$. We need to prove that $I(f) = H + \overline{G} \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ where h and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.2, it is enough to show that

$$\sum_{k=2}^{\infty} (k-\alpha)\left|\frac{e^{-m_1} m_1^{k-1}(k-1)^n}{k!} a_k\right| + \sum_{k=2}^{\infty} (k+\alpha)\left|\frac{e^{-m_2} m_2^{k-1}(k-1)^n}{k!} b_k\right| \leq 1-\alpha. \tag{9}$$

Applying Lemma 2.1, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} (k-\alpha)\left|\frac{e^{-m_1} m_1^{k-1}(k-1)^n}{k!} a_k\right| + \sum_{k=2}^{\infty} (k+\alpha)\left|\frac{e^{-m_2} m_2^{k-1}(k-1)^n}{k!} b_k\right| \\
\leq\ & \frac{1}{2}\left[\sum_{k=2}^{\infty} (k-\alpha)(k+1)\frac{e^{-m_1} m_1^{k-1}(k-1)^n}{k!} + \sum_{k=2}^{\infty} (k+\alpha)(k-1)\frac{e^{-m_2} m_2^{k-1}(k-1)^n}{k!}\right] \\
=\ & \frac{1}{2}\left[\sum_{k=2}^{\infty} \{k(k-1) + k(2-\alpha) - \alpha\}\frac{e^{-m_1} m_1^{k-1}(k-1)^n}{k!} \right. \\
& \left. + \sum_{k=2}^{\infty} \{k(k-1) + k\alpha - \alpha\}\frac{e^{-m_2} m_2^{k-1}(k-1)^n}{k!}\right] \\
=\ & \frac{1}{2}\left[\sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1}(k-1)^n}{(k-2)!} + (2-\alpha)\sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1}(k-1)^n}{(k-1)!} - \alpha\sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1}(k-1)^n}{k!} \right. \\
& \left. + \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1}(k-1)^n}{(k-2)!} + \alpha\sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1}(k-1)^n}{(k-1)!} - \alpha\sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1}(k-1)^n}{k!}\right] \\
=\ & \frac{1}{2}\sum_{i=0}^{n} \binom{n}{k} k^{n-i}\Big[m_1 + m_2 + (2-\alpha)(1-e^{-m_1}) + \alpha(1-e^{-m_2}) \\
& - \frac{\alpha}{m_1}(1-e^{-m_1} - m_1 e^{-m_1}) - \frac{\alpha}{m_2}(1-e^{-m_2} - m_2 e^{-m_2})\Big].
\end{aligned}$$

The last expression is bounded above by $1-\alpha$ by the given hypothesis.
Thus the proof of Theorem 3.2 is established.

Theorem 3.3. *If* $0 \leq \alpha < 1,\ m_1, m_2 > 0$ *and if the inequality*

$$\sum_{i=0}^{n} \binom{n}{k} k^{n-i}\Big[2m_1^2 + m(9-2\alpha)m_1 + (6-5\alpha)(1-e^{-m_1}) - \frac{\alpha}{m_1}(1-e^{-m_1} - m_1 e^{-m_1}) \\ 2m_2^2 + m(2\alpha+3)m_2 - \alpha(1-e^{-m_1}) - \frac{\alpha}{m_2}(1-e^{-m_2} - m_2 e^{-m_2})\Big] \leq 6(1-\alpha)$$

is satisfied, then $I(\mathcal{S}^{*,0}_{\mathcal{H}}) \subset \mathcal{S}^*_{\mathcal{H}}(\alpha)$ *and* $I(\mathcal{C}^0_{\mathcal{H}}) \subset \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

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Proof. Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^{*,0}$ where h and g are given (1.1) with $b_1 = 0$. We need to prove that $I(f) = H + \overline{G} \in \mathcal{S}_{\mathcal{H}}^{*}(\alpha)$ where h and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.2, it is enough to show that

$$\sum_{k=2}^{\infty}(k-\alpha)\left|\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}a_k\right| + \sum_{k=2}^{\infty}(k+\alpha)\left|\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}b_k\right| \leq 1-\alpha. \tag{10}$$

Applying Lemma 2.4, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k-\alpha)\left|\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}a_k\right| + \sum_{k=2}^{\infty}(k+\alpha)\left|\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}b_k\right| \\
\leq\ & \frac{1}{6}\left[\sum_{k=2}^{\infty}(k-\alpha)(2k+1)(k+1)\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!} + \sum_{k=2}^{\infty}(k+\alpha)(2k-1)(k-1)\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}\right] \\
=\ & \frac{1}{6}\left[\sum_{k=2}^{\infty}\{2k(k-1)(k-2)+(9-2\alpha)k(k-1)+(6-5\alpha)k-\alpha\}\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}\right. \\
&\left. + \sum_{k=2}^{\infty}\{2k(k-1)(k-2)+(2\alpha+3)k(k-1)-k\alpha+\alpha\}\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}\right] \\
=\ & \frac{1}{6}\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\left[e^{-m_1}\left\{2\sum_{k=2}^{\infty}\frac{m_1^{k-1}}{(k-3)!} + (9-2\alpha)\sum_{k=2}^{\infty}\frac{m_1^{k-1}}{(k-2)!} + (6-5\alpha)\sum_{k=2}^{\infty}\frac{m_1^{k-1}}{(k-1)!} - \alpha\sum_{k=2}^{\infty}\frac{m_1^{k-1}}{k!}\right\}\right. \\
&\left. + e^{-m_2}\left\{2\sum_{k=2}^{\infty}\frac{m_2^{k-1}}{(k-3)!} + (2\alpha+3)\sum_{k=2}^{\infty}\frac{m_2^{k-1}}{(k-2)!} - \alpha\sum_{k=2}^{\infty}\frac{m_2^{k-1}}{(k-1)!} + \alpha\sum_{k=2}^{\infty}\frac{m_2^{k-1}}{k!}\right\}\right] \\
=\ & \frac{1}{6}\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\left[\left\{2m_1^2+(9-2\alpha)m_1+(6-5\alpha)(1-e^{-m_1}) - \frac{\alpha}{m_1}(1-e^{-m_1}-m_1e^{-m_1})\right\}\right. \\
&\left. - \left\{2m_2^2+(2\alpha+3)m_2-\alpha(1-e^{-m_1}) + \frac{\alpha}{m_2}(1-e^{-m_2}-m_2e^{-m_2})\right\}\right] \\
\leq\ & 1-\alpha,
\end{aligned}$$

by the given hypothesis.
This completes the proof of Theorem 3.3.

Theorem 3.4. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{TS}_{\mathcal{H}}^{*}(\alpha)) \subset \mathcal{TS}_{\mathcal{H}}^{*}(\alpha)$, if and only if the inequality*

$$\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\left[\frac{1}{m_1}(1-e^{-m_1}-m_1e^{-m_1}) + \frac{1}{m_2}(1-e^{-m_2}-m_2e^{-m_2})\right] \leq 1 - \frac{1+\alpha}{1-\alpha}|b_1|, \tag{11}$$

is satisfied.

Proof. Let $f = h+\overline{g} \in \mathcal{TS}_{\mathcal{H}}^{*}(\alpha)$, where h and g are given by (2.2), we need to prove that the integral operator

$$I(f(z)) = z - \sum_{k=2}^{\infty}\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}|a_k|z^k + |b_1|\overline{z} + \sum_{k=2}^{\infty}\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}|b_k|\overline{z^k}$$

is in $\mathcal{TS}_{\mathcal{H}}^{*}(\alpha)$, if and only if,

$$\sum_{k=2}^{\infty}(k-\alpha)\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}|a_k| + (1+\alpha)|b_1| + \sum_{k=2}^{\infty}(k+\alpha)\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}|b_k| \leq 1-\alpha.$$

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By using Lemma 2.3, we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty}(k-\alpha)\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}|a_k| + (1+\alpha)|b_1| + \sum_{k=2}^{\infty}(k+\alpha)\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}|b_k| \\ \leq\ & (1-\alpha)\left[\sum_{k=2}^{\infty}\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!} + \sum_{k=2}^{\infty}\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}\right] + (1+\alpha)|b_1| \\ =\ & (1-\alpha)\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\Big[\frac{1}{m_1}(1-e^{-m_1}-m_1e^{-m_1}) + \frac{1}{m_2}(1-e^{-m_2}-m_2e^{-m_2})\Big] + (1+\alpha)|b_1| \\ \leq\ & 1-\alpha, \end{aligned}$$

by the given condition and this completes the proof of the theorem.

We next explore a sufficient condition which ensure that I maps $\mathcal{C}_{\mathcal{H}}^0$ into $\mathcal{C}_{\mathcal{H}}(\alpha)$.

Theorem 3.5. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{C}_{\mathcal{H}}(\alpha)$, if the inequality*

$$\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\Big[m_1^2 + m_2^2 + (4-\alpha)m_1 + (2-\alpha)m_2\Big] \leq 2(1-\alpha).$$

is satisfied.

Proof. Let $f = h + \overline{g} \in \mathcal{C}_{\mathcal{H}}^0$, where h and g are given by (2.2)with $b_1 = 0$, we need to prove that the integral operator $I(f(z)) = H + \overline{G} \in \mathcal{C}_{\mathcal{H}}(\alpha)$, where H and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.4, it is enough to show that

$$\sum_{k=2}^{\infty}k(k-\alpha)\Big|\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}a_k\Big| + \sum_{k=2}^{\infty}k(k+\alpha)\Big|\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}b_k\Big| \leq 1-\alpha.$$

By applying Lemma 2.1, we have

$$\begin{aligned} &\sum_{k=2}^{\infty}k(k-\alpha)\Big|\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{k!}a_k\Big| + \sum_{k=2}^{\infty}k(k+\alpha)\Big|\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{k!}b_k\Big| \\ \leq\ & \frac{1}{2}\left[\sum_{k=2}^{\infty}(k-\alpha)(k+1)\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{(k-1)!} + \sum_{k=2}^{\infty}(k+\alpha)\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{(k-2)!}\right] \\ =\ & \frac{1}{2}\left[\sum_{k=2}^{\infty}\big\{(k-1)(k-2) + (4-\alpha)(k-1) + 2(1-\alpha)\big\}\frac{e^{-m_1}m_1^{k-1}(k-1)^n}{(k-1)!}\right. \\ & \left. + \sum_{k=2}^{\infty}\big\{(k-2) + (2+\alpha)\big\}\frac{e^{-m_2}m_2^{k-1}(k-1)^n}{(k-2)!}\right] \\ =\ & \frac{1}{2}\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\Big[m_1^2 + (4-\alpha)m_1 + 2(1-\alpha)(1-e^{-m_1}) + m_2^2 + (2-\alpha)m_2\Big] \\ \leq\ & 1-\alpha, \end{aligned}$$

by the given hypothesis. Thus the proof of Theorem 3.5 is established.

The proof of the following theorems ar similar to previous theorems so we state only the results.

Theorem 3.6. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{TS}_{\mathcal{H}}^*(\alpha)) \subset \mathcal{TC}_{\mathcal{H}}(\alpha)$, if and only if the inequality*

$$\sum_{i=0}^{n}\binom{n}{k}k^{n-i}\big[e^{-m_1} + e^{-m_2}\big] \leq 1 + \frac{1+\alpha}{1-\alpha}|b_1|,$$

is satisfied.

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Theorem 3.7. *If* $0 \leq \alpha < 1,\ m_1, m_2 > 0$, *then* $I(\mathcal{TC}_{\mathcal{H}}(\alpha)) \subset \mathcal{TC}_{\mathcal{H}}(\alpha)$, *if and only if the inequality (3.4) is satisfied.*

Theorem 3.8. *If* $0 \leq \alpha < 1,\ m_1, m_2 > 0$ *and if the inequality*

$$\sum_{i=0}^{n} \binom{n}{k} k^{n-i} e^{m_1} \Big[2(m_1^3 + m_2^3) + (15 - 2\alpha)m_1^2 + 3(8 - 3\alpha)m_1 \\ +(2\alpha + 9)m_2^2 + 3(2 + \alpha)m_2 \Big] \leq 6(1 - \alpha),$$

is satisfied, then $I(\mathcal{S}_{\mathcal{H}}^{*;0}) \subset \mathcal{C}_{\mathcal{H}}(\alpha)$ *and* $I(\mathcal{C}_{\mathcal{H}}^{0}) \subset \mathcal{K}_{\mathcal{H}}(\alpha)$.

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