# A Novel Approach to Necessary and Sufficient Conditions for the Diagonalization of Refined Neutrosophic Matrices

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**Abstract**

This work is dedicated to study the conditions of diagonalization in the case of refined neutrosophic matrices, where it presents the necessary and sufficient conditions for the diagonalization of these matrices by finding a relationship with classical diagonalization of matrices. Also, it describes an algorithm to obtain all eigen values and eigen vectors of refined neutrosophic matrices from the classical ones.

**Keywords:** Refined neutrosophic matrix, refined neutrosophic eigen value, refined neutrosophic eigen vector, refined neutrosophic diagonalization

## 1. Introduction

Neutrosophy is a new branch of philosophy founded by F. Smarandache [16] to deal with uncertainty in all fields of human knowledge, where it can be considered as a generalization of intuitionistic fuzzy logic.

Neutrosophic algebraic studies began with the great efforts of Kandasamy and Smarandache in [12], where they studied for the first time neutrosophic rings and neutrosophic groups.

In the literature, we find many neutrosophic algebraic results about spaces [11], modules [6], rings [5,13], number theory [15,17], and other related systems [23-25]

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Neutrosophic matrices were defined in [9] as a useful tool to deal with indeterminacy and as a generalization of fuzzy matrices [10]. The algebraic properties of these matrices were studied in [8], such as their linear transformations and diagonalization problem [27].

In [7], Agboola et. al. presented the idea of splitting the indeterminacy element I into two levels of indeterminacy $I_1, I_2$ with the following property $I_1 I_2 = I_2 I_1 = I_1, {I_1}^2 = I_1, {I_2}^2 = I_2$ . This idea was used to defined refined neutrosophic groups [7], refined neutrosophic rings [1], modules [14,18], and matrices [4].

The invertibility, nilpotency, and idempotency of refined neutrosophic matrices were characterized in [4].

In this work, we study the problem of diagonalization of refined neutrosophic matrices, where we determine an algorithm to find all eigen vectors and values, and we use this idea to determine the necessary and sufficient condition for the diagonalization of refined neutrosophic matrices.

All refined neutrosophic matrices through this study are considered over a refined neutrosophic field $K(I_1, I_2)$.

## 2. Preliminaries

**Definition:** [7]

If $X$ is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2); a , b , c \in X \}$ is called the refined neutrosophic generated by $X$ , $I_1, I_2$ .

**Definition:** [1]

Let $(R, +, .)$ be a ring then $(R(I_1, I_2) , + , .)$ is called a refined neutrosophic ring generated by $R$ ,$I_1, I_2$.

Where ${I_1}^2 = I_1$ , ${I_2}^2 = I_2$ , $I_1 I_2 = I_2 I_1 = I_1$.

If R is an algebraic field, then $R(I_1, I_2)$ is called a refined neutrosophic field.

**Definition:** [4]

Let $A=\begin{bmatrix} a_{11} & \dots. & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$ be an $n \times m$ matاكتب المعادلة هناrix, if $a_{ij} = x + yI_1 + zI_2 \in R_2(I)$, then it is called an refined neutrosophic matrix. Where $R_2(I)$ is an refined neutrosophic field.

**Theorem:** [4]

Let $X$=$A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, then it is invertible if and only if

$A, A + C, A + B + C$ are invertible. The inverse of $X$ is

$$X^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2 .$$

**Definition:** [4]

We defined the determinant of a square $n \times n$ refined neutrosophic matrix as

$$detX = detA + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - detA]I_2.$$

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## 3. Main results

**Definition 3.1:**

Let $L = A + BI_1 + CI_2$ be a refined neutrosophic matrix, and $M = X + YI_1 + ZI_2$ is a strong refined neutrosophic vector, then it is called a refined neutrosophic eigen vector of $L$ if and only if

$LM = (a + bI_1 + cI_2)M$.

The refined neutrosophic number $a + bI_1 + cI_2$ is called a refined neutrosophic eigen value.

**Theorem 3.2:**

Let $L = A + BI_1 + CI_2$ be a refined neutrosophic matrix, then $M = X + YI_1 + ZI_2$ is a refined neutrosophic eigen vector with $a + bI_1 + cI_2$as the corresponding eigen value if and only if $X, X + Z, X + Y + Z$ are eigen vectors of $A, A + C, A + B + C$ respectively. As well as, $a, a + c, a + b + c$ are the corresponding eigen values respectively.

**Proof:**

Suppose that $M = X + YI_1 + ZI_2$ is a refined neutrosophic eigen vector with $a + bI_1 + cI_2$as the corresponding eigen value, then $LM = (a + bI_1 + cI_2)M$. By easy computing, we get

$(AX + I_1[(A + B + C)(X + Y + Z) - (A + C)(X + Z)] + I_2[(A + C)(X + Z) - AX]=$

$aX + I_1[(a + b + c)(X + Y + Z) - (a + c)(X + Z)] + I_2[(a + c)(X + Z) - aX]$. Which is equivalent to

$AX = aX, (A + C)(X + Z) = (a + c)(X + Z), (A + B + C)(X + Y + Z) = (a + b + c)(X + Y + Z)$. Thus

$X, X + Z, X + Y + Z$ are eigen vectors of $A, A + C, A + B + C$ respectively. As well as, $a, a + c, a + b + c$ are the corresponding eigen values respectively.

Conversely, we assume that $X, X + Z, X + Y + Z$ are eigen vectors of $A, A + C, A + B + C$ respectively, with $a, a + c, a + b + c$ are the corresponding eigen values respectively, then by the definition of eigen vectors we can write:

$AX = aX, (A + C)(X + Z) = (a + c)(X + Z), (A + B + C)(X + Y + Z) = (a + b + c)(X + Y + Z)$.

This means that $(AX + I_1[(A + B + C)(X + Y + Z) - (A + C)(X + Z)] + I_2[(A + C)(X + Z) - AX]=$

$aX + I_1[(a + b + c)(X + Y + Z) - (a + c)(X + Z)] + I_2[(a + c)(X + Z) - aX]$. Which implies

$LM = (a + bI_1 + cI_2)M$, thus $M = X + YI_1 + ZI_2$ is a refined neutrosophic eigen vector with $a + bI_1 + cI_2$as the corresponding eigen value.

**Theorem 3.3:**

Eigen values can be gotten by solving the refined neutrosophic equation $\det(L - (a + bI_1 + cI_2)U_{n\times n}) = 0$.

**Proof:**

Firstly, we have:

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$$\det(L-(a+bI_1+cI_2)U_{n\times n}) = \det([A-aU_{n\times n}]+[B-bU_{n\times n}]I_1+[C-cU_{n\times n}]I_2)$$

$=\det(A-aU_{n\times n})+I_1[\det(A+B+C-(a+b+c)U_{n\times n})-\det(A+C-(a+c)U_{n\times n})]+I_2[\det(A+C-(a+c)U_{n\times n})-\det(A-aU_{n\times n})]=0$. This implies that

$\det(A-aU_{n\times n})=\det(A+C-(a+c)U_{n\times n})=\det(A+B+C-(a+b+c)U_{n\times n})=0$.

Hence, $a, a+c, a+b+c$ are eigen values of $A, A+C, A+B+C$ respectively, which is equivalent to that

$a+bI_1+cI_2$ is a refined neutrosophic eigen value of L.

**Example 3.4:**

Consider the following refined neutrosophic matrix $L=\begin{bmatrix}1-I_1+I_2 & 1-I_2\\ -I_1+I_2 & -1-I_1+4I_2\end{bmatrix}$.

(a) L is written as $L=A+BI_1+CI_2$. Where $A=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, B=\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix}, C=\begin{pmatrix}1 & -1\\ 1 & 4\end{pmatrix}$.

(b) We have $A+C=\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}, A+B+C=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$. The set of eigen values of A is $\{1,-1\}$, for $A+C$ it is $\{2,3\}$, and for $A+B+C$ it is $\{1,2\}$.

(c) According to Theorem 3.2, the set of refined neutrosophic eigen values of L is

$$\{1+I_1[1-2]+I_2[2-1], 1+I_1[2-2]+I_2[2-1], 1+I_1[1-3]+I_2[3-1], 1+I_1[2-3]+I_2[3-1], -1+I_1[1-2]+I_2[2+1], -1+I_1[2-2]+I_2[2+1], -1+I_1[1-3]+I_2[3+1], -1+I_1[2-3]+I_2[3+1]\}$$

$=\{1-I_1+I_2, 1+I_2, 1-2I_1+2I_2, 1-I_1+2I_2, -1-I_1+3I_2, -1+3I_2, -1-2I_1+4I_2, -1-I_1+4I_2\}$.

(d) We get the same values by solving the following refined neutrosophic equation:

$\det(L-(a+bI_1+cI_2)U_{n\times n})=(a+bI_1+cI_2)^2+(a+bI_1+cI_2)(2I_1-5I_2)+(-1-4I_1+7I_2)=0$. For the solution, we can use the algebraic algorithm which was introduced in [3].

**Definition 3.5:**

Let $X=A+BI_1+CI_2$ be any refined neutrosophic matrix, then it is called diagonalizable if there exists an invertible refined neutrosophic matrix $Y=F+GI_1+HI_2$ such that $Y^{-1}XY=D$. Where D is a refined neutrosophic diagonal matrix.

**Theorem 3.6:**

Let $X=A+BI_1+CI_2$ be any refined neutrosophic matrix, then it is diagonalizable if and only if $A, A+C, A+B+C$ are diagonalizable.

**Proof:**

Suppose that $A, A+C, A+B+C$ are diagonalizable, then there are three invertible matrices F,G,H such that $F^{-1}AF=D_0, G^{-1}(A+B+C)G=D_1, H^{-1}(A+C)H=D_2$. Where $D_0, D_1, D_2$ are diagonal matrices. Put $Y=F+(G-H)I_1+(H-F)I_2, D=D_0+(D_1-D_2)I_1+(D_2-D_0)I_2$. Now, let us compute $Y^{-1}XY=(F^{-1}+I_1[(G-$

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$H + H - F + F)^{-1} - (H - F + F)^{-1}] + I_2[(H - F + F)^{-1} - F^{-1}])(A + BI_1 + CI_2)\big(F + (G - H)I_1 + (H - F)\big)$
$=$

$(F^{-1} + I_1[G^{-1} - H^{-1}] + I_2[H^{-1} - F^{-1}])(AF + I_1[(A + B + C)(G) - (A + C)H] + I_2[(A + C)H - AF]) = F^{-1}AF + I_1[G^{-1}(A + B + C)G - H^{-1}(A + C)H] + I_2[H^{-1}(A + C)H - F^{-1}AF] = D_0 + (D_1 - D_2)I_1 + (D_2 - D_0)I_2 = D$. This means that X is diagonalizable.

Conversely, we assume that X is diagonalizable, hence there exists an invertible refined neutrosophic matrix $Y = F + GI_1 + HI_2$ and a refined diagonal neutrosophic matrix $D = D_0 + D_1I_1 + D_2I_2$ such that $Y^{-1}XY = D$ (*).

We shall compute equation (*):

$(F^{-1} + I_1[(F + G + H)^{-1} - (F + H)^{-1}] + I_2[(F + H)^{-1} - F^{-1}])(A + I_1[B] + I_2[C])(F + I_1[G] + I_2[H]) = (F^{-1} + I_1[(F + G + H)^{-1} - (F + H)^{-1}] + I_2[(F + H)^{-1} - F^{-1}])(AF + I_1[(A + B + C)(F + G + H) - (A + C)(F + H)] + I_2[(A + C)(F + H) - AF]) =$

$F^{-1}AF + I_1[(F + G + H)^{-1}(A + B + C)(F + G + H) - (F + H)^{-1}(A + C)(F + H)] + I_2[(F + H)^{-1}(A + C)(F + H) - F^{-1}AF] = D_0 + D_1I_1 + D_2I_2$.

This implies that $F^{-1}AF = D_0, (F + H)^{-1}(A + C)(F + H) - F^{-1}AF = D_2, (F + G + H)^{-1}(A + B + C)(F + G + H) - (F + H)^{-1}(A + C)(F + H) = D_1$

Thus $(F + H)^{-1}(A + C)(F + H) = D_0 + D_2, (F + G + H)^{-1}(A + B + C)(F + G + H) = D_0 + D_1 + D_2$. Which means that $A, A + C, A + B + C$ are diagonalizable.

**Remark 3.7:**

If $F, G, H$ are the diagonalization matrices of $A, A + B + C, A + C$ respectively, then $F + (G - H)I_1 + (H - F)I_2$ is the diagonalization matrix of $X$. Also, the corresponding diagonal matrix of $X$ is

$D = D_0 + (D_1 - D_2)I_1 + (D_2 - D_0)I_2$, where $D_0, D_1, D_2$ are the corresponding diagonal matrices of $A, A + B + C, A + C$ respectively. .

**Example 3.8:**

Consider the refined neutrosophic matrix defined in the Example 3.4, we have:

(a) The diagonalization matrix of $A$ is $F = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$, the corresponding diagonal matrix is $D_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that $F^{-1}AF = D_0$.

(b) The diagonalization matrix of $A + B + C$ is $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. It is clear that $G^{-1}(A + B + C)G = D_1$.

(c) The diagonalization matrix of $A + C$ is $H = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. It is clear that $H^{-1}(A + C)H = D_2$ .

(d) The refined neutrosophic diagonalization matrix of L is :

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$Y = F + (G - H)I_1 + (H - F)I_2 = \begin{pmatrix} 1 & 1 - I_2 \\ -2 + I_1 + I_2 & I_2 \end{pmatrix}$. The corresponding diagonal matrix is

$D = D_0 + (D_1 - D_2)I_1 + (D_2 - D_0)I_2 = \begin{pmatrix} -1 - I_1 + 3I_2 & 0 \\ 0 & 1 - I_1 + 2I_2 \end{pmatrix}$.

(e) $F^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}, G^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, H^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, Y^{-1} = F^{-1} + (G^{-1} - H^{-1})I_1 + (H^{-1} - F^{-1})I_2$

$= \begin{pmatrix} I_2 & -\frac{1}{2} + \frac{1}{2}I_2 \\ 1 - I_1 & \frac{1}{2} + \frac{1}{2}I_2 \end{pmatrix}, Y^{-1}LY = D.$

**Conclusion**

In this article, we have studied the necessary and sufficient conditions for the diagonalization of refined neutrosophic matrices. Also, we have determined an algorithm to compute all refined neutrosophic eigen values/vectors which are related to these matrices.

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