A Study of AH-Substructures in n-Refined Neutrosophic Vector Spaces

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Abstract

The aim of this paper is to define and study for the first time AH-substructures in n-refined neutrosophic vector spaces such as weak/strong AH-subspaces, and weak/strong AH-linear transformations between two n-refined neutrosophic vector spaces. Also, this paper introduces some elementary properties of these concepts.

Keywords: n-Refined Neutrosophic vector space , AH-subspace, AHS-subspace, AH-linear transformation.

1. Introduction

Neutrosophy as a new kind of logic, concerns with nature, origin, and scope of neutralities became a rich material in algebra. Many algebraic structures have been defined and handled such as neutrosophic rings, neutrosophic modules, and neutrosophic vector spaces. See [8,9,10,11,12,14]. More generalizations came to light such as refined neutrosophic rings, n-refined neutrosophic rings, and n-refined neutrosophic vector spaces. See [3,5,6,7,15,16].

AH-substructures were defined for the first time in neutrosophic rings [1]. Then they were defined in n-refined neutrosophic rings, and neutrosophic vector spaces in [2,4]. AH-structures consist of similar objects, each object has the same structure. For example in a neutrosophic vector space $V(I) = V + VI$, AH-subspace is a non empty subset with form $T = M + NI$, where M, N are two classical subspaces in V. Also, AHS-linear transformations were defined by similar aspect [4]. AH-substructures illustrate a bridge between neutrosophic structures and classical algebraic structures and help us to use classical methods in neutrosophical studies.

This article defines some AH-substructures in n-refined neutrosophic vector spaces. Concepts such as AH-subspaces, and AH-linear transformations. Also, it presents some interesting properties and theorems concerning these concepts.

2. Preliminaries

Definition 2.1: [15]

Let (R, +, ×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)=$\{a_0 + a_1 I + \cdots + a_n I_n \ ; \ a_i \in R\}$ to be n-refined neutrosophic ring.

Definition 2.2: [15]

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(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = { $a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i . AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal if P_i are two sided ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

Definition 2.3 :[8]

Let (V, +, .) be a vector space over the field K then (V(I),+,.) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4 : [8]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.5: [14]

Let (K, +, .) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic ring but not a field in classical meaning**.**

Definition 2.6 : [14]

Let (V, +, .) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n;\ x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Definition 2.7: [14]

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 2.8: [14]

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Definition 2.9: [4]

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are subspaces of V is called an AH-subspace of V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Definition 2.10: [4]

(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a + bI) = L_V(a) + L_V(b)I.$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I.$

3. Main discussion

Definition 3.1:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K. Consider the set $\{M_i; 0 \le i \le n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n;\ m_i \in M_i\}$ is a weak n-refined AH-subspace of the weak n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a weak n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j.$

Definition 3.2:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Consider the set $\{M_i; 0 \le i \le n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n;\ m_i \in M_i\}$ is a strong n-refined AH-subspace of the strong n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a strong n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j.$

Theorem 3.3:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ be a weak n-refined AH-subspace. Then

(a) $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) If X_i is a bases of M_i, $X = \bigcup_{i=0}^{n} X_iI_i$ is a bases of $M_n(I)$.

(c) $\dim(M_n(I)) = \sum_{i=0}^{n} \dim(M_i)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_iI_i, y = \sum_{i=0}^{n} b_iI_i;\ b_i, a_i \in M_i$ be two arbitrary elements in $M_n(I)$, r be an arbitrary element in K, we have:

$x + y = \sum_{i=0}^{n}(a_i + b_i)I_i \in M_n(I)$, since $a_i + b_i \in M_i$ because M_i is a subspace of V.

$r.x = \sum_{i=0}^{n} ra_iI_i \in M_n(I)$, since $ra_i \in M_i$ because M_i is a subspace of V. Thus $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) Suppose that $X_0 = \left\{x_1^{(0)}, \dots, x_{s_0}^{(0)}\right\}, X_1 = \left\{x_1^{(1)}, \dots, x_{s_1}^{(1)}\right\}, \dots \dots, X_n = \{x_1^{(n)}, \dots, x_{s_n}^{(n)}\}$, let $x = \sum_{i=0}^{n} a_iI_i$ be an arbitrary element of $M_n(I)$, since X_i is a basis of M_i for all i. We can write:

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$a_i = \sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)};\ t_j \in K$, so $x = \sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)} I_n$. This implies that X is a generating set of $M_n(I)$.

Now we prove that X is linearly independent. For our purpose we assume

$\sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)} I_n = 0$, by definition of n-refined vector space we find

$\sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)}$ for all i, hence $t_j^{(i)} = 0$ for all i,j, since each X_i is linearly independent itself. Thus our proof is complete.

(c) It holds directly from (b).

Example 3.4:

Let $V = R^2$ be a vector space over the field R, $V_2(I) = R_2^2(I) = \{(a_0, b_0) + (a_1, b_1)I_1 + (a_2, b_2)I_2;\ a_i, b_i \in R\}$ be the corresponding weak 2-refined neutrosophic vector space over the field R, we have:

(a) $M =< (1,0) >= \{(m, 0); m \in R\}$, $N =< (0,1) >= \{(0, n); n \in R\}$ are two subspaces of $V = R^2$.

(b) $T = M + NI_1 + MI_2 = \{(m, 0) + (0, n)I_1 + (s, 0)I_2; m, n, s \in R\}$ is a weak AH-subspace of $V_2(I)$.

(c) The set $X = \{(1,0), (0,1)I_1, (1,0)I_2\}$ is a bases of T, $\dim(T) = \dim(M) + \dim(N) + \dim(M) = 3$.

(d) $D = N + NI_1 + NI_2 = \{(0, a) + (0, b)I_1 + (0, c)I_2; a, b, c \in R\}$ is a weak AHS-subspace.

Theorem 3.5:

Let V be a vector space with $\dim(V) = n + 1$. Then V is isomorphic to a weak AHS-subspace of the corresponding weak n-refined neutrosophic vector space.

Proof:

Let M be any one dimensional subspace of V, $T = M + MI_1 + \cdots + MI_n$ is a weak AHS-subspace of the weak n-refined neutrosophic vector space $V_n(I)$. As a result of Theorem 3.3 , we find $\dim(T) = n + 1 = \dim(V)$, thus V is isomorphic to T.

Example 3.6:

Let $V = R^3$ be a vector space over the field R, $V_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in V\}$ is the corresponding weak 3-refined neutrosophic vector space, $M =< (1,0,0) >$ is a subspace of V.

$T = M + MI_1 + MI_2 = \{(a, 0,0) + (b, 0,0)I_1 + (c, 0,0)I_2; a, b, c \in R\}$ is a weak AHS-subspace of $V_3(I)$ with $\dim(T) = 3$, this implies $T \cong V$.

Theorem 3.7:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, $M_n(I) = M + MI_1 + \cdots + MI_n$ be a strong n-refined AHS-subspace. Then:

(a) $M_n(I)$ is a submodule of $V_n(I)$.

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(b) If Y is a bases of M, $X = \cup_{i=0}^{n} YI_i$ is a bases of $M_n(I)$.

(c) $\dim(M_n(I)) = \sum_{i=0}^{n} \dim(M) = n.\dim(M)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i$; $b_i, a_i \in M_i$ be two arbitrary elements in $M_n(I)$, $r = \sum_{i=0}^{n} r_i I_i$ be an arbitrary element in $K_n(I)$, we have:

$x + y = \sum_{i=0}^{n}(a_i + b_i)I_i \in M_n(I)$, since $a_i + b_i \in M_i$ because M_i is a subspace of V.

$r.x = \sum_{i,j=0}^{n} r_i a_j I_i I_j \in M_n(I)$, since $r_i a_j \in M$ because M is a subspace of V. Thus $M_n(I)$ is a vector subspace of $V_n(I)$.

(b),(c) They are similar to that of Theorem 3.5 .

Remark 3.8:

If $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, and

$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a strong n-refined AH-subspace, then it is not supposed to be a submodule.

We clarify it by the following example.

Example 3.9:

Let $V = R^2$ be a vector space over R, $V_2(I) = R_2^2(I) = \{(a,b) + (c,d)I_1 + (e,f)I_2; a,b,c,d,e,f \in R\}$ be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field $R_2(I)$.

$M =< 0,1 >, N =< (1,0) >$ are two subspaces of V, $T = M + NI_1 + NI_2$ is a strong AH-subspace of $V_2(I)$.

$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1.I_1 + 1.I_2 \in R_2(I)$,

$r.x = 1.(0,1) + 1.(0,1)I_1 + 1.(0,1)I_2 + 1.(2,0)I_1I_1 + 1.(2,0)I_1 + 1.(1,0)I_1I_2 + 1.(0,1)I_2 + 1.(2,0)I_1I_2 + 1.(2,0)I_2I_2 = (0,1) + [(0,1) + (2,0) + (1,0) + (2,0)]I_1 + [(0,1) + (0,1) + (2,0)]I_2 =$

$(0,1) + (5,1)I_1 + (2,2)I_2$, $r.x$ does not belong to T, thus T is not a submodule.

Definition 3.10:

Let $V_n(I)$ be a weak/strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two weak/strong AH-subspaces of $V_n(I)$, we define:

(a) $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \cdots + (M_n \cap W_n)I_n$.

(b) $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \cdots + (M_n + W_n)I_n$.

Theorem 3.11:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two weak AH-subspaces of $V_n(I)$. Then:

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$M_n(I) \cap W_n(I), M_n(I) + W_n(I)$ are two weak AH-subspaces of $V_n(I)$.

Proof:

Since $M_i \cap W_i, M_i + W_i$ are subspaces of V for all i, we obtain the proof.

Theorem 3.12:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \dots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \dots + W_n I_n$ be two strong AH-subspaces of $V_n(I)$. Then:

(a) $M_n(I) \cap W_n(I)$ is a strong AH-subspaces of $V_n(I)$.

(b) $M_n(I) + W_n(I)$ is not supposed to be a strong AH-subspace of $V_n(I)$.

Proof:

The proof is similar to that of Theorem 3.11 .

Definition 3.13:

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \le i \le n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding weak n-refined neutrosophic vector spaces over the field K respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \dots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a weak AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i,j$, we call f a weak AHS-linear transformation.

Example 3.14:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{ (x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\; x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2;\; x_i, y_i \in R\}$ be the corresponding weak 2-refined neutrosophic vector spaces. We have $g: V \to W; g(a,b,c) = (b,c), h: V \to W; h(a,b,c) = (2a, 0)$

$s: V \to W; s(a,b,c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m,n,q \in V$ is a weak AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3,-1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3,-1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m,n,q \in V$ is a weak AHS-linear transformation.

Definition 3.15:

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Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \le i \le n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a strong AH-linear transformation.

(b) If $f_i = f_j\ for\ all\ i,j$, we call f a strong AHS-linear transformation.

Example 3.16:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{(x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\ x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2;\ x_i, y_i \in R\}$ be the corresponding strong 2-refined neutrosophic vector spaces over the 2-refined neutrosophic field $R_2(I)$. We have $g: V \to W; g(a,b,c) = (b,c), h: V \to W; h(a,b,c) = (2a, 0)$,

$s: V \to W; s(a,b,c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m,n,q \in V$ is a strong AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3,-1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3,-1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m,n,q \in V$ is a strong AHS-linear transformation.

Definition 3.17:

Let $V_n(I), W_n(I)$ be two weak/strong n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak/strong AH-linear transformation. We define:

(a) $AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$.

(b) $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$.

Theorem 3.18:

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

(a) $AH - Ker(f)$ is a weak AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a weak AH-subspace of $W_n(I)$.

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(c) If $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ is a weak AH-subspace of $V_n(I)$, $f(M_n(I))$ is a weak AH-subspace of $W_n(I)$.

Proof:

(a) Since $Ker(f_i)$ is a subspace of V, we find that

$AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$ is a weak AH-subspace of $V_n(I)$.

(b) Since $Im(f_i)$ is a subspace of W, we find that $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$ is a weak AH-subspace of $W_n(I)$.

(c) It is known that $f_i(M_i)$ is a subspace of W, hence

$f\big(M_n(I)\big) = f_0(M_0) + f_1(M_1)I_1 + \cdots + f_n(M_n)I_n$ is a weak AH-subspace of $W_n(I)$.

Theorem 3.19:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

(a) $AH - Ker(f)$ is a strong AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a strong AH-subspace of $W_n(I)$.

(c) If $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ is a strong AH-subspace of $V_n(I)$, $f(M_n(I))$ is a strong AH-subspace of $W_n(I)$.

Proof:

The proof is similar to that of Theorem 3.18 .

Example 3.20:

Let $V_2(I), W_2(I)$ be the two weak 2-refined neutrosophic vector spaces defined in Example 3.16 .

(a) $M =< (1,0,0) >, N =< (0,1,0) >, L =< (0,0,1) >$ are three subspaces of V,

$T = M + NI_1 + LI_2 = \{(a,0,0) + (0,b,0)I_1 + (0,0,c)I_2; a,b,c \in R\}$ is a weak AH-subspace of $V_2(I)$.

Consider $f: V_2(I) \to W_2(I)$ the weak AH-linear transformation defined in Example 3.16 .

(b) $AH - Ker(f) = Ker(g) + Ker(h)I_1 + Ker(s)I_2 = \{(a,0,0) + (0,b,c)I_1 + (d,0,0)I_2; a,b,c,d \in R$.

(c) $AH - Im(f) = Im(g) + Im(h)I_1 + Im(s)I_2 = R^2 + < (1,0) > I_1 + R^2I_2$.

(d) $f(T) = g(M) + h(N)I_1 + s(L)I_2 =< (0,0) > + < (0,0) > I_1 + < (0,1)I_2 =$

$\{(0,0) + (0,0).I_1 + (0,a)I_2; a \in R\}$, which is a weak AH-subspace of $W_2(I)$.

Theorem 3.21:

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Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces over the field K,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

$f(x+y) = f(x) + f(y), f(r.x) = r.f(x)$ for all $x, y \in V_n(I), r \in K$.

Proof:

Let $x = \sum_{i=0}^{n} a_i I_i\,, y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, $r \in K$ be any element in the field K, we have:

$f(x+y) = f(\sum_{i=0}^{n}(a_i + b_i)I_i) = \sum_{i=0}^{n} f_i(a_i + b_i)I_i = \sum_{i=0}^{n} f_i(a_i)I_i + \sum_{i=0}^{n} f_i(b_i)I_i = f(x) + f(y)$.

$f(r.x) = f(\sum_{i=0}^{n} r a_i I_i) = \sum_{i=0}^{n} f_i(r a_i)I_i = r.\sum_{i=0}^{n} f_i(a_i)I_i = r.f(x)$.

Theorem 3.22:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

$f(x+y) = f(x) + f(y), f(r.x) = r.f(x)$ for all $x, y \in V_n(I), r \in K_n(I)$.

Proof:

Let $x = \sum_{i=0}^{n} a_i I_i\,, y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, $r = \sum_{i=0}^{n} r_i I_i \in K_n(I)$ be any element in the n-refined neutrosophic field $K_n(I)$, we have:

$f(x+y) = f(\sum_{i=0}^{n}(a_i + b_i)I_i) = \sum_{i=0}^{n} f_i(a_i + b_i)I_i = \sum_{i=0}^{n} f_i(a_i)I_i + \sum_{i=0}^{n} f_i(b_i)I_i = f(x) + f(y)$.

For the proof of the second proposition we use induction on n. If n=0, the theorem is true clearly.

Suppose that it is true for n-1, we must prove it for n.

$f(r.x) = f\left(\sum_{i,j=0}^{n} r_i a_j I_i I_j\right) = f(\sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n)$, we can write

$\sum_{i,j=0}^{n-1} r_i a_j I_i I_j = m_0 + m_1 I_1 + \cdots + m_{n-1}I_{n-1}$,

$(\sum_{i=0}^{n} r_i I_i)a_n I_n = r_1 a_n I_1 + r_2 a_n I_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$r.x = \sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n = \mathrm{m}_0 + (\mathrm{m}_1 + \mathrm{r}_1\mathrm{a}_\mathrm{n})\mathrm{I}_1 + (\mathrm{m}_2 + \mathrm{r}_2\mathrm{a}_\mathrm{n})\mathrm{I}_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$f(r.x) = f_0(m_0) + f_1(m_1 + r_1 a_n)I_1 + f_2(m_2 + r_2 a_n)I_2 + \cdots f_n(r_0 a_n + r_n a_n)I_n =$

$f_0(m_0) + [f_1(m_1) + r_1 f_1(a_n)]I_1 + \cdots + [r_0 f_n(a_n) + r_n f_n(a_n)]I_n = r.f(x)$.

Theorem 3.23:

Let $V_n(I), W_n(I), U_n(I)$ be three weak n-refined neutrosophic vector spaces over the field K,

$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

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be two weak AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a weak AH-linear transformation between $V_n(I), U_n(I)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i \in V_n(I)$, $fog(x) = f(\sum_{i=0}^{n} g_i(a_i)I_i) = f(g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n)$=

$f_0(g_0(a_0)) + f_1(g_1(a_1))I_1 + \cdots + f_n(g_n(a_n))I_n = \sum_{i=0}^{n}(f_i o g_i)(a_i)I_i$.

(b) Since $f_i o g_i$ is a linear transformation for all i, then we get the proof.

Theorem 3.24:

Let $V_n(I), W_n(I), U_n(I)$ be three strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field K,

$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

be two strong AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a strong AH-linear transformation between $V_n(I), U_n(I)$.

Proof:

The proof is similar to that of Theorem 3.23.

Example 3.25:

(a) Let $V = R^3$ be a vector spaces over the field R, $V_2(I) = R_2^3(I) = \{(x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\ x_i, y_i, z_i \in R\}$,

be the corresponding weak 2-refined neutrosophic vector space. We have $g: V \to V; g(a,b,c) = (2b, 2c, 0)$,

$h: V \to V; h(a,b,c) = (2a, c, c)$,

$s: V \to V; s(a,b,c) = (2b, 3c, a)$ are three linear transformations.

(b) $f: V_2(I) \to V_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$ is a weak AH-linear transformation, $j: V_2(I) \to V_2(I); j(m + nI_1 + qI_2) = g(m) + g(n)I_1 + h(q)I_2; m, n, q \in V$ is a weak AH-linear transformation.

(c) $foj(m + nI_1 + qI_2) = gog(m) + hog(n)I_1 + soh(q)I_2$.

(d) Put $m = (2,1,0), n = (-1,0,0), q = (3,2,2)$, we compute foj as follows:

$foj(m + nI_1 + qI_2) = gog[(2,1,0)] + hog[(-1,0,0)]I_1 + soh[(3,2,2)]I_2$=

$g(2,0,0) + h(0,0,0)I_1 + s(6,2,2)I_2 = (0,0,0) + (0,0,0)I_1 + (4,6,6)I_2$.

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5. Conclusion

In this paper we have defined and studied weak/strong AH-subspaces and weak/strong AH-linear transformations in n-refined neutrosophic vector spaces. Also, we have presented some elementary properties and theorems about these concepts.

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