On Refined Neutrosophic Algebraic Hyperstructures I

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Abstract

Given any algebraic hyperstructure $(X, *, \circ)$, the objective of this paper is to generate a refined neutrosophic algebraic hyperstructure $(X(I_1, I_2), *', \circ')$ from X, I_1 and I_2 and study refined neutrosophic Krasner hyperrings in particular.
Keywords: refined neutrosophic group, refined neutrosophic ring, refined neutrosophic hyperring.

1 Introduction

The concept of refined neutrosophic algebraic structures was introduced by Agboola in.[1] Adeleke et.al in[24,25] introduced and studied refined neutrosophic rings and ideals. In the present paper, we introduce the concept of refined neutrosophic algebraic hyperstructure and we study refined neutrosophic Krasner hyperrings in particular. In this section, we present introduction and some necessary definitions for completeness.

Definition 1.1 ([22])**.** A hypergroup $(X, \circ)$ is called a canonical hypergroup if the following conditions are satisfied:

(i) $(X, \circ)$ is commutative.

(ii) $(X, \circ)$ has a scalar identity that is $\forall x \in X$, there exists $e \in X$ such that

$$x \circ e = e \circ x = x.$$

(iii) Every element of $(X, \circ)$ has a unique inverse that is $\forall x \in X$, there exists a unique $x^{-1} \in X$ such that

$$e \in x \circ x^{-1} \cap x^{-1} \circ x.$$

(iv) $(X, \circ)$ is reversible that is if $x \in y \circ z$, then there exist $y^{-1}, z^{-1} \in X$ such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

Definition 1.2 ([22])**.** An algebraic hyperstructure $(X, +, .)$ where $+$ is a hyperoperation and . is the usual multiplication operation is called a Krasner hyperring if the following conditions are satisfied:

(i) $(X, +)$ is a canonical hypergroup with identity 0.

(ii) $(X, .)$ is a semigroup with 0 as a bilateral absorbing element that is

$$x.0 = 0.x = 0 \quad \forall x \in X.$$

(iii) . is distributive over $+$.

A Krasner hyperring $(X, +, .)$ is said to be commutative with unit element if $(X, .)$ is a commutative semigroup with unit element.

Definition 1.3 ([22])**.** (i) Let $(X, +, .)$ be a Krasner hyperring and let A be a nonempty subset of X. A is called a subhyperring of X if $(A, +, .)$ is a hyperring in its own right.

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(ii) A is called a normal subhyperring of X if and only if $x + A - x \subseteq A$ for all $x \in X$.

(iii) A is called a hyperideal of X if for all $x \in X$, $a \in A$, we have $a.x, x.a \in A$.

Definition 1.4 ([22]). Let $(X_1, +, .)$ and $(X_2, +', .')$ be two Krasner hyperrings. The mapping $\phi : X_1 \rightarrow X_2$ is called a good(strong) homomorphism if $\forall x, y \in X_1$, we have the following:

(i) $\phi(x + y) = \phi(x) +' \phi(y)$ and

(ii) $\phi(x.y) = \phi(x).'\phi(y)$.

The kernel of ϕ denoted by $Ker\phi = \{x \in X_1 : \phi(x) = 0\}$.

For full details about algebraic hyperstructures and applications, the readers should see [[9–12,15–21,23,30–38,40,41]].

The concept of neutrosophic logic was introduced by Florentin Smarandache in 1995. Neutrosophic logic is an extension and generalization of fuzzy logic and intuitionistic fuzzy logic of Lofti Zadeh and Atanassov respectively. In neutrosophic logic, each proposition is approximated to have the percentage of truth in the subset T, percentage of indeterminacy in the subset I and the percentage of falsity in the subset F where $T, I, F \subseteq]^-0, 1^+[$. For simplicity and practical applications, T, I, F are taken as single-valued numbers from the interval $[0, 1]$ with $0 \leq T + I + F \leq 3$. In neutrosophy, if $< A >$ is an item or a concept, then only the triads $(< A >, < neutA >, < antiA >)$ that makes sense in the real world. The neutrosophication of a concept into a (T, I, F)-concept, or more generally into a refined $(T_1, T_2, \cdots, I_1, I_2, \cdots, F_1, F_2, \cdots)$-concept is possible if the triad $(< concept >, < neutconcept >, < anticoncept >$, or more generally its corresponding refined concept triad, makes sense in the real world. If the indeterminacy factor I is refined into two indeterminacies I_1 and I_2 in the sense of Smarandache in,[27] then we can take I_1 = contradiction (true (T) and false (F)), I_2 = ignorance (true (T) or false (F)) so that logically we have

$$I_1^2 = I_1, I_2^2 = I_2, I_1 I_2 = I_2 I_1 = I_1.$$

For full details about neutrosophy, neutrosphic logic and neutrosophic algebraic structures, the readers should see [[26–28,39]]

Definition 1.5 ([24]). Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2. $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 1.6 ([24]). Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $J(I_1, I_2)$ be a nonempty subset of $R(I_1, I_2)$. $J(I_1, I_2)$ is called a refined neutrosophic subring of $R(I_1, I_2)$ if $(J(I_1, I_2), +, .)$ is itself a refined neutrosophic ring. It is essential that $J(I_1, I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1, I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1, I_2)$.

Definition 1.7 ([25]). Let $R(I_1, I_2)$ be a refined neutrosophic ring and let J be a nonempty subset of $R(I_1, I_2)$. J is called a refined neutrosophic ideal of $R(I_1, I_2)$ if the following conditions hold:

(i) J is a refined neutrosophic subring of $R(I_1, I_2)$.

(ii) For every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr, rx \in J$.

If J is a pseudo refined neutrosophic subring of $R(I_1, I_2)$ and for every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr \in J$, then J is called a pseudo refined neutrosophic ideal of $R(I_1, I_2)$.

Definition 1.8 ([25]). Let J be a refined neutrosophic ideal of the refined neutrosophic ring $R(I_1, I_2)$. The set $R(I_1, I_2)/J$ is defined by

$$R(I_1, I_2)/J = \{r + J : r \in R(I_1, I_2)\}.$$

If $\bar{x} = r_1 + J$ and $\bar{y} = r_2 + J$ are two arbitrary elements of $R(I_1, I_2)/J$ and $\oplus, \odot$ are two binary operations on $R(I_1, I_2)/J$ defined by

$$\begin{aligned} \bar{x} \oplus \bar{y} &= (x + y) + J, \\ \bar{x} \odot \bar{y} &= (xy) + J. \end{aligned}$$

It can be shown that $(R(I_1, I_2)/J, \oplus, \odot)$ is a refined neutrosophic ring with the additive identity J.
$(R(I_1, I_2)/J, \oplus, \odot)$ is called a refined quotient neutrosophic ring.

Definition 1.9 ([25]). Let $(R(I_1, I_2), +, .)$ and and $(S(I_1, I_2), +, .)$ be two refined neutrosophic rings. The mapping $\phi : (R(I_1, I_2), +, .) \rightarrow (S(I_1, I_2), +, .)$ is called a refined neutrosophic ring homomorphism if the following conditions hold:

(i) $\phi(x + y) = \phi(x) + \phi(y)$.

(ii) $\phi(x.y) = \phi(x).\phi(y)$.

(iii) $\phi(I_k) = I_k \;\; \forall x, y \in R(I_1, I_2)$ and $k = 1, 2$.

The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in S(I_1, I_2) : y = \phi(x) \text{ for some } x \in R(I_1, I_2)\}.$$

The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in R(I_1, I_2) : \phi(x) = (0, 0, 0)\}.$$

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of ϕ are similarly defined as in the classical cases. For full details about neutrosophic and refined neutrosophic algebraic structures and algebraic hyperstructures, the readers should see [1–8, 13, 14, 24, 25, 29]

2 Development of Refined Neutrosophic Algebraic Hyperstructures

In this section, we present the development of refined neutrosophic algebraic hyperstructures. Refined neutrosophic Krasner hyperrings are studied in particular and some of their basic properties are presented.

Suppose that $(X, *, \circ)$ is any given algebraic hyperstructure with $*$ as a hyperoperation on X and $\circ$ is an operation on X. Let $X(I_1, I_2) =< X, I_1, I_2 >$ be a refined neutrosophic set generated by X, I_1, I_2 defined by

$$X(I_1, I_2) = \{(x, yI_1, zI_2) : x, y, z \in X\}.$$

Suppose that

$$*' : X(I_1, I_2) \times X(I_1, I_2) \rightarrow 2^{X(I_1,I_2)} \quad \text{and} \quad \circ' : X(I_1, I_2) \times X(I_1, I_2) \rightarrow X(I_1, I_2)$$

are respectively a hyperoperation and an operation on $X(I_1, I_2)$ and such that $\forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ with $a, b, c, d, e, f \in X$, we define

$$\begin{aligned}
(a, bI_1, cI_2)*'(d, eI_1, fI_2) &= \{((x, yI_1, zI_2) : x \in a * d, y \in (a * e) \\
&\quad *(b * d) * (b * e) * (b * f) * (c * e), \\
&\quad z \in (a * f) * (c * d) * (c * f)\} \qquad (1) \\
(a, bI_1, cI_2)\circ'(d, eI_1, fI_2) &= ((x, yI_1, zI_2) \quad \text{such that} \\
&\quad x = a \circ d, y = (a \circ e) \circ (b \circ d) \\
&\quad \circ(b \circ e) \circ (b \circ f) \circ (c \circ e), \\
&\quad z = (a \circ f) \circ (c \circ d) \circ (c \circ f). \qquad (2)
\end{aligned}$$

With the composition of the elements of $X(I_1, I_2)$ according to equations (1) and (2), we call the algebraic structure $(X(I_1, I_2), *', \circ')$ a refined neutrosophic algebraic hyperstructure. $(X(I_1, I_2), *', \circ')$ is named according to the algebraic laws satisfied by $*'$ and $\circ'$.

If $(X(I_1, I_2), *', \circ')$ and $(Y(I_1, I_2), *'', \circ'')$ are two given refined neutrosophic algebraic hyperstructures, then the mapping $\phi : (X(I_1, I_2), *', \circ') \rightarrow (Y(I_1, I_2), *'', \circ'')$ is called a refined neutrosophic good(strong) homomorphism if
$\forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, the following conditions hold:

(i) $\phi((a, bI_1, cI_2)*'(d, eI_1, fI_2)) = \phi((a, bI_1, cI_2))*''\phi((d, eI_1, fI_2)))$,

(ii) $\phi((a, bI_1, cI_2)\circ'(d, eI_1, fI_2)) = \phi((a, bI_1, cI_2))\circ''\phi((d, eI_1, fI_2)))$,

(iii) $\phi(I_i) = I_i$ for $i = 1, 2$.

In all that follows in the sequel, all hyperrings $(X, +, .)$ will be Krasner hyperrings.

Definition 2.1. Let $(X, +, .)$ be any hyperring and let $X(I_1, I_2) =< X, I_1, I_2 >$ be a set generated by X, I_1 and I_2. The triple $(X(I_1, I_2), +, .)$ is called a refined neutrosophic hyperring. It is said to be commutative if for all $x, y \in X(I_1, I_2)$, we have $x.y = y.x$.

For all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define the composition of elements with respect to the hyperoperation $+$ and ordinary operation . in $X(I_1, I_2)$ as follows.

$$\begin{aligned} (a, bI_1, cI_2) + (d, eI_1, fI_2) &= \{(x, yI_1, zI_2) : x \in a + d, y \in b + e, \\ &\quad z \in c + f\}. \qquad (3) \\ (a, bI_1, cI_2).(d, eI_1, fI_2) &= (x, yI_1, zI_2) \text{ such that } x = ad, \\ &\quad y \in (ae + bd + be + bf + ce), \\ &\quad z \in (af + cd + cf). \qquad (4) \end{aligned}$$

Definition 2.2. (i) Let $(X(I_1, I_2), +, .)$ be a refined neutrosophic hyperring and let $A(I_1, I_2)$ be a nonempty subset of $X(I_1, I_2)$. Then $A(I_1, I_2)$ is called a refined neutrosophic subhyperring of $X(I_1, I_2)$ if $(A(I_1, I_2), +, .)$ is a refined neutrosophic hyperring in its own right. $-A(I_1, I_2)$ is defined as

$$-A(I_1, I_2) = \{-a : a \in A(I_1, I_2)\}.$$

(iii) $A(I_1, I_2)$ is called a refined neutrosophic hyperideal of $X(I_1, I_2)$ if for all $x \in X(I_1, I_2), a \in A(I_1, I_2)$, we have $a.x, x.a \in A(I_1, I_2)$.

Definition 2.3. Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be two refined neutrosophic hyperideals of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$.

(i) The sum $A(I_1, I_2) + B(I_1, I_2)$ is defined by

$$A(I_1, I_2) + B(I_1, I_2) = \{x : x \in a + b \text{ for some } a \in A(I_1, I_2), b \in B(I_1, I_2)\}.$$

(ii) The product $A(I_1, I_2)B(I_1, I_2)$ is defined by

$$A(I_1, I_2)B(I_1, I_2) = \left\{ x : x \in \sum_{i}^{n} a_i b_i, a_i \in A(I_1, I_2), b_i \in B(I_1, I_2)\}, n \in \mathbb{Z}^+ \right\}.$$

Definition 2.4. Let $A_1(I_1, I_2), A_2(I_1, I_2), \cdots, A_n(I_1, I_2)$ be refined neutrosophic hyperideals of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$.

$$\mathbb{A}(I_1, I_2) = A_1(I_1, I_2) \oplus A_2(I_1, I_2) \oplus \cdots \oplus A_n(I_1, I_2) = \oplus \sum_{i=1}^{n} A_i(I_1, I_2)$$

is called a direct sum if for each element x in $\mathbb{A}(I_1, I_2)$, we have

$$x \in \sum_{i}^{n} a_i \text{ (uniquely), with } a_i \in A_i(I_1, I_2).$$

Definition 2.5. Let $(X_1(I_1, I_2), +, .)$ and $X_2(I_1, I_2), +', .')$ be two refined neutrosophic hyperrings. The mapping $\phi : X_1(I_1, I_2) \rightarrow X_2(I_1, I_2)$ is called a refined neutrosophic good(strong) homomorphism if $\forall x, y \in X_1(I_1, I_2)$, we have the following:

(i) $\phi(x + y) = \phi(x) +' \phi(y)$ and

(ii) $\phi(x.y) = \phi(x).'\phi(y)$.

(iii) $\phi(I_i) = I_i$ for all $i = 1, 2$.

The kernel of ϕ denoted by $Ker\phi = \{x \in X_1(I_1, I_2) : \phi(x) = 0\}$. The image of ϕ denoted by $Im\phi = \{y \in X_2(I_1, I_2) : y = \phi(x) \text{ for some } x \in X_1(I_1, I_2)\}$.

Example 2.6. Let $(\mathbb{R}(I_1, I_2), +, .)$ be a refined neutrosophic ring of real numbers and let $J(I_1, I_2)$ be a refined neutrosophic ideal of $\mathbb{R}(I_1, I_2)$. Let $\mathbb{R}(I_1, I_2)/J(I_1, I_2)$ be a refined neutrosophic quotient ring defined by

$$\mathbb{R}(I_1, I_2)/J(I_1, I_2) = \{x + J(I_1, I_2) : x \in \mathbb{R}(I_1, I_2)\}.$$

Let $\oplus$ and $\odot$ be additive hyperoperation and ordinary multiplicative operation respectively defined on $\mathbb{R}(I_1, I_2)/J(I_1, I_2)$ as follows:

$$\begin{aligned}(x + J(I_1, I_2)) \oplus (y + J(I_1, I_2)) &= \{z + J(I_1, I_2) : z \in (x + y)J(I_1, I_2)\},\\ (x + J(I_1, I_2)) \odot (y + J(I_1, I_2)) &= xy + J(I_1, I_2).\end{aligned}$$

Then $(\mathbb{R}(I_1, I_2)/J(I_1, I_2), \oplus, \odot)$ is a refined neutrosophic hyperring.

Theorem 2.7. *Any refined neutrosophic hyperring is a hyperring.*

Proof. Suppose that $(R(I_1, I_2), +, .)$ is a refined neutrosophic hyperring. It is clear that:

(1) $(R(I_1, I_2), +)$ is a canonical hypergroup.

(2) $(R(I_1, I_2), .)$ is a semigroup and $0 = (0, 0I_1, 0I_2)$ is a bilateral absorbing element.

(3) We establish the distributive laws. Let $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2), z = (g, hI_1, iI_2)$ be arbitrary elements in $R(I_1, I_2)$ with $a, b, c, d, e, f, g, h, i \in R$.

$$\begin{aligned}x(y + z) &= \{(p, qI_1, rI_2) : p \in a(d + g), q \in (a(e + h) + b(d + g) + b(e + h)\\ &\quad +b(f + i) + c(e + h)), r \in (a(f + i) + c(d + g) + c(f + i))\}, \text{and}\\ xy + xz &= \{(u, vI_1, wI_2) : u \in a(d + g), v \in (a(e + h) + b(d + g) + b(e + h)\\ &\quad +b(f + i) + c(e + h)), w \in (a(f + i) + c(d + g) + c(f + i))\}.\end{aligned}$$

This shows that $x(y+z) = xy+xz$. Similarly, it can be shown that $(y+z)x = yx+zx$ and $\therefore .$ is distributive over $+$. The proof is complete. □

Lemma 2.8. *Let $(X(I_1, I_2), +, .)$ be a refined neutrosophic hyperring. Then for all $x, y, z, w \in X(I_1, I_2)$, we have*

(i) $-(-x) = x$.

(ii) $-(x + y) = -x - y$.

(iii) $(x + y)(z + w) = xz + xw + yz + yw$.

Lemma 2.9. *Let $A(I_1, I_2)$ be a nonempty subset of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$. $A(I_1, I_2)$ is a refined neutrosophic hyperideal of $X(I_1, I_2)$ if and only if:*

(i) $x, y \in A(I_1, I_2)$ implies that $x - y \subseteq A(I_1, I_2)$,and

(ii) $x \in A(I_1, I_2)$ and $r \in X(I_1, I_2)$ imply that $xr, rx \in A(I_1, I_2)$.

Theorem 2.10. *Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be refined neutrosophic hyperideals of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$ and let $\{A(I_1, I_2)_i\}_{i=1}^n$ be a family of refined neutrosophic hyperideals of $(X(I_1, I_2), +, .)$. Then:*

(i) $\bigcap_{i=1}^n A(I_1, I_2)_i$ is a refined neutrosophic hyperideal of $X(I_1, I_2)$.

(ii) $\oplus \sum_{i=1}^n A_i(I_1, I_2)$ is a refined neutrosophic hyperideal of $X(I_1, I_2)$.

(iii) $A(I_1, I_2) + B(I_1, I_2)$ is a refined neutrosophic hyperideal of $X(I_1, I_2)$.

(iv) $A(I_1, I_2) + B(I_1, I_2)$ is a refined neutrosophic hyperideal of $X(I_1, I_2)$.

Lemma 2.11. *Let $A(I_1, I_2)$ be a refined neutrosophic hyperideal of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$. Then*

(i) $A(I_1, I_2) + A(I_1, I_2) = A(I_1, I_2)$.

(ii) $x + A(I_1, I_2) = A(I_1, I_2)$ for all $x \in A(I_1, I_2)$.

Proof. (i) Let $y = (p, qI_1, rI_2)$. Then

$$\begin{aligned} y &\in A(I_1, I_2) + A(I_1, I_2) \\ &\Leftrightarrow y \in u + u \quad \text{for some} \quad u = (a, bI_1, cI_2) \in A(I_1, I_2) \\ &\Leftrightarrow y \in A(I_1, I_2) \quad \text{since} \quad u + u \subseteq A(I_1, I_2) \\ &\therefore \quad A(I_1, I_2) + A(I_1, I_2) = A(I_1, I_2). \end{aligned}$$

(ii) Let $y = (p, qI_1, rI_2)$. Then

$$\begin{aligned} y &\in x + A(I_1, I_2) \\ &\Leftrightarrow y \in x + u \quad \text{for some} \quad u = (a, bI_1, cI_2) \in A(I_1, I_2) \\ &\Leftrightarrow y \in A(I_1, I_2) \quad \text{since} \quad x + u \subseteq A(I_1, I_2) \end{aligned}$$

$\therefore\ x + A(I_1, I_2) = A(I_1, I_2).$ □

Theorem 2.12. *Let $A(I_1, I_2)$ be a refined neutrosophic hyperideal of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$. Then*

(i) $(A(I_1, I_2) + x) + (A(I_1, I_2) + y) = A(I_1, I_2) + x + y$ for all $x, y \in X(I_1, I_2)$.

(ii) $A(I_1, I_2) + x = A(I_1, I_2) + y$ for all $y \in A(I_1, I_2) + x$.

(iii) $A(I_1, I_2) + x + y = A(I_1, I_2) + z$ if $z \in x + y$ for all $x, y, z \in X(I_1, I_2)$.

Proof. (i) Let $v = (p, qI_1, rI_2)$. Then

$$\begin{aligned} v &\in (A(I_1, I_2) + x) + (A(I_1, I_2) + y) \\ &\Leftrightarrow v \in (u + u) + x + y \quad \text{for some} \quad u = (a, bI_1, cI_2) \in A(I_1, I_2) \\ &\Leftrightarrow v \in A(I_1, I_2) + x + y \quad \text{since} \quad u + u \subseteq A(I_1, I_2) \\ &\therefore \quad (A(I_1, I_2) + x) + (A(I_1, I_2) + y) = A(I_1, I_2) + x + y. \end{aligned}$$

(ii) Suppose that $y \in A(I_1, I_2) + x$. Let $v = (p, qI_1, rI_2)$. Then

$$\begin{aligned} v &\in A(I_1, I_2) + x \\ &\Leftrightarrow v \in A(I_1, I_2) + A(I_1, I_2) + x \\ &\Leftrightarrow v \in A(I_1, I_2) + y \\ &\therefore \quad A(I_1, I_2) + x = A(I_1, I_2) + y. \end{aligned}$$

(iii) Suppose that $z \in x + y$ for all $x, y, z \in X(I_1, I_2)$. Let $v = (p, qI_1, rI_2)$. Then

$$\begin{aligned} v &\in A(I_1, I_2) + x + y \\ &\Leftrightarrow v \in u + (x + y) \quad \text{for some} \quad u = (a, bI_1, cI_2) \in A(I_1, I_2) \\ &\Leftrightarrow v \in u + z \\ &\Leftrightarrow v \in A(I_1, I_2) + z. \end{aligned}$$

$\therefore A(I_1, I_2) + x + y = A(I_1, I_2) + z.$ □

Theorem 2.13. *Let $A(I_1, I_2)$ be a refined neutrosophic hyperideal of a refined neutrosophic hyperring $(X(I_1, I_2), +, .)$. Then $A(I_1, I_2)$ is normal in $X(I_1, I_2)$ only if A is normal in X.*

Proof. Suppose that A is normal in X. Let $u = (x, yI_1, zI_2) \in X(I_1, I_2)$. Then there exists $v = (a, bI_1, cI_2) \in A(I_1, I_2)$ such that

$$\begin{aligned} u + A(I_1, I_2) - u &= u + v - u \\ &= (x, yI_1, zI_2) + (a, bI_1, cI_2) - (x, yI_1, zI_2) \\ &= \{(p, qI_1, rI_2) : p \in x + a - x, q \in y + b - y, \\ &\quad r \in z + c - z\} \\ &= \{(p, qI_1, rI_2) : p \in x + A - x \subseteq A, \\ &\quad q \in y + A - y \subseteq A, r \in z + A - z \subseteq A\} \\ &= \{(p, qI_1, rI_2) : p \in A, q \in A, r \in A\} \\ &\subseteq A(I_1, I_2) \end{aligned}$$

which shows that $A(I_1, I_2)$ is normal in $X(I_1, I_2)$. □

Example 2.14. Let $X_1(I_1, I_2)$ and $X_2(I_1, I_2)$ be two refined neutrosophic hyperrings and let $X(I_1, I_2) = X_1(I_1, I_2) \oplus X_2(I_1, I_2)$ be their direct sum. Let $\phi_i : X(I_1, I_2) \rightarrow X_i(I_1, I_2)$ with $i = 1, 2$ be the projections defined by

$$\phi_1(x, y) = x, \phi_2(x, y) = y \quad \forall\, (x, y) \in X(I_1, I_2).$$

Then ϕ_i are refined neutrosophic good(strong) homomorphisms.

Theorem 2.15. *Let $\phi : X_1(I_1, I_2) \rightarrow X_2(I_1, I_2)$ be a refined neutrosophic good(strong) hyperring homomorphism from $(X_1(I_1, I_2), +, .)$ into $X_2(I_1, I_2), +', .')$. Then*

(i) $\phi((0, 0I_1, 0I_2)) = (0, 0I_1, 0I_2)$.

(ii) $\phi(-x) = -\phi(x)$ *for all* $x \in X_1(I_1, I_2)$.

(iii) $Ker\phi$ *is not a refined neutrosophic hyperideal of* $X_1(I_1, I_2)$.

Proof. (i) and (ii) are clear.

(iii) It should be noted that only the element $(0, 0I_1, 0I_2) \in X_1(I_1, I_2)$ is the only element in $Ker\phi$. Since $\phi(I_i) = I_i$ for all $i = 1, 2$, it follows that no other element of $X_1(I_1, I_2)$ can be in the kernel of ϕ . Hence, $Ker\phi$ is not closed under $-$ and consequently, it cannot be a refined neutrosophic hyperideal of $X_1(I_1, I_2)$.

However, it can be shown that $Ker\phi$ is a subhyperring of $X_1(I_1, I_2)$ but not a refined neutrosophic subhyperring of $X_1(I_1, I_2)$. □

3 Conclusion

We have in this paper introduced the new concept of refined neutrosophic algebraic hyperstructures. We have studied refined neutrosophic Krasner hyperrings in particular. Their refined neutrosophic hypersubstructures were studied and their basic properties were presented. It was established that every refined neutrosophic Krasner hyperring is a Krasner hyperring. It was also established that the kernel of a refined neutrosophic Krasner hyperring homomorphism cannot be be a refined neutrosophic hyperideal but it can be a subhyperring. In our next paper to be titled Refined Neutrosophic Algebraic Hyperstructures II, we will study refined neutrosophic multiplicative hyperrings, their hypersubstructures and their properties.

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