Applications of Neutrosophic $\mathcal{N}$-Structures in Ternary Semirings: A Study on Neutrosophic Ternary $\mathcal{N}$-Subsemirings

Thammarat Panityakul[1] **, Ronnason Chinram**[1,*]
[1]Division of Computational Science, Faculty of Science, Prince of Songkla University, Songkhla 90110, Thailand

Emails: thammarat.p@psu.ac.th; ronnason.c@psu.ac.th

Abstract

In this paper, we apply neutrosophic $\mathcal{N}$-structures in ternary semirings. We consider ternary neutrosophic $\mathcal{N}$-subsemirings of ternary semirings. We investigate the conditions for neutrosophic $\mathcal{N}$-structures to be neutrosophic ternary $\mathcal{N}$-subsemirings. In addition, we show the relation between ternary subsemirins and neutrosophic ternary $\mathcal{N}$-subsemirins. Finally, we showed that the homomorphic preimage of the neutrosophic ternary $\mathcal{N}$-subsemirings is a neutrosophic ternary $\mathcal{N}$-subsemirings and the onto homomorphic image of the neutrosophic ternary $\mathcal{N}$-subsemiring is also a neutrosophic ternary $\mathcal{N}$-subsemirings.

Keywords: Neutrosophic $\mathcal{N}$-structures; Ternary semirings; Neutrosophic ternary $\mathcal{N}$-subsemirings; Homomorphism; Homomorphic image

1 Introduction

In 1965, Zadeh[14] introduced the concept of the degree of membership/truth (T) and the notion of the fuzzy set as an extension of the classical notion of sets. Fuzzy sets permitted the gradual assessment of the membership of elements and described with the membership function valued of for each element in the unit closed interval [0, 1]. Later, Atanassov[1] introduced the degree of nonmembership/falsehood (F) and defined the intuitionistic fuzzy set by using the degree of membership/truth (T) and the degree of nonmembership/falsehood (F) in 1986. Neutrosophy, means knowledge of neutral, is a branch of philosophy introduced as a theory of generalization of dialectic in 1995 by Smarandache. Smarandache proposed the term neutrosophic because neutrosophic originally comes from neutrosophy. Smarandache[12] introduced the concept of neutrosophic logics and introduced the degree of indeterminancy/neuterality (I). Moreover, Smarandache proposed the neutrosophic set on three components: (T, I, F) = (truth, indeterminacy, falsehood). Neutrosophic set theory is very important in many application areas since indeterminacy is quantified explicitly and the truth membership function, indeterminacy membership function and falsity membership functions are independent.

In 2009, Jun et al.[3] first introduced a negative-valued function and defined $\mathcal{N}$-structures. Later, Khan et al.[5] investigated the notion of neutrosophic $\mathcal{N}$-structures and their applications in semigroups in 2017. Next, Jun et al.[3,4] considered neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras. Neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras were proposed by Song et al.[13] Rangsuk et al.[11] applied neutrosophic $\mathcal{N}$-structures in UP-algebras in 2017.

The literature of ternary algebraic system was introduced by Lehmer[6] in 1932. Lehmer[6] investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. Later, in compatibility of Lister's generalization of ternary rings[7] were first introduced. Dutta and Kar[2] concocted the thought of ternary semirings. Later, some algebraic properties of this algebraic structure were studied, for examples: In 2020, Murugadas[9] introduced the notion of strong prime ideals in ternary semiring and an m-system corresponding to strong prime ideals in ternary semiring and investigated some results in completely prime ideals in ternary semiring. In 2022, the notions of hybrid ideals and k-hybrid ideals in a ternary semiring were introduced in Muhiuddin et al.[8] In 2024, Pandiselvi, and Anbalagan[10] investigated the characterization of g-inverses in ternary semirings.

The purpose of this paper is to investigate the extension of the results of neutrosophic $\mathcal{N}$-structures to ternary semirings. Some basic notations and definitions will be presented in section 2. In section 3, we will investigate the results of neutrosophic $\mathcal{N}$-structures in ternary semirings. Section 4 contains a brief summary of this paper.

2 Preliminaries

The aim of this section is to review some notations and definitions of ternary semirings and neutrosophic $\mathcal{N}$-structures.

2.1 Ternary semirings

In this subsection, we recall some notions of ternary semirings in the same fashion as found in Dutta and Kar.[2]

Definition 2.1. [2] A non-empty set S together with a binary operation, called the addition, and the ternary multiplication operation from $S \times S \times S \to S$, written as $(a, b, c) \mapsto [abc]$, is said to be a *ternary semiring* if S is an additive commutative semigroup satisfying the following conditions:

(i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,

(ii) $[(a + b)cd] = [acd] + [bcd]$,

(iii) $[a(b + c)d] = [abd] + [acd]$,

(iv) $[ab(c + d)] = [abc] + [abd]$

for all $a, b, c, d, e \in S$.

Definition 2.2. [2] A nonempty subset I of a ternary semiring S is called a *ternary subsemiring* of S if $a+b \in I$ and $[abc] \in I$ for all $a, b, c \in I$.

Definition 2.3. [2] Let S_1 and S_2 be two ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a *homomorphism* if

$$f(a + b) = f(a) + f(b) \text{ and } f([abc]) = [f(a)f(b)f(c)]$$

for all $a, b, c \in S_1$.

2.2 Neutrosophic $\mathcal{N}$-structures

In this subsection, we recall some notion of Neutrosophic $\mathcal{N}$-structures. For the sake of completeness, we state some definitions in the same fashion as found in Khan et al.[5] which are used throughout this paper.

Let $\mathcal{F}(S, [-1, 0])$ denote the collection of functions from a nonempty set S to a closed interval $[-1, 0]$. A function in $\mathcal{F}(S, [-1, 0])$ is called a negative-valued function (briefly, an $\mathcal{N}$-function) from S to $[-1, 0]$. By an $\mathcal{N}$-structure, we mean an ordered pair (S, f) where $f \in \mathcal{F}(S, [-1, 0])$.

Definition 2.4. [5] A *neutrosophic $\mathcal{N}$-structure* over a nonempty set S is defined to be the structure

$$S_N := \frac{S}{(T_N, I_N, F_N)} = \left\{ \frac{a}{(T_N(a), I_N(a), F_N(a))} \mid a \in S \right\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on S which are called the truth membership function, the indeterminacy membership function and the falsity membership function on S, respectively.

Let $\{x_i \mid i \in \Lambda\}$ be a family of real numbers. We define

$$\bigvee\{x_i \mid i \in \Lambda\} := \begin{cases} \max\{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{x_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{x_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{x_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

For any two real numbers a and b, we use $a \vee b$ and $a \wedge b$ instead of $\vee\{a, b\}$ and $\wedge\{a, b\}$, respectively.

Definition 2.5. [5] Let $S_N = \frac{S}{(T_N, I_N, F_N)}$ and $S_M = \frac{S}{(T_M, I_M, F_M)}$ be any two neutrosophic $\mathcal{N}$-structures over a nonempty set S.

(1) S_N is a *neutrosophic $\mathcal{N}$-substructure* of S_M over S, denoted by $S_N \subseteq S_M$, if it satisfies the conditions

$$T_N(a) \leq T_M(a), I_N(a) \geq I_M(a), F_N(a) \leq F_M(a)$$

for all $a \in S$

We note that $S_N = S_M$ if and only if $S_N \subseteq S_M$ and $S_M \subseteq S_N$.

(2) The *union* of S_N and S_M, denoted it briefly by $S_{N \cup M}$, is defined to be a neutrosophic $\mathcal{N}$-structure

$$S_{N \cup M} = \frac{S}{(T_{N \cup M}, I_{N \cup M}, F_{N \cup M})}$$

where

$$T_{N \cup M}(a) = \bigwedge \{T_N(a), T_M(a)\},$$
$$I_{N \cup M}(a) = \bigvee \{I_N(a), I_M(a)\},$$
$$F_{N \cup M}(a) = \bigwedge \{F_N(a), F_M(a)\}.$$

(3) The *intersection* of S_N and S_M, written it simply as $S_{N \cap M}$, is defined to be a neutrosophic $\mathcal{N}$-structure

$$S_{N \cap M} = \frac{S}{(T_{N \cap M}, I_{N \cap M}, F_{N \cap M})}$$

where

$$T_{N \cap M}(a) = \bigvee \{T_N(a), T_M(a)\},$$
$$I_{N \cap M}(a) = \bigwedge \{I_N(a), I_M(a)\},$$
$$F_{N \cap M}(a) = \bigvee \{F_N(a), F_M(a)\}.$$

Definition 2.6. [5] Let $S_N := \frac{S}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over a nonempty set S. The *complement* of S_N, denoted by S_{N^c}, is defined to be a neutrosophic $\mathcal{N}$-structure

$$S_{N^c} := \frac{S}{(T_{N^c}, I_{N^c}, F_{N^c})}$$

over S, where

$$T_{N^c}(a) = -1 - T_N(a),\ I_{N^c}(a) = -1 - I_N(a),\ F_{N^c}(a) = -1 - F_N(a),$$

for all $a \in S$.

Definition 2.7. [5] Let S_N be a neutrosophic $\mathcal{N}$-structure over a nonempty set S and three real numbers $\alpha, \beta, \gamma \in [-1, 0]$. Define the sets

$$T_N^\alpha = \{a \in S \mid T_N(a) \leq \alpha\},$$
$$I_N^\beta = \{a \in S \mid I_N(a) \geq \beta\},$$
$$F_N^\gamma = \{a \in S \mid F_N(a) \leq \gamma\}.$$

We call a set

$$S_N(\alpha, \beta, \gamma) = \{a \in S \mid T_N(a) \leq \alpha, I_N(a) \geq \beta, F_N(a) \leq \gamma\}$$

an (α, β, γ)-*level set* of S_N.

For the convenience, we note that $S_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

3 Main results

In this section, we discuss on neutrosophic $\mathcal{N}$-ternary subsemirings, the (α, β, γ)-level set, the intersection of neutrosophic $\mathcal{N}$-ternary subsemirings, neutrosophic $\mathcal{N}$-products, homomorphic preimage of the neutrosophic $\mathcal{N}$-ternary subsemiring and onto homomorphic image of the neutrosophic $\mathcal{N}$-ternary subsemiring. Throughout this section, we denote a ternary semiring S as the universe of discourse unless otherwise specified.

Definition 3.1. Let S_N be a neutrosophic $\mathcal{N}$-structure over a ternary semiring S. Then S_N is said to be a *neutrosophic $\mathcal{N}$-ternary subsemiring* of S if it satisfies the following conditions:

$$T_N(a+b) \leq \bigvee\{T_N(a), T_N(b)\},$$
$$I_N(a+b) \geq \bigwedge\{I_N(a), I_N(b)\},$$
$$F_N(a+b) \leq \bigvee\{F_N(a), F_N(b)\},$$
$$T_N([abc]) \leq \bigvee\{T_N(a), T_N(b), T_N(c)\},$$
$$I_N([abc]) \geq \bigwedge\{I_N(a), I_N(b), I_N(c)\},$$
$$F_N([abc]) \leq \bigvee\{F_N(a), F_N(b), F_N(c)\},$$

for all $a, b, c \in S$.

Theorem 3.2. *Let S_N be a neutrosophic $\mathcal{N}$-structure over a ternary semiring S and let α, β, γ be any three real numbers on the closed interval $[-1, 0]$. If S_N is a neutrosophic $\mathcal{N}$-ternary subsemiring of S, then the nonempty (α, β, γ)-level set of S_N is a ternary subsemiring of S.*

Proof. Let S_N be a neutrosophic $\mathcal{N}$-ternary subsemiring of a ternary semiring S and let $a, b, c \in S$. Thus

$$T_N(a+b) \leq \bigvee\{T_N(a), T_N(b)\},$$
$$I_N(a+b) \geq \bigwedge\{I_N(a), I_N(b)\},$$
$$F_N(a+b) \leq \bigvee\{F_N(a), F_N(b)\},$$
$$T_N([abc]) \leq \bigvee\{T_N(x), T_N(b), T_N(c)\},$$
$$I_N([abc]) \geq \bigwedge\{I_N(x), I_N(b), I_N(c)\},$$
$$F_N([abc]) \leq \bigvee\{F_N(x), F_N(b), F_N(c)\},$$

Assume that $S_N(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$. Let $a, b, c \in S_N(\alpha, \beta, \gamma)$. Thus

$$T_N(a) \leq \alpha, I_N(a) \geq \beta, F_N(a) \leq \gamma,$$
$$T_N(b) \leq \alpha, I_N(b) \geq \beta, F_N(b) \leq \gamma,$$
$$T_N(c) \leq \alpha, I_N(c) \geq \beta, F_N(c) \leq \gamma.$$

This implies that

$$
\begin{aligned}
T_N(a+b) &\leq \bigvee\{T_N(a), T_N(b)\} \leq \alpha,\\
I_N(a+b) &\geq \bigwedge\{I_N(a), I_N(b)\} \geq \beta,\\
F_N(a+b) &\leq \bigvee\{F_N(a), F_N(b)\} \leq \gamma,\\
T_N([abc]) &\leq \bigvee\{T_N(a), T_N(b), T_N(c)\} \leq \alpha,\\
I_N([abc]) &\geq \bigwedge\{I_N(a), I_N(b), I_N(c)\} \geq \beta,\\
F_N([abc]) &\leq \bigvee\{F_N(a), F_N(b), F_N(c)\} \leq \gamma.
\end{aligned}
$$

Hence, $a+b, [abc] \in S_N(\alpha, \beta, \gamma)$. Therefore, $S_N(\alpha, \beta, \gamma)$ is a ternary subsemiring of S. □

Theorem 3.3. *Let S_N be a neutrosophic $\mathcal{N}$-structure over a ternary semiring S. If $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} are ternary subsemirings of S for all $\alpha, \beta, \gamma \in [-1, 0]$, then S_N is a neutrosophic $\mathcal{N}$-ternary subsemiring of S.*

Proof. We will prove this theorem by contradiction. We first assume that there exist $a, b \in S$ such that $T_N(a+b) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(a+b) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$, but $a+b \notin T_N^{t_\alpha}$, which is a contradiction. Therefore

$$T_N(a+b) \leq \bigvee\{T_N(a), T_N(b)\}$$

for all $a, b \in S$.

Next, assume that there are $a, b \in S$ such that $I_N(a+b) < \bigwedge\{I_N(a), I_N(b)\}$. Then $a, b \in I_N^{t_\beta}$ and $a+b \notin I_N^{t_\beta}$ for $t_\beta := \bigwedge\{I_N(a), I_N(b)\}$. This is a contradiction. Hence

$$I_N(a+b) \geq \bigwedge\{I_N(a), I_N(b)\}$$

for all $a, b \in S$.

Assume that there exist $a, b \in S$ such that $F_N(a+b) > \bigvee\{F_N(a), F_N(b)\}$. Then $F_N(a+b) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}$ for some $t_\gamma \in [-1, 0)$. Hence $a, b \in F_N^{t_\gamma}$, but $a+b \notin F_N^{t_\gamma}$, which is a contradiction. Therefore

$$F_N(a+b) \leq \bigvee\{F_N(a), F_N(b)\}$$

for all $a, b \in S$.

Now, we suppose that there are $a, b \in S$ such that $T_N([abc]) > \bigvee\{T_N(a), T_N(b), T_N(c)\}$. Then $T_N([abc]) > t_\gamma \geq \bigvee\{T_N(a), T_N(b), F_N(c)\}$ for some $s_\alpha \in [-1, 0)$. This implies that $a, b, c \in T_N^{s_\alpha}$, but $[abc] \notin T_N^{s_\alpha}$, which is a contradiction. Therefore

$$T_N([abc]) \leq \bigvee\{T_N(a), T_N(b), T_N(c)\}$$

for all $a, b, c \in S$.

Now assume that there are $a, b, c \in S$ such that $I_N([abc]) < \bigwedge\{I_N(a), I_N(b), I_N(c)\}$. Then $a, b, c \in I_N^{s_\beta}$ and $[abc] \notin I_N^{s_\beta}$ for $t_\beta := \bigwedge\{I_N(a), I_N(b), I_N(c)\}$. This is a contradiction. Hence

$$I_N([abc]) \geq \bigwedge\{I_N(a), I_N(b), I_N(c)\}$$

for all $a, b, c \in S$.

Finally, suppose that there are $a, b, c \in S$ such that $F_N([abc]) > \bigvee\{F_N(a), F_N(b), F_N(c)\}$. Then $F_N([abc]) > s_\gamma \geq \bigvee\{F_N(a), F_N(b), F_N(c)\}$ for some $s_\gamma \in [-1, 0)$. This implies that $a, b, c \in T_N^{s_\gamma}$, but $[abc] \notin T_N^{s_\gamma}$, which is a contradiction. Therefore

$$F_N([abc]) \leq \bigvee\{F_N(a), F_N(b), F_N(c)\}$$

for all $a, b, c \in S$.

Therefore S_N is a neutrosophic $\mathcal{N}$-ternary subsemiring of S. □

Theorem 3.4. *Let* $S_N := \dfrac{S}{(T_N, I_N, F_N)}$ *and* $S_M := \dfrac{S}{(T_M, I_M, F_M)}$ *be any two neutrosophic* $\mathcal{N}$*-ternary subsemirings of* S*. The intersection of* S_N *and* S_M*,* $S_{N\cap M}$*, is also a neutrosophic* $\mathcal{N}$*-ternary subsemiring of* S*.*

Proof. Let $a, b, c \in S$. We have

$$\begin{aligned} T_{N\cap M}(a+b) &= \bigvee\{T_N(a+b), T_M(a+b)\} \\ &\leq \bigvee\{\bigvee\{T_N(a), T_N(b)\}, \bigvee\{T_M(a), T_M(b)\}\} \\ &= \bigvee\{\bigvee\{T_N(a), T_M(a)\}, \bigvee\{T_N(b), T_M(b)\}\} \\ &= \bigvee\{T_{N\cap M}(a), T_{N\cap M}(b)\}, \end{aligned}$$

$$\begin{aligned} I_{N\cap M}(a+b) &= \bigwedge\{I_N(a+b), I_M(a+b)\} \\ &\geq \bigwedge\{\bigwedge\{I_N(a), I_N(b)\}, \bigwedge\{I_M(a), I_M(b)\}\} \\ &= \bigwedge\{\bigwedge\{I_N(a), I_M(a)\}, \bigwedge\{I_N(b), I_M(b)\}\}\} \\ &= \bigwedge\{I_{N\cap M}(a), I_{N\cap M}(b)\}, \end{aligned}$$

$$\begin{aligned} F_{N\cap M}(a+b) &= \bigvee\{F_N(a+b), F_M(a+b)\} \\ &\leq \bigvee\{\bigvee\{F_N(a), F_N(b)\}, \bigvee\{F_M(a), F_M(b)\}\} \\ &= \bigvee\{\bigvee\{F_N(a), F_M(a)\}, \bigvee\{F_N(b), F_M(b)\}\} \\ &= \bigvee\{F_{N\cap M}(a), F_{N\cap M}(b)\}, \end{aligned}$$

$$\begin{aligned} T_{N\cap M}([abc]) &= \bigvee\{T_N([abc]), T_M([abc])\} \\ &\leq \bigvee\{\bigvee\{T_N(a), T_N(b), T_N(c)\}, \bigvee\{T_M(a), T_M(b), T_M(c)\}\} \\ &= \bigvee\{\bigvee\{T_N(a), T_M(a)\}, \bigvee\{T_N(b), T_M(b)\}, \bigvee\{T_N(c), T_M(c)\}\} \\ &= \bigvee\{T_{N\cap M}(a), T_{N\cap M}(b), T_{N\cap M}(c)\}, \end{aligned}$$

$$\begin{aligned} I_{N\cap M}([abc]) &= \bigwedge\{I_N([abc]), I_M([abc])\} \\ &\geq \bigwedge\{\bigwedge\{I_N(a), I_N(b), I_N(c)\}, \bigwedge\{I_M(a), I_M(b), I_M(c)\}\} \\ &= \bigwedge\{\bigwedge\{I_N(a), I_M(a)\}, \bigwedge\{I_N(b), I_M(b)\}, \bigwedge\{I_N(c), I_M(c)\}\} \\ &= \bigwedge\{I_{N\cap M}(a), I_{N\cap M}(b), I_{N\cap M}(c)\}, \end{aligned}$$

and

$$\begin{aligned} F_{N\cap M}([abc]) &= \bigvee\{F_N([abc]), F_M([abc])\} \\ &\leq \bigvee\{\bigvee\{F_N(a), F_N(b), F_N(c)\}, \bigvee\{F_M(a), F_M(b), F_M(c)\}\} \\ &= \bigvee\{\bigvee\{F_N(a), F_M(a)\}, \bigvee\{F_N(b), F_M(b)\}, \bigvee\{F_N(c), F_M(c)\}\} \\ &= \bigvee\{F_{N\cap M}(a), F_{N\cap M}(b), F_{N\cap M}(c)\} \end{aligned}$$

for all $a, b, c \in S$. Thus $S_{N\cap M}$ is a neutrosophic $\mathcal{N}$-ternary subsemiring of S. □

By Mathematical Induction, we have the next following corollary.

Corollary 3.5. *Let $\{S_{N_i} \mid i \in \mathbb{N}\}$ be a family of neutrosophic $\mathcal{N}$-ternary subsemiring of a ternary semiring S. Then the intersection of S_{N_i}, denoted by $S_{\bigcap N_i}$, is also a neutrosophic $\mathcal{N}$-ternary subsemiring of S.*

Let $S_N := \dfrac{S}{(T_N, I_N, F_N)}$ and $S_M := \dfrac{S}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over a ternary semiring S. The *neutrosophic additive $\mathcal{N}$-product* of S_N and S_M is defined to be a neutrosophic $\mathcal{N}$-structure over S

$$\begin{aligned} S_N \bigoplus S_M &:= \frac{S}{(T_{N+M}, I_{N+M}, F_{N+M})} \\ &= \left\{ \frac{a}{(T_{N+M}(a), I_{N+M}(a), F_{N+M}(a))} \mid a \in S \right\} \end{aligned}$$

where

$$T_{N+M}(x) = \begin{cases} \bigwedge\limits_{x=a+b} \{T_N(a) \vee T_M(b)\} & \text{if } a, b \in S \text{ such that } x = a + b, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{N+M}(x) = \begin{cases} \bigvee\limits_{x=a+b} \{I_N(a) \wedge I_M(b)\} & \text{if } a, b \in S \text{ such that } x = a + b, \\ -1 & \text{otherwise,} \end{cases}$$

$$F_{N+M}(x) = \begin{cases} \bigwedge\limits_{x=a+b} \{F_N(a) \vee F_M(b)\} & \text{if } a, b \in S \text{ such that } x = a + b, \\ 0 & \text{otherwise.} \end{cases}$$

For any $a \in S$, the element $\dfrac{a}{(T_{N+M}(a), I_{N+M}(a), F_{N+M}(a))}$ is simply denoted by

$$(S_N \bigoplus S_M)(a) := (T_{N+M}(a), I_{N+M}(a), F_{N+M}(a)).$$

Let $S_N := \dfrac{S}{(T_N, I_N, F_N)}, S_M := \dfrac{S}{(T_M, I_M, F_M)}$ and $S_Q := \dfrac{S}{(T_Q, I_Q, F_Q)}$ be neutrosophic $\mathcal{N}$-structures over a ternary semiring S. The *neutrosophic ternary multiplicative $\mathcal{N}$-product* of S_N, S_M and S_Q is defined to be a neutrosophic $\mathcal{N}$-structure over S

$$\begin{aligned} [S_N S_M S_Q] &:= \frac{S}{(T_{[NMQ]}, I_{[NMQ]}, F_{[NMQ]})} \\ &= \left\{ \frac{a}{(T_{[NMQ]}(a), I_{[NMQ]}(a), F_{[NMQ]}(a))} \mid a \in S \right\} \end{aligned}$$

where

$$T_{[NMQ]}(x) = \begin{cases} \bigwedge\limits_{x=[abc]} \{T_N(a) \vee T_M(b) \vee T_Q(c)\} & \text{if } a, b, c \in S \text{ such that } x = [abc], \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{[NMQ]}(x) = \begin{cases} \bigvee\limits_{x=[abc]} \{I_N(a) \wedge I_M(b) \wedge I_Q(c)\} & \text{if } a, b, c \in S \text{ such that } x = [abc], \\ -1 & \text{otherwise,} \end{cases}$$

$$F_{[NMQ]}(x) = \begin{cases} \bigwedge\limits_{x=[abc]} \{F_N(a) \vee F_M(b) \vee F_Q(c)\} & \text{if } a, b, c \in S \text{ such that } x = [abc], \\ 0 & \text{otherwise.} \end{cases}$$

For any $a \in S$, the element $\dfrac{a}{(T_{[NMQ]}(a), I_{[NMQ]}(a), F_{[NMQ]}(a))}$ is simply denoted by

$$[S_N S_M S_Q](a) := (T_{[NMQ]}(a), I_{[NMQ]}(a), F_{[NMQ]}(a)).$$

Theorem 3.6. *A neutrosophic $\mathcal{N}$-structure S_N over a ternary semiring S is a neutrosophic ternary $\mathcal{N}$-ternary subsemiring of S if and only if $S_N \bigoplus S_N \subseteq S_N$ and $[S_N S_N S_N] \subseteq S_N$.*

Proof. To show the necessity condition, we assume that S_N is a neutrosophic $\mathcal{N}$-ternary subsemiring of S. First, let x be an element of S. If $x \neq a+b$ for all $a, b \in S$, then it is clear that $S_N \bigoplus S_N \subseteq S_N$. Suppose that there are $a, b \in S$ such that $x = a+b$.

$$\begin{aligned} T_{N+N}(x) &= \bigwedge_{x=a+b} \{T_N(a) \vee T_N(b)\} \geq \bigwedge_{x=a+b} T_N(a+b) = T_N(x). \\ I_{N+N}(x) &= \bigvee_{x=a+b} \{I_N(a) \wedge I_N(b)\} \leq \bigvee_{x=a+b} I_N(a+b) = I_N(x), \\ F_{N+N}(x) &= \bigwedge_{x=a+b} \{F_N(a) \vee F_N(b)\} \geq \bigwedge_{x=a+b} F_N(a+b) = F_N(x). \end{aligned}$$

Hence $S_N \bigoplus S_N \subseteq S_N$.

Next, let x be an element of S. If $x \neq [abc]$ for all $a, b, c \in S$, then it is clear that $[S_N S_N S_N] \subseteq S_N$. Suppose that there are $a, b, c \in S$ such that $x = [abc]$.

$$\begin{aligned} T_{[NNN]}(x) &= \bigwedge_{x=[abc]} \{T_N(a) \vee T_N(b) \vee T_N(c)\} \geq \bigwedge_{x=[abc]} T_N([abc]) = T_N(x). \\ I_{[NNN]}(x) &= \bigvee_{x=[abc]} \{I_N(a) \wedge I_N(b) \wedge I_N(c)\} \leq \bigvee_{x=[abc]} I_N([abc]) = I_N(x), \\ F_{[NNN]}(x) &= \bigwedge_{x=[abc]} \{F_N(a) \vee F_N(b) \vee F_N(c)\} \geq \bigwedge_{x=[abc]} F_N([abc]) = F_N(x). \end{aligned}$$

Hence $[S_N S_N S_N] \subseteq S_N$.

Conversely, let S_N be any neutrosophic $\mathcal{N}$-structure over S such that $S_N \bigoplus S_N \subseteq S_N$ and $[S_N S_N S_N] \subseteq S_N$. Let a, b be any three elements of S and let $x = a+b$. Then

$$\begin{aligned} T_N(a+b) = T_N(x) \leq T_{N+N}(x) &= \bigwedge_{x=a+b} \{T_N(a) \vee T_N(b)\} \leq T_N(a) \vee T_N(b), \\ I_N(a+b) = I_N(x) \geq I_{N+N}(x) &= \bigvee_{x=a+b} \{I_N(a) \wedge I_N(b)\} \geq I_N(a) \wedge I_N(b), \\ F_N(a+b) = F_N(x) \leq F_{N+N}(x) &= \bigwedge_{x=a+b} \{F_N(a) \vee F_N(b))\} \leq F_N(a) \vee F_N(b). \end{aligned}$$

Next, let a, b, c be any three elements of S and let $x = [abc]$. Then

$$\begin{aligned} T_N([abc]) = T_N(x) \leq T_{[NNN]}(x) &= \bigwedge_{x=[abc]} \{T_N(a) \vee T_N(b) \vee T_N(c)\} \leq T_N(a) \vee T_N(b) \vee T_N(c), \\ I_N([abc]) = I_N(x) \geq I_{[NNN]}(x) &= \bigvee_{x=[abc]} \{I_N(a) \wedge I_N(b) \wedge I_N(c)\} \geq I_N(a) \wedge I_N(b) \wedge I_N(c), \\ F_N([abc]) = F_N(x) \leq F_{[NNN]}(x) &= \bigwedge_{x=[abc]} \{F_N(a) \vee F_N(b) \vee F_N(c)\} \leq F_N(a) \vee F_N(b) \vee F_N(c). \end{aligned}$$

Therefore, S_N is a neutrosophic $\mathcal{N}$-ternary subsemiring of S. □

Let S_1 and S_2 be any nonempty sets and $f : S_1 \to S_2$ be a function. If $(S_2)_M := \dfrac{S_2}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structure over Y, then the *preimage* of $(S_2)_M$ under f

$$f^{-1}((S_2)_M) := \frac{S_1}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

is defined to be a neutrosophic $\mathcal{N}$-structure over S_1 where

$$f^{-1}(T_M)(a) = T_M(f(a)), f^{-1}(I_M)(a) = I_M(f(a)) \text{ and } f^{-1}(F_M)(a) = F_M(f(a))$$

for all $a \in S_1$.

Theorem 3.7. *Let S_1, S_2 be ternary semirings and $f : S_1 \to S_2$ be a homomorphism. If $(S_2)_M := \frac{S_2}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-ternary subsemiring of S_2, then the preimage of $(S_2)_M$ under f,*

$$f^{-1}((S_2)_M) := \frac{a}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

is a neutrosophic $\mathcal{N}$-ternary subsemiring of S_1.

Proof. Let $a, b, c \in S_1$. Then

$$\begin{aligned} f^{-1}(T_M)(a+b) &= T_M(f(a+b)) = T_M(f(a)+f(b)) \\ &\leq \bigvee\{T_M(f(a)), T_M(f(b)))\} \\ &= \bigvee\{f^{-1}(T_M)(a), f^{-1}(T_M)(b)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M)(a+b) &= I_M(f(a+b)) = I_M(f(a)+f(b)) \\ &\geq \bigwedge\{I_M(f(a)), I_M(f(b))\} \\ &= \bigwedge\{f^{-1}(I_M)(a), f^{-1}(I_M)(b)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(F_M)(a+b) &= F_M(f(a+b)) = F_M(f(a)+f(b)) \\ &\leq \bigvee\{F_M(f(a)), F_M(f(b))\} \\ &= \bigvee\{f^{-1}(F_M)(a), f^{-1}(F_M)(b)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(T_M)([abc]) &= T_M(f([abc])) = T_M([f(a)f(b)f(c)]) \\ &\leq \bigvee\{T_M(f(a)), T_M(f(b)), T_M(f(c))\} \\ &= \bigvee\{f^{-1}(T_M)(a), f^{-1}(T_M)(b), f^{-1}(T_M)(c)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M)([abc]) &= I_M(f([abc])) = I_M([f(a)f(b)f(c)]) \\ &\geq \bigwedge\{I_M(f(a)), I_M(f(b)), I_M(f(c))\} \\ &= \bigwedge\{f^{-1}(I_M)(a), f^{-1}(I_M)(b), f^{-1}(I_M)(c)\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M)([abc]) &= F_M(f([abc])) = F_M([f(a)f(b)f(c)]) \\ &\leq \bigvee\{F_M(f(a)), F_M(f(b)), F_M(f(c))\} \\ &= \bigvee\{f^{-1}(F_M)(a), f^{-1}(F_M)(b), f^{-1}(F_M)(c)\}. \end{aligned}$$

Therefore $f^{-1}((S_2)_M)$ is a neutrosophic $\mathcal{N}$-ternary subsemiring of S_1, which completes the proof. □

Let S_1 and S_2 be any nonempty sets and $f : S_1 \to S_2$ be a onto function. If $(S_1)_N := \frac{S_1}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-structure over S_1, then the *image* of $(S_1)_N$ under f

$$f((S_1)_N) := \frac{S_2}{(f(T_N), f(I_N), f(F_N))}$$

is defined to be a neutrosophic $\mathcal{N}$-structure over S_2 where

$$f(T_N)(a) = \bigwedge_{x\in f^{-1}(a)} T_N(x),$$
$$f(I_N)(a) = \bigvee_{x\in f^{-1}(a)} I_N(x),$$
$$f(F_N)(a) = \bigwedge_{x\in f^{-1}(a)} F_N(x).$$

for all $a \in S_2$.

Theorem 3.8. *Let S_1 and S_2 be ternary semirings and $f : S_1 \to S_2$ be an onto homomorphism. Let $(S_1)_N := \dfrac{S_1}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure of S_1 such that*

$$T_N(s_0) = \bigwedge_{z\in A} T_N(c), \quad I_N(s_0) = \bigvee_{z\in A} I_N(c), \quad F_N(s_0) = \bigwedge_{z\in A} F_N(c)$$

for all $A \subseteq X$ and some $s_0 \in A$. If $(S_1)_N$ is a neutrosophic $\mathcal{N}$-structure of S_1, then the image of $(S_1)_N$ under f defined by

$$f((S_1)_N) := \frac{S_2}{(f(T_N), f(I_N), f(F_N))}$$

is a neutrosophic $\mathcal{N}$-ternary subsemiring of S_2.

Proof. Let $a, b, c \in S_2$. Then $f^{-1}(a) \neq \emptyset, f^{-1}(b) \neq \emptyset$ and $f^{-1}(c) \neq \emptyset$ in S. It follows that there exist $S_a \in f^{-1}(a), S_b \in f^{-1}(b)$ and $S_c \in f^{-1}(c)$ such that

$$T_N(s_a) = \bigwedge_{u\in f^{-1}(a)} T_N(u), \quad I_N(s_a) = \bigvee_{u\in f^{-1}(a)} I_N(u), \quad F_N(s_a) = \bigwedge_{u\in f^{-1}(a)} F_N(u),$$
$$T_N(s_b) = \bigwedge_{v\in f^{-1}(b)} T_N(v), \quad I_N(s_b) = \bigvee_{v\in f^{-1}(b)} I_N(v), \quad F_N(s_b) = \bigwedge_{v\in f^{-1}(b)} F_N(v),$$
$$T_N(s_c) = \bigwedge_{w\in f^{-1}(c)} T_N(w), \quad I_N(s_c) = \bigvee_{w\in f^{-1}(c)} I_N(w), \quad F_N(s_c) = \bigwedge_{w\in f^{-1}(c)} F_N(w).$$

Hence

$$\begin{aligned} f(T_N)(a+b) &= \bigwedge_{x\in f^{-1}(a+b)} T_N(a) \leq T_N(s_a + s_b) \\ &\leq \bigvee\{T_N(s_a), T_N(s_b)\} \\ &= \bigvee\{\bigwedge_{u\in f^{-1}(a)} T_N(u), \bigwedge_{v\in f^{-1}(b)} T_N(v)\} \\ &= \bigvee\{f(T_N)(a), f(T_N)(b)\}, \end{aligned}$$

$$\begin{aligned} f(I_N)(a+b) &= \bigvee_{x\in f^{-1}(a+b)} I_N(a) \geq I_N(s_a + s_b) \\ &\geq \bigwedge\{I_N(S_a), I_N(S_b)\} \\ &= \bigwedge\{\bigvee_{u\in f^{-1}(a)} I_N(u), \bigvee_{v\in f^{-1}(b)} I_N(v)\} \\ &= \bigwedge\{f(I_N)(a), f(I_N)(b)\}, \end{aligned}$$

$$
\begin{aligned}
f(F_N)(a+b) &= \bigwedge_{x\in f^{-1}(a+b)} F_N(a) \leq F_N(s_a+s_b) \\
&\leq \bigvee\{F_N(s_a), F_N(s_b)\} \\
&= \bigvee\{\bigwedge_{u\in f^{-1}(a)} F_N(u), \bigwedge_{v\in f^{-1}(b)} F_N(v)\} \\
&= \bigvee\{f(F_N)(a), f(F_N)(b)\},
\end{aligned}
$$

$$
\begin{aligned}
f(T_N)([abc]) &= \bigwedge_{x\in f^{-1}([abc])} T_N(a) \leq T_N([s_as_bs_c]) \\
&\leq \bigvee\{T_N(s_a), T_N(s_b), T_N(s_c)\} \\
&= \bigvee\{\bigwedge_{u\in f^{-1}(a)} T_N(u), \bigwedge_{v\in f^{-1}(b)} T_N(v), \bigwedge_{w\in f^{-1}(c)} T_N(w)\} \\
&= \bigvee\{f(T_N)(a), f(T_N)(b), f(T_N)(c)\},
\end{aligned}
$$

$$
\begin{aligned}
f(I_N)([abc]) &= \bigvee_{x\in f^{-1}([abc])} I_N(a) \geq I_N([s_as_bs_c]) \\
&\geq \bigwedge\{I_N(s_a), I_N(s_b), I_N(s_c)\} \\
&= \bigwedge\{\bigvee_{u\in f^{-1}(a)} I_N(u), \bigvee_{v\in f^{-1}(b)} I_N(v), \bigvee_{w\in f^{-1}(c)} I_N(w)\} \\
&= \bigwedge\{f(I_N)(a), f(I_N)(b), f(I_N)(c)\},
\end{aligned}
$$

and

$$
\begin{aligned}
f(F_N)([abc]) &= \bigwedge_{x\in f^{-1}([abc])} F_N(a) \leq F_N([s_as_bs_c]) \\
&\leq \bigvee\{F_N(s_a), F_N(s_b), F_N(s_c)\} \\
&= \bigvee\{\bigwedge_{u\in f^{-1}(a)} F_N(u), \bigwedge_{v\in f^{-1}(b)} F_N(v), \bigwedge_{w\in f^{-1}(c)} F_N(w)\} \\
&= \bigvee\{f(F_N)(a), f(F_N)(b), f(F_N)(c)\}.
\end{aligned}
$$

Then $f(S_N)$ is a neutrosophic $\mathcal{N}$-ternary subsemiring of S_2. □

4 Conclusions

In this paper, we applied neutrosophic $\mathcal{N}$-structure to ternary semirings. We also investigated the notion of neutrosophic $\mathcal{N}$-ternary subsemirings and showed some properties. Moreover, the conditions for neutrosophic $\mathcal{N}$-structure to be neutrosophic $\mathcal{N}$-ternary subsemiring have been investigated. We also defined the neutrosophic additive $\mathcal{N}$-product and neutrosophic ternary multiplicative $\mathcal{N}$-product. Finally, we showed that the homomorphic preimage of the neutrosophic $\mathcal{N}$-ternary subsemiring is a neutrosophic $\mathcal{N}$-ternary subsemiring and the onto homomorphic image of the neutrosophic $\mathcal{N}$-ternary subsemiring is a neutrosophic $\mathcal{N}$-ternary subsemiring.

In our future study, we will apply these notions/results to other types of neutrosophic $\mathcal{N}$-structures in ternary semirings. We will also apply neutrosophic $\mathcal{N}$-structures to other algebraic structures.

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