# Lagrange's theorem based on neutrosophic sets

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**Abstract**

This paper explores the fundamental concepts of sub-level subgroups, element orders, normalizers, and centralizers within the framework of neutrosophic group theory. Additionally, it examines quotient groups and the index of a subgroup, extending classical algebraic structures to a neutrosophic setting. Finally, a generalized formulation of Lagrange's theorem is presented, demonstrating its applicability in the neutrosophic environment and highlighting its implications for uncertain and indeterminate group structures.

**Keywords:** Neutrosophic set; Neutrosophic subgroup; Neutrosophic order; Neutrosophic quotient group

## 1 Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 as a means of handling uncertainty in real-world problems.[15] Since then, various generalizations have been proposed, including intuitionistic fuzzy sets,[3] Pythagorean fuzzy sets,[14] and neutrosophic sets.[12] Neutrosophic sets, introduced by Smarandache, extend classical fuzzy set theory by incorporating three membership degrees: truth, indeterminacy, and falsehood. This framework has been widely applied in decision-making, artificial intelligence, and algebraic structures.

In group theory, the study of uncertainty-based algebraic structures has gained significant attention. Rosenfeld[10] was among the first to investigate fuzzy subgroups, laying the foundation for subsequent studies on fuzzy algebraic structures. Ajmal and Prajapati[1] expanded this concept by introducing fuzzy cosets and normal subgroups, further enhancing the applicability of fuzzy logic in abstract algebra. More recently, Bhunia et al.[4] explored the properties of neutrosophic subgroups, while Cetkin and Aygun[7] developed fundamental theorems for neutrosophic algebraic structures. Thiruveni and Solairaju[13] studied neutrosophic Q-fuzzy subgroups. Al-Odhari[2] introduced the characteristics of neutrosophic subgroups.

Lagrange's theorem, one of the cornerstone results in classical group theory, states that the order of a subgroup must divide the order of the group. This theorem has been extensively studied in various algebraic settings, including fuzzy subgroups,[9] Pythagorean fuzzy subgroups,[6] and $(\alpha, \beta)$-Pythagorean fuzzy environment.[5] However, its generalization to neutrosophic subgroups remains an active area of research. Iampan et al.[8] recently explored neutrosophic subgroups and their normal properties, laying the groundwork for further theoretical development.

This study extends classical group-theoretic concepts to the neutrosophic setting, examining sub-level subgroups, the order of elements, normalizers, centralizers, quotient groups, and subgroup indices. By integrating these structures within neutrosophic group theory, we provide a broader perspective on algebraic uncertainty and indeterminacy. Furthermore, we establish a neutrosophic extension of Lagrange's theorem, demonstrating its relevance in handling group structures characterized by imprecise and incomplete information.

## 2 Preliminaries

In this section, we present fundamental definitions and key concepts related to neutrosophic sets and neutrosophic subgroups, which serve as the foundation for our study. We begin by recalling the definition of a neutrosophic set, characterized by three membership functions representing truth, indeterminacy, and falsehood. Building on this, we introduce the notion of neutrosophic subgroups within a group structure, extending classical subgroup properties to the neutrosophic framework. Furthermore, we explore essential properties such as neutrosophic cosets, normality conditions, and related algebraic structures. These preliminaries provide the necessary groundwork for our subsequent analysis of neutrosophic subgroups and their role in the generalization of Lagrange's theorem.

**Definition 2.1.** [12] Let $X$ be a nonempty set. The *neutrosophic set* on $X$ is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle : x \in X\}, \tag{2.1}$$

where $\mu : X \to [0, 1]$ is a truth membership function, $\gamma : X \to [0, 1]$ is an indeterminate membership function, and $\psi : X \to [0, 1]$ is a false membership function. The neutrosophic fuzzy set in (2.1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

**Definition 2.2.** [7] Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group $(C, \circ)$. Then $A$ is a neutrosophic subgroup of $C$ if

(1) $\mu_A(x \circ y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\gamma_A(x \circ y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in C$,

(2) $\mu_A(x^{-1}) \geq \mu_A(x)$, $\gamma_A(x^{-1}) \geq \gamma_A(x)$, and $\gamma_A(x^{-1}) \leq \gamma_A(x)$ for all $x \in C$.

**Proposition 2.3.** [7] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group $(C, \circ)$. Then $A$ is a neutrosophic subgroup of $C$ if and only if $\mu_A(x \circ y^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y^{-1}) \geq \gamma_A(x) \wedge \gamma_A(y)$, and $\gamma_A(x \circ y^{-1}) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in C$.*

**Definition 2.4.** [7] Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then for $x \in C$, the neutrosophic left coset of $A$ is the neutrosophic set $xA = (x\mu_A, x\gamma_A, x\psi_A)$, defined by $(x\mu_A)(u) = \mu_A(x^{-1} \circ u)$, $(x\gamma_A)(u) = \gamma_A(x^{-1} \circ u)$, and $(x\psi_A)(u) = \psi_A(x^{-1} \circ u)$ and the neutrosophic right coset of $A$ is the neutrosophic set $Ax = (\mu_A x, \gamma_A x, \psi_A x)$, defined by $(\mu_A x)(u) = \mu_A(u \circ x^{-1})$, $(\gamma_A x)(u) = \gamma_A(u \circ x^{-1})$, and $(\psi_A x)(u) = \psi_A(u \circ x^{-1})$ for all $u \in C$.

**Definition 2.5.** [7] Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then $A$ is a neutrosophic normal subgroup on $C$ if every neutrosophic left coset of $A$ is a neutrosophic right coset of $A$ on $C$. Equivalently, $xA = Ax$ for all $x \in C$.

**Proposition 2.6.** [7] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then $A$ is a neutrosophic normal subgroup on $C$ if and only if $\mu_A(x \circ y) = \mu_A(y \circ x)$, $\gamma_A(x \circ y) = \gamma_A(y \circ x)$, and $\psi_A(x \circ y) = \psi_A(y \circ x)$ for all $x, y \in C$.*

**Proposition 2.7.** [7] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then $A$ is a neutrosophic normal subgroup of $C$ if and only if $\mu_A(k \circ u \circ k^{-1}) = \mu_A(u)$, $\gamma_A(k \circ u \circ k^{-1}) = \gamma_A(u)$, and $\psi_A(k \circ u \circ k^{-1}) = \psi_A(u)$ for all $u, k \in C$.*

## 3 Properties of neutrosophic subgroups

In this section, we define neutrosophic subgroups as extensions of fuzzy subgroups and traditional neutrosophic subgroups. Next, we investigate whether the union and intersection of two neutrosophic subgroups of a group $(C, \circ)$ are themselves neutrosophic subgroups of $C$.

**Theorem 3.1.** [8] *The intersection of a family of neutrosophic subgroups of a group $(C, \circ)$ is also a neutrosophic subgroup of C.*

**Proposition 3.2.** [8] *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $(C, \circ)$, then $\mu_A(x^k) \geq \mu_A(x)$, $\gamma_A(x^k) \geq \gamma_A(x)$, and $\psi_A(x^k) \leq \psi_A(x)$ for all $x \in C$ and $k \in \mathbb{N}$. Here $x^k = x \circ x \circ \cdots \circ x \ (k \ times)$.*

**Proposition 3.3.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. If $\mu(x) \neq \mu(y)$, $\gamma(x) \neq \gamma(y)$, and $\psi(x) \neq \psi(y)$, then $\mu_A(x \circ y) = \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x \circ y) = \gamma_A(x) \wedge \gamma_A(y)$, and $\psi_A(x \circ y) = \psi_A(x) \vee \wedge_A(y)$, respectively, for all $x, y \in C$.*

**Proposition 3.4.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$ with $e$ as the identity element and $x \in C$. Then*

$$(\forall y \in C) \begin{pmatrix} \mu_A(x) = \mu_A(e) \Rightarrow \mu_A(x \circ y) = \mu_A(y), \\ \gamma_A(x) = \gamma_A(e) \Rightarrow \gamma_A(x \circ y) = \gamma_A(y), \\ \psi_A(x) = \psi_A(e) \Rightarrow \psi_A(x \circ y) = \psi_A(y) \end{pmatrix}.$$

**Theorem 3.5.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then the set $N = \{x \in C : \mu_A(e) = \mu_A(x), \gamma_A(e) = \gamma_A(x), \psi_A(e) = \psi_A(x)\}$ forms a subgroup of C, where e is the identity element of C.*

**Definition 3.6.** [8] Let $C$ be a crisp set. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on the set $C$. For $\alpha, \beta, \delta \in [0, 1]$, the set $\psi_{(\alpha,\beta,\delta)} = \{x \in C : \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \delta\}$ is called a neutrosophic level subset of the neutrosophic set $A$ of $C$, where $0 \leq \alpha + \beta + \delta \leq 1$.

**Proposition 3.7.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be two neutrosophic sets on the universal set C. Then*

(1) $A_{(\alpha,\beta,\delta)} \subseteq A_{(\epsilon,\omega,\sigma)}$ *if $\epsilon \leq \alpha$, $\omega \leq \beta$ and $\delta \leq \sigma$ for $\epsilon, \beta, \alpha, \omega, \delta, \sigma \in [0, 1]$,*

(2) $A \subseteq B \Rightarrow A_{(\alpha,\beta,\delta)} \subseteq B_{(\alpha,\beta,\delta)}$ *for $\alpha, \beta, \delta \in [0, 1]$.*

**Proposition 3.8.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $(C, \circ)$. Then, the neutrosophic level subset $A_{(\alpha,\beta,\delta)}$ forms a subgroup of C, where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$, and $\delta \geq \psi_A(e)$, and e is the identity element of C.*

**Definition 3.9.** [8] The subgroup $A_{(\alpha,\beta,\delta)}$ of the group $(C, \circ)$ is called the neutrosophic level subgroup of the neutrosophic subgroup $A = (\mu_A, \gamma_A, \psi_A)$.

**Proposition 3.10.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a group $(C, \circ)$. If the neutrosophic level subgroup $A_{(\alpha,\beta,\delta)}$ is a subgroup of C, where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$, and $\delta \geq \psi_A(e)$, then $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of C.*

**Proposition 3.11.** [8] *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of a group $(C, \circ)$. Then, the neutrosophic level subset $A_{(\alpha,\beta,\delta)}$ forms a normal subgroup of C, where $\alpha \leq \mu_A(e)$, $\beta \leq \gamma_A(e)$ and $\delta \geq \psi_A(e)$, and e is the identity element in C.*

**Theorem 3.12.** [8] *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let $f : C_1 \to C_2$ be a surjective homomorphism and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of $C_1$. Then $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic subgroup of $C_2$.*

**Theorem 3.13.** [8] *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let f be a bijective homomorphism from $C_1$ to $C_2$ and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of $C_2$. Then $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic subgroup of $C_1$.*

**Theorem 3.14.** [8] *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let $f : C_1 \to C_2$ be a surjective homomorphism and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of $C_1$. Then $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A))$ is a neutrosophic normal subgroup of $C_2$.*

**Theorem 3.15.** [8] *Let $(C_1, \circ_1)$ and $(C_2, \circ_2)$ be two groups. Let $f$ be a bijective homomorphism from $C_1$ to $C_2$ and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of $C_2$. Then $f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A), f^{-1}(\psi_A))$ is a neutrosophic normal subgroup of $C_1$.*

**Theorem 3.16.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$, then $\Gamma(x) = \{y \in C : \mu_A(y) \geq \mu_A(x), \gamma_A(y) \geq \gamma_A(x), \psi_A(y) \leq \psi_A(x)\}$ is a subgroup of $C$.*

*Proof.* We have $\Gamma(x) = \{y \in C : \mu_A(y) \geq \mu_A(x), \gamma_A(y) \geq \gamma_A(x), \psi_A(y) \leq \psi_A(x)\}$, where $x \in C$. Then $\Gamma(x) \subset C$ as $x \in \Gamma(x)$. Also, $e \in \Gamma(x)$ as $\mu_A(e) \geq \mu_A(x)$, $\gamma_A(e) \geq \gamma_A(x)$ and $\psi_A(e) \leq \psi_A(x)$. Let $p, q \in \Gamma(x)$. Then

$$\mu_A(pq^{-1}) \geq \mu_A(p) \wedge \mu_A(q^{-1}) = \mu_A(p) \wedge \mu_A(q) \geq \mu_A(x),$$
$$\gamma_A(pq^{-1}) \geq \gamma_A(p) \wedge \gamma_A(q^{-1}) = \gamma_A(p) \wedge \gamma_A(q) \geq \gamma_A(x),$$
$$\psi_A(pq^{-1}) \leq \psi_A(p) \vee \psi_A(q^{-1}) = \psi_A(p) \vee \psi_A(q) \leq \psi_A(x).$$

Thus, $pq^{-1} \in \Gamma(x)$. Therefore, $\Gamma(x)$ is a subgroup of $C$. □

**Definition 3.17.** Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. Then the subgroup $\Gamma(x)$ is a neutrosophic sub-level subgroup of $C$ corresponding to $m$.

**Definition 3.18.** Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. Then the neutrosophic order of $x$ in $A$ is denoted by $O(x)_A$ and defined by the order of the neutrosophic sub-level subgroup of $x$ in $C$. Therefore, $O(x)_A = O(\Gamma(x))$ for all $x \in C$.

**Example 3.19.** Consider the group $(\mathbb{Z}_4, +_4)$. We define a neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ as follows:

| C | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mu_A$ | 0.9 | 0.6 | 0.6 | 0.8 |
| $\gamma_A$ | 0.8 | 0.5 | 0.5 | 0.7 |
| $\psi_A$ | 0.2 | 0.7 | 0.7 | 0.4 |

Clearly, $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup on $\mathbb{Z}_4$. Then neutrosophic order of the elements of $\mathbb{Z}_4$ in $A$ is presented by $O(0)_A = O(\Gamma(0)) = 2$, $O(1)_A = O(\Gamma(1)) = 4$, $O(2)_A = O(\Gamma(2)) = 4$, $O(3)_A = O(\Gamma(3)) = 2$. We see that $O(0)_A \neq O(0)$ and $O(0)_A = O(3)_A = 2$.

**Remark 3.20.** The neutrosophic order of an element in neutrosophic subgroup may not always be the same to the element's order in the group.

**Proposition 3.21.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$, then $O(e)_A \leq O(x)_A$ for all $x \in C$, where e is the identity element of the group $C$.*

*Proof.* Let $O(e)_A = s$, where $s \in \mathbb{Z}^+$. Suppose $\Gamma(e) = \{x_1, x_2, \ldots, x_s\}$, where $x_i \neq x_j$ for all $i, j$. Then $\mu_A(x_1) = \mu_A(x_2) = \ldots = \mu_A(x_s) = \mu_A(e)$, $\gamma_A(x_1) = \gamma_A(x_2) = \ldots = \gamma_A(x_s) = \gamma_A(e)$ and $\psi_A(x_1) = \psi_A(x_2) = \ldots = \psi_A(x_s) = \psi_A(e)$. As $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup on $C$, $\mu_A(e) \geq \mu_A(x)$, $\gamma_A(e) \geq \gamma_A(x)$ and $\psi_A(e) \leq \psi_A(x)$ for all $x \in C$. So, $x_1, x_2, \ldots, x_s \in \Gamma(u)$. Then $\Gamma(e) \subset \Gamma(u)$. Thus, $O(\Gamma(e)) \leq O(\Gamma(x))$ for all $x \in C$. Therefore, $O(e)_A \leq O(x)_A$ for all $x \in C$. □

The next result represents a relationship between the order and neutrosophic order of an element in a group.

**Theorem 3.22.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$, then $O(x)$ divides $O(x)_A$ for all $x \in C$.*

*Proof.* Let $x \in C$ and $O(x) = k$, where $k \in \mathbb{Z}^+$. Then $x^k = e$. Consider $D = \langle x \rangle$ as a subgroup of C. Now, $\mu_A(x) \geq \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$, $\gamma_A(x) \geq \gamma_A(x) \wedge \gamma_A(x) = \gamma_A(x)$ and $\psi_A(x) \leq \psi_A(x) \vee \psi_A(x) = \psi_A(x)$. Then, by induction, $\mu_A(x^p) \geq \mu_A(x)$, $\gamma_A(x^p) \geq \gamma_A(x)$ and $\psi_A(x^p) \leq \psi_A(x)$ for all $p \in \mathbb{Z}^+$. So, $x, x^2, \ldots, x^k \in \Gamma(x)$. Consequently, $D \subset \Gamma(x)$. Therefore, $D$ is a subgroup of $\Gamma(x)$. Thus, by Lagrange's theorem, $O(D)|O(\Gamma(x))$. Therefore, $O(x)|O(x)_A$. Since $x$ is any element of $C$, $O(x)|O(x)_A$ for all $x \in C$. □

We will now construct a relationship between the neutrosophic order of an element of a group in neutrosophic subgroup and the group's order.

**Theorem 3.23.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$, then neutrosophic order of each element of $C$ in $A$ divides the order of $C$.*

*Proof.* According to the definition, $O(x)_A = O(\Gamma(x))$ for all $x \in C$. From Theorem 3.16, $\Gamma(x)$ is a subgroup of $C$. Therefore, by Lagrange's theorem, the order of $\Gamma(x)$ divides the order of $C$. That is $O(\Gamma(x))|O(C)$. This represents that $O(x)_A|O(C)$ for all $x \in C$. Hence, the neutrosophic order of each element of $C$ in $A$ divides the order of $C$. □

**Theorem 3.24.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$, then $O(x)_A = O(x^{-1})_A$ for all $x \in C$.*

*Proof.* Let $x \in C$. Then $O(x)_A = O(\Gamma(x))$. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup on $C$, $\mu_A(x) = \mu_A(x^{-1})$, $\gamma_A(x) = \gamma_A(x^{-1})$ and $\psi_A(x) = \psi_A(x^{-1})$. Then $\Gamma(x) = \{y \in C : \mu_A(y) \geq \mu_A(x^{-1}), \gamma_A(y) \geq \gamma_A(x^{-1}), \psi_A(y) \leq \psi_A(x^{-1})\} = \Gamma(x^{-1})$. Hence, $O(\Gamma(x)) = O(\Gamma(x^{-1}))$. That is, $O(x)_A = O(x^{-1})_A$. Therefore, $O(x)_A = O(x^{-1})_A$ for all $x \in C$. □

Now, we will introduce the neutrosophic order of a neutrosophic subgroup of a group.

**Definition 3.25.** Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$. Then the neutrosophic order of the neutrosophic subgroup $A$ is denoted by $O(A)$ and is defined by $O(A) = \bigvee\{O(x)_A : x \in C\}$.

**Example 3.26.** Consider the neutrosophic subgroup $A$ on $\mathbb{Z}_4$ in Example 3.19. The neutrosophic order of the elements of $\mathbb{Z}_4$ in $A$ is presented by $O(0)_A = 2, O(1)_A = 4, O(2)_A = 4$ and $O(3)_A = 2$. Therefore, $O(A) = \bigvee\{O(x)_A : x \in \mathbb{Z}_4\} = 4$.

**Theorem 3.27.** *The neutrosophic order of each neutrosophic subgroup of a group is the same as the group's order.*

*Proof.* Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. Without loss of generality, we assume that $\mu_A(y) \geq \mu_A(x)$, $\gamma_A(y) \geq \gamma_A(x)$ and $\psi_A(y) \leq \psi_A(x)$ for all $y \in C$. Since $\Gamma(x) = \{y \in C : \mu_A(y) \geq \mu_A(x), \gamma_A(y) \geq \gamma_A(x), \psi_A(y) \leq \psi_A(x)\}$, $\Gamma(x) = C$. Also, $|\Gamma(x)| \geq |\Gamma(y)|$ for all $y \in C$.

Consequently, $O(A) = O(x)_A$. Again, $O(x)_A = O(\Gamma(x))$. Therefore, $O(A) = O(C)$. Hence, the neutrosophic order of a neutrosophic subgroup of a group is the same as group's order. □

**Remark 3.28.** For a neutrosophic subgroup of a group $C$, the neutrosophic order of an element of $C$ divides the neutrosophic order of that neutrosophic subgroup.

**Theorem 3.29.** *Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$ such that $O(x)_A = s$. If $\gcd(s, t) = 1$, then $\mu_A(x^t) = \mu_A(x)$, $\gamma_A(x^t) = \gamma_A(x)$ and $\psi_A(x^t) = \psi_A(x)$.*

*Proof.* Since $O(x)_A = s$, then $x^s = e$. Also, $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup on $C$, then $\mu_A(x^t) \geq \mu_A(x)$, $\gamma_A(x^t) \geq \gamma_A(x)$ and $\psi_A(x^t) \leq \psi_A(x)$. Since $\gcd(s, t) = 1$, there exist $a$ and $b$ such that $as + bt = 1$. Then

$$\mu_A(x) = \mu_A(x^{as+bt}) \geq \mu_A(x^{as}) \wedge \mu_A(x^{bt}) \geq \mu_A(e) \wedge \mu_A(x^t) = \mu_A(x^t),$$

$$\gamma_A(x) = \gamma_A(x^{as+bt}) \geq \gamma_A(x^{as}) \wedge \gamma_A(x^{bt}) \geq \gamma_A(e) \wedge \gamma_A(x^t) = \gamma_A(x^t),$$

$$\psi_A(x) = \psi_A(x^{as+bt}) \leq \psi_A(x^{as}) \vee \psi_A(x^{bt}) \leq \psi_A(e) \vee \psi_A(x^t) = \psi_A(x^t).$$

Hence, $\mu_A(x^t) = \mu_A(x)$, $\gamma_A(x^t) = \gamma_A(x)$ and $\psi_A(x^t) = \psi_A(x)$. □

**Theorem 3.30.** *Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. If $\mu_A(x^t) = \mu_A(e)$, $\gamma_A(x^t) = \gamma_A(e)$ and $\psi_A(x^t) = \psi_A(e)$, then $t|O(x)_A$, where $t \in \mathbb{Z}$.*

*Proof.* Let $O(x)_A = s$. We can suppose that $q$ is the smallest integer for which $\mu_A(x^q) = \mu_A(e)$, $\gamma_A(x^q) = \gamma_A(e)$ and $\psi_A(x^q) = \psi_A(e)$ holds. By division algorithm, there exist $a, b \in \mathbb{Z}$ such that $s = at + b$ where $0 \leq b < t$. Now,

$$\begin{aligned}
\mu_A(x^b) &= \mu_A(x^{s-at}) \\
&\geq \mu_A(x^s) \wedge \mu_A((x^{-1})^{at}) \\
&= \mu_A(x^s) \wedge \mu_A(x^{at}) \\
&= \mu_A(e) \wedge \mu_A((x^t)^a) \\
&\geq \mu_A(e) \wedge \mu_A(x^t) \\
&= \mu_A(e),
\end{aligned}$$

$$\begin{aligned}
\gamma_A(x^b) &= \gamma_A(x^{s-at}) \\
&\geq \gamma_A(x^s) \wedge \gamma_A((x^{-1})^{at}) \\
&= \gamma_A(x^s) \wedge \gamma_A(x^{at}) \\
&= \gamma_A(e) \wedge \gamma_A((x^t)^a) \\
&\geq \gamma_A(e) \wedge \gamma_A(x^t) \\
&= \gamma_A(e),
\end{aligned}$$

$$\begin{aligned}
\psi_A(x^b) &= \psi_A(x^{s-at}) \\
&\leq \psi_A(x^s) \vee \psi_A((x^{-1})^{at}) \\
&= \psi_A(x^s) \vee \psi_A(x^{at}) \\
&= \psi_A(e) \vee \psi_A((x^t)^a) \\
&\leq \psi_A(e) \vee \psi_A(x^t) \\
&= \psi_A(e).
\end{aligned}$$

Thus, $\mu_A(x^b) = \mu_A(e)$, $\gamma_A(x^b) = \gamma_A(e)$ and $\psi_A(x^b) = \psi_A(e)$. This contradicts the minimality of $q$ as $0 \leq b < t$. Therefore, $b = 0$, so $s = at$. Hence, $t|O(x)_A$. □

**Theorem 3.31.** *Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. If $O(x)_A = s$, then $O(x^v)_A = \dfrac{s}{\gcd(s, v)}$, where $v \in \mathbb{Z}$.*

*Proof.* Let $O(x^v)_A = a$ and $\gcd(s, v) = g$. Since $O(x)_A = s$, by Theorem 3.30, $x^s = e$. Now, $\mu_A((x^v)^{\frac{s}{g}}) = \mu_A((x^s)^{\frac{v}{g}}) = \mu_A(e^{\frac{v}{g}}) = \mu_A(e)$. Similarly, $\gamma_A((x^v)^{\frac{s}{g}}) = \gamma_A(e)$ and $\psi_A((x^v)^{\frac{s}{g}}) = \psi_A((x^s)^{\frac{v}{g}}) = \psi_A(e^{\frac{v}{g}}) = \psi_A(e)$. It follows from Theorem 3.30, $sg$ divides $a$. Since $\gcd(s, v) = g$, there exist $p, q \in \mathbb{Z}$ such that $sp + vq = g$. Therefore,

$$\begin{aligned}
\mu_A(x^{ga}) &= \mu_A(x^{(ps+vq)a}) \\
&= \mu_A(x^{psa}x^{vqa}) \\
&\geq \mu_A((x^s)^{pa}) \wedge \mu_A((x^{va})^q) \\
&\geq \mu_A(x^s) \wedge \mu_A((x^v)^a) \\
&= \mu_A(e) \wedge \mu_A(e) \\
&= \mu_A(e).
\end{aligned}$$

But the only option is $\mu_A(x^{ga}) = \mu_A(e)$.
Also

$$\begin{aligned}
\gamma_A(x^{ga}) &= \gamma_A(x^{(ps+vq)a}) \\
&= \gamma_A(x^{psa}x^{vqa}) \\
&\geq \gamma_A((x^s)^{pa}) \wedge \gamma_A((x^{va})^q) \\
&\geq \gamma_A(x^s) \wedge \gamma_A((x^v)^a) \\
&= \gamma_A(e) \wedge \gamma_A(e) \\
&= \gamma_A(e).
\end{aligned}$$

But the only option is $\gamma_A(x^{ga}) = \gamma_A(e)$.

$$\begin{aligned}
\psi_A(x^{ga}) &= \psi_A(x^{(ps+vq)a}) \\
&= \psi_A(x^{psa}x^{vqa}) \\
&\leq \psi_A((x^s)^{pa}) \vee \psi_A((x^{va})^q) \\
&\leq \psi_A(x^s) \vee \psi_A((x^v)^a) \\
&= \psi_A(e) \vee \psi_A(e) \\
&= \psi_A(e).
\end{aligned}$$

But the only option is $\psi_A(x^{ga}) = \psi_A(e)$. Thus, by Theorem 3.30, $ga|s$, that is $a|\frac{s}{g}$. Therefore, $a = \frac{s}{g}$. Hence, $O(x^v)_A = \dfrac{s}{\gcd(s, v)}$. □

**Theorem 3.32.** *Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$ and $x \in C$. If $O(x)_A = z$ and $g \cong h(mod\ z)$, then $O(x^g)_A = O(x^h)_A$, where $g, h, z \in \mathbb{Z}$.*

*Proof.* Let $O(x^g)_A = l_1$ and $O(x^h)_A = l_2$. Since $g \cong h(mod\ z)$, then $g = wz + h$, where $w \in \mathbb{Z}$. Then

$$\begin{aligned} \mu_A((x^g)^{l_2}) &= \mu_A((x^{zw+h})^{l_2}) \\ &= \mu_A(x^{wzl_2}x^{hl_2}) \\ &\geq \mu_A((x^z)^{wl_2}) \wedge \mu_A((x^h)^{l_2}) \\ &= \mu_A(e) \wedge \mu_A(e) \\ &= \mu_A(e). \end{aligned}$$

But the only option is $\mu_A((x^g)^{l_2}) = \mu_A(e)$.
Also

$$\begin{aligned} \gamma_A((x^g)^{l_2}) &= \gamma_A((x^{zw+h})^{l_2}) \\ &= \gamma_A(x^{wzl_2}x^{hl_2}) \\ &\geq \gamma_A((x^z)^{wl_2}) \wedge \gamma_A((x^h)^{l_2}) \\ &= \gamma_A(e) \wedge \gamma_A(e) \\ &= \gamma_A(e). \end{aligned}$$

But the only option is $\mu_A((x^g)^{l_2}) = \mu_A(e)$.

$$\begin{aligned} \psi_A((x^g)^{l_2}) &= \psi_A((x^{zw+h})^{l_2}) \\ &= \psi_A(x^{wzl_2}x^{hl_2}) \\ &\leq \psi_A((x^z)^{wl_2}) \vee \psi_A((x^h)^{l_2}) \\ &= \psi_A(e) \vee \psi_A(e) \\ &= \psi_A(e). \end{aligned}$$

But the only option is $\psi_A((x^g)^{l_2}) = \psi_A(e)$. Hence, by Theorem 3.30, $l_2|l_1$. In the same manner, we can prove that $l_1|l_2$. Thus, $l_1 = l_2$. Hence, $O(x^g)_A = O(x^h)_A$, where $g, h \in \mathbb{Z}$. □

**Theorem 3.33.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a group $C$ and $x \in C$, then $O(x)_A = O(yxy^{-1})_A$ for all $y \in C$.*

*Proof.* Let $y \in C$. Since $A$ is a neutrosophic normal subgroup on the group $C$, $\mu_A(x) = \mu_A(yxy^{-1})$, $\gamma_A(x) = \gamma_A(yxy^{-1})$ and $\psi_A(x) = \psi_A(yxy^{-1})$. Then the neutrosophic sub-level subgroup corresponding to $x$ is equal to $yxy^{-1}$. Hence, $\Gamma(x) = \Gamma(yxy^{-1})$. Consequently, $O(\Gamma(x)) = O(\Gamma(yxy^{-1}))$. Since $y$ is any element of $C$, $O(x)_A = O(yxy^{-1})_A$ for all $y \in C$. □

**Theorem 3.34.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a group $C$, then $O(xy)_A = O(yx)_A$ for all $x, y \in C$.*

*Proof.* Let $x, y \in C$. Then

$$\mu_A(xy) = \mu_A((y^{-1}y)(xy)) = \mu_A(y^{-1}(yx)y),$$

$$\gamma_A(xy) = \gamma_A((y^{-1}y)(xy)) = \gamma_A(y^{-1}(yx)y),$$

$$\psi_A(xy) = \psi_A((y^{-1}y)(xy)) = \psi_A(y^{-1}(yx)y).$$

Hence, $\Gamma(xy) = \Gamma(y^{-1}(xy)(y^{-1})^{-1})$, and hence $O(xy)_A = O(y^{-1}(yx)(y^{-1})^{-1})_A$. It follows from Theorem 3.33 that $O(y(yx)y^{-1})_A = O(yx)_A$. Since $x$ and $y$ are any elements of $C$, $O(xy)_A = O(yx)_A$ for all $x, y \in G$. □

**Theorem 3.35.** *Suppose $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a commutative group $C$ and $m, n$ are two elements of $C$ such that* $\gcd(O(x)_A, O(y)_A) = 1$. *If $\mu_A(xy) = \mu_A(e)$, $\gamma_A(xy) = \gamma_A(e)$ and $\psi_A(xy) = \psi_A(e)$, then $O(x)_A = O(y)_A = 1$.*

*Proof.* Assume $O(x)_A = p$ and $O(y)_A = q$. So, we get $\gcd(p, q) = 1$. Now, $\mu_A(x^q y^q) = \mu_A((xy)^q) \geq \mu_A(xy) = \mu_A(e)$. But the only option is $\mu_A(x^q y^q) = \mu_A(e)$. Also, $\mu_A(x^q) = \mu_A(x^q y^q v^{-q}) \geq \mu_A(x^q y^q) \wedge \mu_A((y^{-1})^q) = \mu_A(e) \wedge \mu_A(e) = \mu_A(e)$. So, we get $\mu_A(x^q) = \mu_A(e)$. Also, $\gamma_A(x^q y^q) = \gamma_A((xy)^q) \leq \gamma_A(xy) = \gamma_A(e)$. But the only option is $\gamma_A(x^q y^q) = \gamma_A(e)$. Also, $\gamma_A(x^q) = \gamma_A(x^q y^q v^{-q}) \leq \gamma_A(x^q y^q) \vee \gamma_A((y^{-1})^q) = \gamma_A(e) \vee \gamma_A(e) = \gamma_A(e)$. So, we get $\gamma_A(x^q) = \gamma_A(e)$. Now, $\psi_A(x^q y^q) = \psi_A((xy)^q) \geq \psi_A(xy) = \psi_A(e)$. But the only option is $\psi_A(x^q y^q) = \psi_A(e)$. Also, $\psi_A(x^q) = \psi_A(x^q y^q v^{-q}) \geq \psi_A(x^q y^q) \wedge \psi_A((y^{-1})^q) = \psi_A(e) \wedge \psi_A(e) = \psi_A(e)$. So, we get $\psi_A(x^q) = \psi_A(e)$. It follows from Theorem 3.32 that $q|p$. Again $\gcd(p, q) = 1$, thus $q = 1$. Similarly, we can present that $p = 1$. Hence, $O(x)_A = O(y)_A = 1$. □

**Theorem 3.36.** *Generators of a cyclic group have same neutrosophic order in a neutrosophic subgroup.*

*Proof.* Assume $C$ is a cyclic group of order $k$. Let $x, y$ be any two generators of $C$. Then $x^k = e = y^k$. Since $x$ is a generator, $y = x^p$ for some $p \in \mathbb{Z}^+$. Then $k$ and $p$ are co-prime, so $\gcd(k, p) = 1$. It follows from Theorem 3.29 that $O(x)_A = O(x^p)_A = O(y)_A$. For an infinite cyclic group it has only two generator. If $x$ is a generator of $C$, then $x^{-1}$ is the only other generator. It follows from Theorem 3.24 that $O(x)_A = O(x^{-1})_A$. Hence, any generators of a cyclic group have the same neutrosophic order in a neutrosophic subgroup. □

## 4 Neutrosophic normal subgroups

The concepts of neutrosophic normalizer and neutrosophic centralizer are developed in this section. We also look into a number of algebraic properties of it.

**Definition 4.1.** Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$. Then neutrosophic normalizer of $A$ is denoted by $\delta(A)$ and defined by $\delta(A) = \{m : x \in C, \mu_A(x) = \mu_A(mxx^{-1}), \gamma_A(x) = \gamma_A(mxx^{-1})$, and $\psi_A(x) = \psi_A(mxx^{-1})\}$ for all $x \in G$.

**Example 4.2.** Consider the group $C = (\mathbb{Z}, +)$. Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic set on $\mathbb{Z}$, which is presented by
$\mu_A(a) = \begin{cases} 0.87 & \text{when } a \in 2\mathbb{Z} \\ 0.62 & \text{elsewhere,} \end{cases}$ $\gamma_A(a) = \begin{cases} 0.87 & \text{when } a \in 2\mathbb{Z} \\ 0.62 & \text{elsewhere,} \end{cases}$ $\psi_A(a) = \begin{cases} 0.31 & \text{when } a \in 2\mathbb{Z} \\ 0.68 & \text{elsewhere.} \end{cases}$
We can clearly verify that $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup on $\mathbb{Z}$. Then the neutrosophic normalizer of $A$ is $\delta(A) = \mathbb{Z}$.

**Theorem 4.3.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a finite group $C$, then the neutrosophic normalizer $\delta(A)$ forms a subgroup of $C$.*

*Proof.* Let $x, y \in \delta(A)$. Then

$$(\forall p \in C) \begin{pmatrix} \mu_A(p) = \mu_A(xpx^{-1}) \\ \gamma_A(p) = \gamma_A(xpx^{-1}) \\ \psi_A(p) = \psi_A(xpx^{-1}) \end{pmatrix} \tag{4.1}$$

and

$$(\forall q \in C) \begin{pmatrix} \mu_A(q) = \mu_A(yqy^{-1}) \\ \gamma_A(q) = \gamma_A(yqy^{-1}) \\ \psi_A(q) = \psi_A(yqy^{-1}) \end{pmatrix}. \tag{4.2}$$

Clearly, $e \in \delta(A)$, so $\delta(A)$ is a non-empty finite subset of $C$. To show $\delta(A)$ is a subgroup of $C$, we need to show $xy \in \delta(A)$. Put $p = yqy^{-1}$ in (4.1), we get

$$\begin{pmatrix} \mu_A(yqy^{-1}) = \mu_A(xyqy^{-1}x^{-1}) \\ \gamma_A(yqy^{-1}) = \gamma_A(xyqy^{-1}x^{-1}) \\ \psi_A(yqy^{-1}) = \psi_A(xyqy^{-1}x^{-1}) \end{pmatrix}. \tag{4.3}$$

It follows from (4.2) and (4.3) that $\mu_A(q) = \mu_A(xyqy^{-1}x^{-1})$, $\gamma_A(q) = \gamma_A(xyqy^{-1}x^{-1})$ and $\mu_A(q) = \mu_A(xyqy^{-1}x^{-1})$. Hence, $\mu_A(q) = \mu_A((xy)q(xy)^{-1})$, $\gamma_A(q) = \gamma_A((xy)q(xy)^{-1})$ and $\mu_A(q) = \mu_A((xy)q(xy)^{-1})$. Therefore, $xy \in \delta(A)$. Hence, $\delta(A)$ forms a subgroup of $C$. □

**Proposition 4.4.** *Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$. Then $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of $C$ if and only if $\delta(A) = C$.*

*Proof.* We have $\delta(A) = \{m : x \in C, \mu_A(p) = \mu_A(mpx^{-1}), \gamma_A(x) = \gamma_A(mpx^{-1})$, and $\psi_A(p) = \psi_A(mpx^{-1})\}$ for all $p \in C$. Therefore, $\delta(A) \subset C$. Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup on $C$. Then $\mu_A(x) = \mu_A(yxy^{-1})$, $\gamma_A(x) = \gamma_A(yxy^{-1})$ and $\psi_A(x) = \psi_A(yxy^{-1})$ for all $x, y \in C$. This presents that $C \subset \delta(A)$. Hence, $\delta(A) = C$.

Conversely, let $\delta(A) = C$. Then $\mu_A(x) = \mu_A(yxy^{-1})$, $\gamma_A(x) = \gamma_A(yxy^{-1})$ and $\psi_A(x) = \psi_A(yxy^{-1})$ for all $x, y \in C$. Hence, $A = (\mu_A, \gamma_A, \psi_A)$ forms a neutrosophic normal subgroup on $C$. □

**Theorem 4.5.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$, then $A$ forms a neutrosophic normal subgroup on the group $\delta(A)$.*

*Proof.* Let $x, y \in \delta(A)$. Then $\mu_A(w) = \mu_A(mwx^{-1})$, $\gamma_A(w) = \gamma_A(mwx^{-1})$, and $\psi_A(w) = \psi_A(mwx^{-1})$ for all $w \in C$. As $\delta(A)$ forms a subgroup of $C$, then $nx \in \delta(A)$. Putting $w = yx$ in the above relation, we have $\mu_A(yx) = \mu_A(xymx^{-1})$, $\gamma_A(yx) = \gamma_A(xymx^{-1})$ and $\psi_A(yx) = \psi_A(xymx^{-1})$. This presents that $\mu_A(yx) = \mu_A(xy)$, $\gamma_A(yx) = \gamma_A(xy)$ and $\psi_A(yx) = \psi_A(xy)$. Hence, $A$ forms a neutrosophic normal subgroup on the group $\delta(A)$. □

**Definition 4.6.** Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$. Then neutrosophic centalizer of $A$ is denoted by $\omega(\psi)$ and defined by $\omega(\psi) = \{m : x \in C, \mu_A(xy) = \mu_A(yx), \gamma_A(xy) = \gamma_A(yx), \psi_A(xy) = \psi_A(yx)\}$ for all $y \in C$.

**Example 4.7.** From Example 3.19, consider the neutrosophic subgroup $A$ on the group $\mathbb{Z}_4$. Then the neutrosophic centralizer of $A$ is $\omega(A) = \mathbb{Z}_4$.

**Theorem 4.8.** *The neutrosophic centralizer of a neutrosophic subgroup of a group forms a subgroup of the group.*

*Proof.* Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of a group $C$. Then the neutrosophic centralizer of $A$ is given by $\omega(A) = \{m : x \in C, \mu_A(xy) = \mu_A(yx), \gamma_A(xy) = \gamma_A(yx), \psi_A(xy) = \psi_A(yx)\}$ for all $y \in C$. Let $s, t \in \omega(A)$. Then for all $r \in C$, we get $\mu_A((st)r) = \mu_A(s(tr)) = \mu_A((tr)s) = \mu_A(t(rs)) = \mu_A((rs)t) = \mu_A(r(st))$. Thus, $\mu_A((st)r) = \mu_A(r(st))$ for all $r \in C$. Similarly, we get $\gamma_A((st)r) = \gamma_A(r(st))$ for all $r \in C$ and $\psi_A((st)r) = \psi_A(r(st))$ for all $r \in C$. This gives that $st \in \omega(A)$. Also, for all $p \in C$, we get $\mu_A(s^{-1}p) = \mu_A((p^{-1}s)^{-1}) = \mu_A(p^{-1}s) = \mu_A(sp^{-1}) = \mu_A((ps^{-1})^{-1}) = \mu_A(ps^{-1})$. Thus, $\mu_A(s^{-1}p) = \mu_A(ps^{-1})$ for all $p \in C$. Similarly, we get $\gamma_A(s^{-1}p) = \gamma_A(ps^{-1})$ and $\psi_A(s^{-1}p) = \psi_A(ps^{-1})$ for all $p \in C$. Hence, for $s \in \omega(A)$, we have $s^{-1} \in \omega(A)$. Hence, $\omega(A)$ forms a subgroup of $C$. □

## 5 Lagrange's Theorem in neutrosophic subgroups

This section revolves around the development of theories for Lagrange's theorem in neutrosophic subgroups.

**Theorem 5.1.** *Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a finite group $C$ and $\Delta$ is the set of all neutrosophic cosets of $A$ on $C$. Then $\Delta$ constructs a group with the composition $xA \circ yA = (xy)_A$ for all $x, y \in C$.*

*Proof.* To prove $(\Delta, \circ)$ constructs a group with the composition $xA \circ yA = (xy)_A$ for all $x, y \in C$, we need to verify that $\circ$ is well defined. Let $m, n, p, q \in C$ such that $mA = pA$ and $nA = qA$. Therefore, $m\mu_A(x) = p\mu_A(x), m\gamma_A(x) = p\gamma_A(x), m\psi_A(x) = p\mu_A(x)$ and $n\mu_A(x) = q\mu_A(x), n\gamma_A(x) = q\gamma_A(x), n\psi_A(x) = q\psi_A(x)$ for all $x \in C$. This presents that for all $x \in C$

$$\mu_A(x^{-1}x) = \mu_A(p^{-1}x), \gamma_A(x^{-1}x) = \gamma_A(p^{-1}x), \psi_A(x^{-1}x) = \psi_A(p^{-1}x) \quad (5.1)$$

$$\mu_A(y^{-1}x) = \mu_A(q^{-1}x), \gamma_A(y^{-1}x) = \gamma_A(q^{-1}x, \psi_A(y^{-1}x) = \psi_A(q^{-1}x)). \quad (5.2)$$

We need to verify that $mA \circ nA = pA \circ qA$. So, $(xy)_A = (pq)_A$. We get $(xy)\mu_A(x) = \mu_A(y^{-1}x^{-1}x)$ and $(pq)\mu_A(x) = \mu_A(q^{-1}p^{-1}x)$ for all $x \in C$. Then

$$\begin{aligned} \mu_A(y^{-1}x^{-1}x) &= \mu_A(y^{-1}x^{-1}pp^{-1}x) \\ &= \mu_A(y^{-1}x^{-1}pqq^{-1}p^{-1}x) \\ &\geq \mu_A(y^{-1}x^{-1}pq) \wedge \mu_A(q^{-1}p^{-1}x). \end{aligned}$$

So,

$$\mu_A(y^{-1}x^{-1}x) \geq \mu_A(y^{-1}x^{-1}pq) \wedge \mu_A(q^{-1}p^{-1}x). \tag{5.3}$$

Replace $x$ with $x^{-1}pq$ in (5.2), then $\mu_A(y^{-1}x^{-1}pq) = \mu_A(q^{-1}x^{-1}pq)$. As $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup on $C$, then $\mu_A(q^{-1}x^{-1}pq) = \mu_A(x^{-1}p)$. Replace $x$ with $p$ in (5.1), we get $\mu_A(x^{-1}p) = \mu_A(p^{-1}p) = \mu_A(e)$. Consequently, $\mu_A(y^{-1}x^{-1}pq) = \mu_A(e)$. It follows from (5.3) that $\mu_A(y^{-1}x^{-1}x) \geq \mu_A(q^{-1}p^{-1}x)$. Similarly, $\mu_A(q^{-1}p^{-1}x) \geq \mu_A(y^{-1}x^{-1}x)$. Then $\mu_A(y^{-1}x^{-1}x) = \mu_A(q^{-1}p^{-1}x)$ for all $x \in C$. Also, we can verify that $\gamma_A(y^{-1}x^{-1}x) = \gamma_A(q^{-1}p^{-1}x)$ and $\psi_A(y^{-1}x^{-1}x) = \psi_A(q^{-1}p^{-1}x)$for all $x \in C$. This presents that $(xy)\mu_A(x) = (pq)\mu_A(x)$, $(xy)\gamma_A(x) = (pq)\gamma_A(x)$ and $(xy)\psi_A(x) = (pq)\psi_A(x)$ for all $x \in C$. Consequently, $(xy)_A = (pq)_A$. Hence, $\circ$ is well defined on $\Delta$. Clearly, $\Delta$'s identity element is $eA$. Also, $x^{-1}A \in \Delta$ is the inverse of $mA$ in $\Delta$. That is $(mA) \circ (x^{-1}A) = eA$. Therefore, $(\Delta, \circ)$ constructs a group with the composition $mA \circ nA = (xy)_A$ for all $x, y \in C$. □

**Definition 5.2.** The index of $A$ is denoted by $[C : A]$ and defined by $[C : A] = O(\Delta)$.

**Example 5.3.** Consider the group $C = (\mathbb{Z}_4, +_4)$. From Example 3.19, take the neutrosophic subgroup $A$ on $\mathbb{Z}_4$. We can clearly show that $A$ is a neutrosophic normal subgroup on the group $C = (\mathbb{Z}_4, +_4)$. Then the set of all neutrosophic cosets of $A$ is $\Delta = \{0A, 1A, 2A, 3A\}$. Now, $(1f)(1) = \mu_A(1^{-1} +_4 1) = \mu_A(3 +_4 1) = \mu_A(0) = 0.9025$, $(2f)(1) = \mu_A(2^{-1} +_4 1) = \mu_A(2 +_4 1) = \mu_A(3) = 0.7225$ and $(3f)(1) = \mu_A(3^{-1} +_4 1) = \mu_A(1 +_4 1) = \mu_A(2) = 0.4225$. Thus, $(1f)(1) \neq (2f)(1) \neq (3f)(1)$. This presents that $1A \neq 2A \neq 3A$. Therefore, the index of $A$ is $[C : A] = O(\Delta) = 4$.

**Theorem 5.4.** *Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a finite group $C$. Then a neutrosophic set $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ on $\Delta$ defined by $\mu^*_A(m\mu_A) = \mu_A(x)$, $\gamma^*_A(m\gamma_A) = \gamma_A(x)$ and $\psi^*_A(m\psi_A) = \psi_A(x)$ constructs a neutrosophic subgroup on $(\Delta, \circ)$ for all $x \in C$.*

*Proof.* Let $mA, nA \in \Delta$, where $x, y \in C$. Then $\mu^*_A((mf) \circ (nf)) = \mu^*_A((xy)f) = \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) = \mu^*_A(mf) \wedge \mu^*_A(nf)$. Therefore, $\mu^*_A((mf) \circ (nf)) \geq \mu^*_A(mf) \wedge \mu^*_A(nf)$. Similarly, we get $\gamma^*_A((mg) \circ (ng)) \geq \gamma^*_A(mg) \vee \gamma^*_A(ng)$ and $\psi^*_A((mf) \circ (nf)) \leq \psi^*_A(mf) \wedge \psi^*_A(nf)$. Also, $\mu^*_A(x^{-1}f) = \mu_A(x^{-1}) = \mu_A(x) = \mu^*_A(mf)$. Similarly, $\gamma^*_A(x^{-1}g) = \gamma^*_A(mg)$ and $\psi^*_A(x^{-1}g) = \psi^*_A(mg)$. Therefore, $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ constructs a neutrosophic subgroup on $(\Delta, \circ)$. □

**Definition 5.5.** The neutrosophic subgroup $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ on the group $(\Delta, \circ)$ is referred to as neutrosophic quotient group on $A$.

**Example 5.6.** From Example 5.3, consider the neutrosophic normal subgroup $A$ on the group $(\mathbb{Z}_4, +_4)$. Then the set of all neutrosophic cosets of $A$ is $\Delta = \{0A, 1A, 2A, 3A\}$. We create a neutrosophic set $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ on $\Delta$ by $\mu^*_A(mf) = f(x)$ and $f(mg) = f(x)$. Then $\mu^*_A(0f) = f(0) = 0.95, \mu^*_A(1f) = f(1) = 0.65, \mu^*_A(2f) = f(2) = 0.65, \mu^*_A(3f) = f(3) = 0.85, \gamma^*_A(0g) = f(0) = 0.25, \gamma^*_A(1g) = f(1) = 0.75, \gamma^*_A(2g) = f(2) = 0.75, \gamma^*_A(3g) = f(3) = 0.45$ and $\psi^*_A(0g) = f(0) = 0.25, \psi^*_A(1g) = f(1) = 0.75, \psi^*_A(2g) = f(2) = 0.75, \psi^*_A(3g) = f(3) = 0.45$. We can clearly verify that $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ is a neutrosophic subgroup on $\Delta$. Hence, $A^* = (\mu^*_A, \gamma^*_A, \psi^*_A)$ is the neutrosophic quotient group on $A$.

**Theorem 5.7.** *Assume $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a finite group $C$ and constructs a function $\kappa : C \to \Delta$ by $\kappa(x) = m\psi$ for all $x \in C$. Then $\kappa$ forms a group homomorphism with kernel* $\ker(\kappa) = \{x \in C : \mu_A(x) = \mu_A(e), \gamma_A(x) = \gamma_A(e), \psi_A(x) = \psi_A(e)\}$.

*Proof.* Let $x, y \in C$. Then $\kappa(xy) = (xy)_A = (m\psi) \circ (m\psi) = \kappa(x) \circ \kappa(y)$. This presents that $\kappa : C \to \Delta$ forms a group homomorphism. The kernel of $\kappa$ is given by

$$\begin{aligned} \ker(\kappa) &= \{x \in C : \kappa(x) = e\psi\} \\ &= \{x \in C : m\psi = e\psi\} \\ &= \{x \in C : m\psi(y) = e\psi(y) \ \forall\, y \in C\} \\ &= \{x \in C : mf(y) = ef(y), mf(y) = ef(y) \ \forall y \in C\} \\ &= \{x \in C : \mu_A(x^{-1}n) = \mu_A(y), \gamma_A(x^{-1}n) = \gamma_A(y), \psi_A(x^{-1}n) = \psi_A(y) \ \forall y \in C\} \\ &= \{x \in C : \mu_A(x) = \mu_A(e), \gamma_A(x) = \gamma_A(e), \psi_A(x) = \psi_A(e)\} \end{aligned}$$

Hence, $\ker(\kappa) = \{x \in C : \mu_A(x) = \mu_A(e), \gamma_A(x) = \gamma_A(e), \psi_A(x) = \psi_A(e)\}$. □

**Remark 5.8.** It is clear that $\ker(\kappa)$ forms a subgroup of $C$.

**Theorem 5.9.** *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic normal subgroup of a finite group C, then $[C : \psi]|O(C)$.*

*Proof.* $\Delta = \{m\psi : x \in C\}$, the set of all neutrosophic cosets of $A$ on $C$ is finite as $C$ is finite. Theorem 5.7 proves that $\kappa : C \to \Delta$ defined by $\kappa(x) = m\psi$ for all $x \in C$ is a group homomorphism. We define $H = \{x \in C : m\psi = e\psi\} = \ker(\kappa)$, which is a subgroup of $C$. Now, $C$ is now decomposed as union of left cosets modulo $p$ by $C = x_1H \cup x_2H \cup x_3H \cup \ldots \cup x_pH$, where $x_pH = H$. We need to verify that there exists a one-one relation between $\Delta$'s elements and cosets $x_iH$ of $C$. We consider an element $p \in H$ and coset $x_iH$ of $C$. Then $\kappa(x_ip) = x_ip\psi = (x_i\psi) \circ (p\psi) = (x_i\psi) \circ (e\psi) = (x_i\psi)$. This represents that $\kappa$ maps elements of $x_iH$ to $x_i\psi$. We now define a mapping $\kappa : \{x_iH : 1 \leq i \leq p\} \to \Delta$ by $\kappa(x_iH) = x_iA$. Let $m_a\psi = m_b\psi$. Then $x_b^{-1}m_a\psi = e\psi$. Therefore, $x_b^{-1}m_a \in H$. This presents that $m_aH = m_bH$. Hence, $\kappa(m_iH) = m_i\psi$ is a one-one map. As a result, we can establish that the number of distinct cosets is the same as $\Delta$'s cardinality. That is, $[C : H] = [C : \psi]$. Since $[C : H]|O(C)$, we have $[C : \psi]|O(C)$. □

## 6 Conclusion

This study extends the classical Lagrange's theorem to the neutrosophic domain, introducing novel concepts such as sub-level subgroups and neutrosophic indices. By defining algebraic structures through neutrosophic sets, this research provides a more nuanced framework for analyzing group properties in the presence of uncertainty. The results offer potential applications in areas where incomplete or indeterminate data necessitates a departure from traditional crisp set theory, opening avenues for future exploration in neutrosophic abstract algebra and its practical implications.

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## References

[1] N. Ajmal and A. S. Prajapati, Fuzzy cosets and fuzzy normal subgroups, Inf. Sci., 64(1) (1992), 17-25. https://doi.org/10.1016/0020-0255(92)90107-J

[2] A. Al-Odhari, Characteristics neutrosophic subgroups of axiomatic neutrosophic groups, Neutrosophic Optimization and Intelligent Systems, 3 (2024), 32-40. https://doi.org/10.61356/j.nois.2024.3265

[3] K. T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst., 20(1) (1986), 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3

[4] S. Bhunia, G. Ghorai and Q. Xin, On the characterization of neutrosophic subgroups, AIMS Math., 6(1) (2021), 962-978. https://doi.org/10.3934/math.2021058

[5] S. Bhunia, G. Ghorai and Q. Xin. On the fuzzification of Lagrange's theorem in $(\alpha, \beta)$-Pythagorean fuzzy environment, AIMS Math., 6(9) (2021), 9290-9308. https://doi.org/10.3934/math.2021540

[6] S. Bhunia and G. Ghorai, An approach to Lagrange's theorem in Pythagorean fuzzy subgroups, Kragujevac J. Math., 48(6) (2024), 893-906.

[7] V. Cetkin and H. Aygun, An approach to neutrosophic subgroup and its fundamental properties, J. Intell. Fuzzy Syst., 29 (2015), 1941-1947. https://doi.org/10.3233/IFS-151672

[8] A. Iampan, C. Sivakumar and N. Rajesh, Neutrosophic subgroups and neutrosophic normal subgroups of groups, Int. J. Neutrosophic Sci., 26(1) (2025), 283-292. https://doi.org/10.54216/IJNS.260124

[9] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, Fuzzy Group Theory, Springer-Verlag, New York, 2005.

[10] A. Rosenfeld, Fuzzy Groups, J. Math. Anal. Appl. 35(3) (1971), 512-517. https://doi.org/10.1016/0022-247X(71)90199-5

[11] R. L. Roth, A history of Lagrange's theorem on groups, Math. Mag., 74(2) (2001), 99-108. https://doi.org/10.1080/0025570X.2001.11953045

[12] F. Smarandache, A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics (fourth edition), American Research Press, Rehoboth, 2005.

[13] S. Thiruveni and A. Solairaju, Neutrosophic Q-fuzzy subgroups, Int. J. Math. And Appl., 6(1) (2018), 859-866.

[14] R. R. Yager, Pythagorean fuzzy subsets, 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 2013, 57-61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[15] L. A. Zadeh, Fuzzy sets, Inf. Control, 8(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X