Continuous Maps and Irresolute Maps Via Fuzzy Hypersoft θ Open set

P. Revathi[1,2], **B. Premamalini**[2,*], **K. Chitirakala**[3,*], **A. Vadivel**[2,4]
[1]Government Polytechnic College, Kuduveli, Chidambaram - 608 305, India
[2]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India
[3]Department of Mathematics, M.Kumarasamy College of Engineering, Karur - 639 113, India
[4]PG and Research Department of Mathematics, Arignar Anna Government Arts College, Namakkal - 637 002, India

Emails: revathimathsau@gmail.com; premamalinips@gmail.com; chitrakalalaksana@gmail.com; avmaths@gmail.com

Abstract

The purpose of this paper is to introduce and study fuzzy hypersoft θ continuous maps, fuzzy hypersoft θ semi continuous maps, fuzzy hypersoft θ pre continuous maps and fuzzy hypersoft θ irresolute maps in fuzzy hypersoft topological spaces with examples. Further, we derived some useful results and properties related to them.

Keywords: Fuzzy hypersoft θ continuous maps; Fuzzy hypersoft θ semi continuous maps; Fuzzy hypersoft θ pre continuous maps; fuzzy hypersoft θ irresolute maps

1 Introduction

The real-world decision-making problems in medical diagnosis, engineering, economics, management computer science, artificial intelligence, social sciences, environmental science and sociology contain more uncertain and inadequate data. Traditional mathematical methods cannot deal with these kinds of problems due to imprecise data. To deal with the problems with uncertainty, Zadeh[16] introduced the fuzzy set in 1965 which contains the membership value in [0,1]. A fuzzy set is a set where each element of the universe belongs to it but with some value or degree of belongingness which lies between 0 and 1 and such values are called the membership value of an element in that set. The topological structure on fuzzy set was undertaken by Chang[7] as fuzzy topological space. Molodstov[9] introduced a new mathematical tool, soft set theory in 1999 to deal with uncertainties in which a soft set is a collection of approximate descriptions of an object. A soft set is a parameterized family of subsets where parameters are the properties, attributes or characteristics of the objects. The soft set theory has several applications in different fields such as decision-making, optimization, forecasting, data analysis etc. Shabir and Naz[13] presented soft topological spaces.

Smarandache[14] extended the notion of a soft set to a hypersoft set and then to plithogenic set by replacing a function with a multi-argument function described in the cartesian product with a different set of attributes. This new concept of hypersoft set is more flexible than the soft set and more suitable in decision-making issues involving a different kinds of attributes. Saeed et al.[10,11] studied the fundamentals of hypersoft set theory by introducing aggregate operators, relations, functions, matrices and operations on hypersoft matrices. Abbas et al.[1] defined the basic operations on hypersoft sets and hypersoft point in the fuzzy, intuitionistic and neutrosophic environments. Ajay and Charisma[3] introduced fuzzy hypersoft topology, intuitionistic hypersoft topology and neutrosophic hypersoft topology. Neutrosophic hypersoft topology is the generalized framework which generalizes intuitionistic hypersoft topology and fuzzy hypersoft topology. Ajay et al.[4] defined fuzzy hypersoft semi-open sets and developed an application in multiattribute group decision-making. The concept of contra continuous function in general topology was introduced by Dontchev[8] in 1996. Vadivel et al.[?] introduced generalized fuzzy contra e-continuous functions in fuzzy topological spaces. Ahsan et al.[2] studied a theoretical and analytical approach for fundamental framework of composite mappings on fuzzy hypersoft classes.

Saha[12] defined δ-open sets and continuous maps in fuzzy topological spaces. The δ-open sets, e-open sets in neutrosophic, neutrosophic soft, fuzzy hypersoft, neutrosophic hypersoft topological spaces are introduced and its maps, separation axioms and compact spaces are studied recently.

The class of sets namely, θ open sets are playing more important role in topological spaces, because of their applications in various fields of Mathematics and other real fields. In 1968 Velicko[15] defined θ open set in H-closed Topological Spaces. In,[5,6] Caldas et al. studied various kinds of θ open sets and their properties in topological spaces. The concept of θ-open sets in fuzzy hypersoft topological spaces is introdued recently.

In this paper, we develop the concept of fuzzy hypersoft θ continuity in fuzzy hypersoft topological spaces and some of their properties are analyzed with examples. Added to that, fuzzy hypersoft θ semi continuous maps, fuzzy hypersoft θ pre continuous maps and fuzzy hypersoft θ irresolute maps are developed and the relation between them are discussed.

2 Preliminaries

Definition 2.1. [16] Let $\mathfrak{M}$ be an initial universe. A function λ from $\mathfrak{M}$ into the unit interval I is called a fuzzy set in $\mathfrak{M}$. For every $\mathfrak{m} \in \mathfrak{M},\ \lambda(\mathfrak{m}) \in I$ is called the grade of membership of $\mathfrak{m}$ in λ. Some authors say that λ is a fuzzy subset of $\mathfrak{M}$ instead of saying that λ is a fuzzy set in $\mathfrak{M}$. The class of all fuzzy sets from $\mathfrak{M}$ into the closed unit interval I will be denoted by $I^{\mathfrak{M}}$.

Definition 2.2. [9] Let $\mathfrak{M}$ be an initial universe, Q be a set of parameters and $\mathcal{P}(\mathfrak{M})$ be the power set of $\mathfrak{M}$. A pair $(\tilde{H}, Q)$ is called the a soft set over $\mathfrak{M}$ where $\tilde{H}$ is a mapping $\tilde{H} : Q \to \mathcal{P}(\mathfrak{M})$. In other words, the soft set is a parametrized family of subsets of the set $\mathfrak{M}$.

Definition 2.3. [14] Let $\mathfrak{M}$ be an initial universe and $\mathcal{P}(\mathfrak{M})$ be the power set of $\mathfrak{M}$. Consider $q_1, q_2, q_3, ..., q_n$ for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $Q_1,\ Q_2,\ ...,\ Q_n$ with $Q_i \cap Q_j = \emptyset$, for $i \neq j$ and $i, j \in \{1, 2, ..., n\}$. Then the pair $(\tilde{H},\ Q_1 \times Q_2 \times ... \times Q_n)$ where $\tilde{H} : Q_1 \times Q_2 \times ... \times Q_n \to \mathcal{P}(\mathfrak{M})$ is called a hypersoft set over $\mathfrak{M}$.

Definition 2.4. [1] Let $\mathfrak{M}$ be an initial universal set and $Q_1, Q_2, ..., Q_n$ be pairwise disjoint sets of parameters. Let $\mathcal{P}(\mathfrak{M})$ be the set of all fuzzy sets of $\mathfrak{M}$. Let E_i be the nonempty subset of the pair Q_i for each $i = 1, 2, ..., n$. A fuzzy hypersoft set (briefly, $FHSs$) over $\mathfrak{M}$ is defined as the pair $(\tilde{H}, E_1 \times E_2 \times ... \times E_n)$ where $\tilde{H} : E_1 \times E_2 \times ... \times E_n \to \mathcal{P}(\mathfrak{M})$ and $\tilde{H}(E_1 \times E_2 \times ... \times E_n) = \{(q, \langle \mathfrak{m}, \mu_{\tilde{H}(q)}(\mathfrak{m})\rangle : \mathfrak{m} \in \mathfrak{M}) : q \in E_1 \times E_2 \times ... \times E_n \subseteq Q_1 \times Q_2 \times ... \times Q_n\}$ where $\mu_{\tilde{H}(q)}(\mathfrak{m})$ is the membership value such that $\mu_{\tilde{H}(q)}(\mathfrak{m}) \in [0, 1]$.

Definition 2.5. [1] Let $\mathfrak{M}$ be an universal set and $(\tilde{H}, \wedge_1)$ and $(\tilde{G}, \wedge_2)$ be two $FHSs$'s over $\mathfrak{M}$. Then $(\tilde{H}, \wedge_1)$ is the fuzzy hypersoft subset of $(\tilde{G}, \wedge_2)$ if $\mu_{\tilde{H}(q)}(\mathfrak{m}) \leq \mu_{\tilde{G}(q)}(\mathfrak{m})$.

It is denoted by $(\tilde{H}, \wedge_1) \subseteq (\tilde{G}, \wedge_2)$.

Definition 2.6. [1] Let $\mathfrak{M}$ be an universal set and $(\tilde{H}, \wedge_1)$ and $(\tilde{G}, \wedge_2)$ be $FHSs$'s over $\mathfrak{M}$. $(\tilde{H}, \wedge_1)$ is equal to $(\tilde{G}, \wedge_1)$ if $\mu_{\tilde{H}(q)}(\mathfrak{m}) = \mu_{\tilde{G}(q)}(\mathfrak{m})$.

Definition 2.7. [1] A $FHSs$ $(\tilde{H}, \wedge)$ over the universe set $\mathfrak{M}$ is said to be null fuzzy hypersoft set if $\mu_{\tilde{H}(q)}(\mathfrak{m}) = 0, \forall q \in \wedge$ and $\mathfrak{m} \in \mathfrak{M}$. It is denoted by $\tilde{0}_{(\mathfrak{M},Q)}$.

A $FHSs$ $(\tilde{G}, \wedge)$ over the universal set $\mathfrak{M}$ is said to be absolute fuzzy hypersoft set if $\mu_{\tilde{H}(q)}(\mathfrak{m}) = 1\ \forall q \in \wedge$ and $\mathfrak{m} \in \mathfrak{M}$. It is denoted by $\tilde{1}_{(\mathfrak{M},Q)}$.

Clearly, $\tilde{0}^c_{(\mathfrak{M},Q)} = \tilde{1}_{(\mathfrak{M},Q)}$ and $\tilde{1}^c_{(\mathfrak{M},Q)} = \tilde{0}_{(\mathfrak{M},Q)}$.

Definition 2.8. [1] Let $\mathfrak{M}$ be an universal set and $(\tilde{H}, \wedge)$ be $FHSs$ over $\mathfrak{M}$. $(\tilde{H}, \wedge)^c$ is the complement of $(\tilde{H}, \wedge)$ if $\mu^c_{\tilde{H}(q)}(\mathfrak{m}) = \tilde{1}_{(\mathfrak{M},Q)} - \mu_{\tilde{H}(q)}(\mathfrak{m})$ where $\forall q \in \wedge$ and $\forall \mathfrak{m} \in \mathfrak{M}$. It is clear that $((\tilde{H}, \wedge)^c)^c = (\tilde{H}, \wedge)$.

Definition 2.9. [1] Let $\mathfrak{M}$ be the universal set and $(\tilde{H}, \wedge_1)$ and $(\tilde{G}, \wedge_2)$ be $FHSs$'s over $\mathfrak{M}$. Extended union $(\tilde{H}, \wedge_1) \cup (\tilde{G}, \wedge_2)$ is defined as

$$\mu((\tilde{H}, \wedge_1) \cup (\tilde{G}, \wedge_2)) = \begin{cases} \mu_{\tilde{H}(q)}(\mathfrak{m}) & \text{if } q \in \wedge_1 - \wedge_2 \\ \mu_{\tilde{G}(q)}(\mathfrak{m}) & \text{if } q \in \wedge_2 - \wedge_1 \\ max\{\mu_{\tilde{H}(q)}(\mathfrak{m}), \mu_{\tilde{G}(q)}(\mathfrak{m})\} & \text{if } q \in \wedge_1 \cap \wedge_2 \end{cases}$$

Definition 2.10. [1,3] Let $\mathfrak{M}$ be the universal set and $(\tilde{H}, \wedge_1)$ and $(\tilde{G}, \wedge_2)$ be $FHSs$'s over $\mathfrak{M}$. Extended intersection $(\tilde{H}, \wedge_1) \cap (\tilde{G}, \wedge_2)$ is defined as

$$\mu((\tilde{H}, \wedge_1) \cap (\tilde{G}, \wedge_2)) = \begin{cases} \mu_{\tilde{H}(q)}(\mathfrak{m}) & ifq \in \wedge_1 - \wedge_2 \\ \mu_{\tilde{G}(q)}(\mathfrak{m}) & ifq \in \wedge_2 - \wedge_1 \\ min\{\mu_{\tilde{H}(q)}(\mathfrak{m}), \mu_{\tilde{G}(q)}(\mathfrak{m})\} & ifq \in \wedge_1 \cap \wedge_2 \end{cases}$$

Definition 2.11. [3] Let $(\mathfrak{M}, Q)$ be the family of all $FHSs$'s over the universe set $\mathfrak{M}$ and $\tau \subseteq FHSs(\mathfrak{M}, Q)$. Then τ is said to be a fuzzy hypersoft topology (briefly, $FHSt$) on $\mathfrak{M}$ if

(i) $\tilde{0}_{(\mathfrak{M},Q)}$ and $\tilde{1}_{(\mathfrak{M},Q)}$ belongs to τ

(ii) the union of any number of $FHSs$'s in τ belongs to τ

(iii) the intersection of finite number of $FHSs$'s in τ belongs to τ.

Then $(\mathfrak{M}, Q, \tau)$ is called a fuzzy hypersoft toplogical space (briefly, $FHSts$) over $\mathfrak{M}$. Each member of τ is said to be fuzzy hypersoft open set (briefly, $FHSos$). A $FHSs$ $(\tilde{H}, \wedge)$ is called a fuzzy hypersoft closed set (briefly, $FHScs$) if its complement $(\tilde{H}, \wedge)^c$ is $FHSos$.

Definition 2.12. [3] Let $(\mathfrak{M}, Q, \tau)$ be a $FHSts$ over $\mathfrak{M}$ and $(\tilde{H}, \wedge)$ be a $FHSs$ in $\mathfrak{M}$. Then,

(i) the fuzzy hypersoft interior (briefly, $FHSint$) of $(\tilde{H}, \wedge)$ is defined as $FHSint(\tilde{H}, \wedge) = \cup\{(\tilde{G}, \wedge) : (\tilde{G}, \wedge) \subseteq (\tilde{H}, \wedge)$ where $(\tilde{G}, \wedge)$ is $FHSos\}$.

(ii) the fuzzy hypersoft closure (briefly, $FHScl$) of $(\tilde{H}, \wedge)$ is defined as $FHScl(\tilde{H}, \wedge) = \cap\{(\tilde{G}, \wedge) : (\tilde{G}, \wedge) \supseteq (\tilde{H}, \wedge)$ where $(\tilde{G}, \wedge)$ is $FHScs\}$.

Definition 2.13. [4] Let $(\mathfrak{M}, Q, \tau)$ be a $FHSts$ over $\mathfrak{M}$ and $(\tilde{H}, \wedge)$ be a $FHSs$ in $\mathfrak{M}$. Then, $(\tilde{H}, \wedge)$ is called the fuzzy hypersoft semiopen set (briefly, $FHSSos$) if $(\tilde{H}, \wedge) \subseteq FHScl(int(\tilde{H}, \wedge))$.

A $FHSs$ $(\tilde{H}, \wedge)$ is called a fuzzy hypersoft semiclosed set (briefly, $FHSScs$) if its complement $(\tilde{H}, \wedge)^c$ is a $FHSSos$.

Definition 2.14. [2] Let $(\mathfrak{M}, L)$ and $(\mathfrak{N}, M)$ be classes of $FHSs$'s over $\mathfrak{M}$ and $\mathfrak{N}$ with attributes L and M respectively. Let $\omega : \mathfrak{M} \to \mathfrak{N}$ and $\nu : L \to M$ be mappings. Then a FHS mappings $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ is defined as follows, for a $FHSs$ $(\tilde{H}, \wedge)_A$ in $(\mathfrak{M}, L)$, $f(\tilde{H}, \wedge)_A$ is a $FHSs$ in $(\mathfrak{N}, M)$ obtained as follows, for $\beta \in \nu(L) \subseteq M$ and $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{h}(\tilde{H}, \wedge)_A(\beta)(\mathfrak{n}) = \bigcup_{\alpha \in \nu^{-1}(\beta) \bigcap A, s \in \omega^{-1}(\mathfrak{n})} (\alpha)\mu_s \mathfrak{h}(\tilde{H}, \wedge)_A$ is called a fuzzy hypersoft image of a $FHSs$ $(\tilde{H}, \wedge)$. Hence $((\tilde{H}, \wedge)_A, \mathfrak{h}(\tilde{H}, \wedge)_A) \in \mathfrak{h}$, where $(\tilde{H}, \wedge)_A \subseteq (\mathfrak{M}, L), \mathfrak{h}(\tilde{H}, \wedge)_A \subseteq (\mathfrak{N}, M)$.

Definition 2.15. [2] If $\mathfrak{h} : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping, then FHS class $(\mathfrak{M}, L)$ is called the domain of $\mathfrak{h}$ and the FHS class $(\tilde{G}, \wedge) \in (\mathfrak{N}, M) : (\tilde{G}, \wedge) = \mathfrak{h}(\tilde{H}, \wedge))$ for some $(\tilde{H}, \wedge) \in (\mathfrak{M}, L)$ is called the range of $\mathfrak{h}$. The FHS class $(\mathfrak{N}, M)$ is called co-domain of $\mathfrak{h}$.

Definition 2.16. [2] If $\mathfrak{h} : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping and $(\tilde{G}, \wedge)_B$, a $FHSs$ in $(\mathfrak{N}, M)$ where $\omega : \mathfrak{M} \to \mathfrak{N}$, $\nu : L \to M$ and $B \subseteq M$. Then $\mathfrak{h}^{-1}(\tilde{G}, \wedge)_B$ is a $FHSs$ in $(\mathfrak{M}, L)$ defined as follows, for $\alpha \in \nu^{-1}(B) \subseteq L$ and $\mathfrak{m} \in \mathfrak{M}$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)_B(\alpha)(\mathfrak{m}) = (\nu(\alpha))\mu_p(\mathfrak{m})\mathfrak{h}^{-1}(\tilde{G}, \wedge)_B$ is called a FHS inverse image of $(\tilde{G}, \wedge)_B$.

Definition 2.17. [2] Let $\mathfrak{h} = (\omega, \nu)$ be a FHS mapping of a FHS class $(\mathfrak{M}, L)$ into a FHS class $(\mathfrak{N}, M)$. Then

(i) $\mathfrak{h}$ is said to be a one-one (or injection) FHS mapping if for both $\omega : \mathfrak{M} \to \mathfrak{N}$ and $\nu : L \to M$ are one-one.

(ii) $\mathfrak{h}$ is said to be a onto (or surjection) FHS mapping if for both $\omega : \mathfrak{M} \to \mathfrak{N}$ and $\nu : L \to M$ are onto.

If $\mathfrak{h}$ is both one-one and onto, then $\mathfrak{h}$ is called a one-one onto (or bijective) correspondance of FHS mapping.

Definition 2.18. [2] If $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ and $g = (m, n) : (\mathfrak{N}, M) \to (P, N)$ are two FHS mappings, then their composite $\mathfrak{g} \circ \mathfrak{h}$ is a FHS mapping of $(\mathfrak{M}, L)$ into (P, N) such that for every $(\tilde{H}, \wedge)_A \in (\mathfrak{M}, L)$, $(\mathfrak{g} \circ \mathfrak{h})(\tilde{H}, \wedge)_A = \mathfrak{g}(\mathfrak{h}(\tilde{H}, \wedge)_A)$. For $\beta \in n(M) \subseteq N$ and $p \in P$, it is defined as $\mathfrak{g}(\mathfrak{h}(\tilde{H}, \wedge)_A(\beta)(p) = \bigcup_{\alpha \in n^{-1}(\beta) \bigcap \mathfrak{h}(A), s \in m^{-1}(p)} (\alpha)\mu_s$.

Definition 2.19. [2] Let $\mathfrak{h} = (\omega, \nu)$ be a FHS mapping where $\omega : \mathfrak{M} \to \mathfrak{M}$ and $\nu : L \to L$. Then $\mathfrak{h}$ is said to be a FHS identity mapping if for both ω and ν are identity mappings.

Definition 2.20. [2] A one-one onto FHS mapping $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ is called FHS invertable mapping. Its FHS inverse mapping is denoted by $\mathfrak{h}^{-1} = (\omega^{-1}, \nu^{-1}) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$.

3 Fuzzy Hypersoft θ Continuous Maps

In this section, fuzzy hypersoft θ continuous maps are introduced and its related properties are discusssed.

Definition 3.1. Consider any two FHSts $(\mathfrak{M}, L, \tau)$ and $(\mathfrak{N}, M, \sigma)$. A map $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ is said to be $FHS\theta$ (resp. $\theta\mathcal{S}$, $\delta\mathcal{S}$, $\delta\mathcal{P}$, $\theta\mathcal{P}$) continuous (in short, $FHS\theta Cts$ (resp. $FHS\theta\mathcal{S}Cts$, $FHS\delta\mathcal{S}Cts$, $FHS\delta\mathcal{P}Cts$ & $FHS\theta\mathcal{P}Cts$)) if the inverse image of each $FHSos$ in $(\mathfrak{N}, M, \sigma)$ is a $FHS\theta os$ (resp. $FHS\theta\mathcal{S}os$, $FHS\delta\mathcal{S}os$, $FHS\delta\mathcal{P}os$ & $FHS\theta\mathcal{P}os$) in $(\mathfrak{M}, L, \tau)$.

Example 3.2. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H}_1, \wedge_1)$, $(\tilde{H}_2, \wedge_1)$, $(\tilde{H}_3, \wedge_3)$, $(\tilde{H}_4, \wedge_3)$, $(\tilde{H}_5, \wedge_3)$, $(\tilde{H}_6, \wedge_3)$, $(\tilde{H}_7, \wedge_3)$, $(\tilde{H}_8, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H}_1, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_2, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_3, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.9}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_4, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.9}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_5}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_6}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.9}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_7}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_8}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.9}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.6}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_1), (\tilde{H_3}, \wedge_2), (\tilde{H_4}, \wedge_3), (\tilde{H_5}, \wedge_3), (\tilde{H_6}, \wedge_3), (\tilde{H_7}, \wedge_3), (\tilde{H_8}, \wedge_3), \}$ is $FHSts$.

Let the $FHSs's$ $(\tilde{G_1}, \wedge_1), (\tilde{G_2}, \wedge_1)$ over the universe $\mathfrak{N}$ be

$$(\tilde{G_1}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\} \rangle \end{Bmatrix}$$

$$(\tilde{G_2}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.3}, \frac{\mathfrak{n}_2}{0.2}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.3}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_1), (\tilde{G_2}, \wedge_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$
$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_1) = (\tilde{H_1}, \wedge_1), \mathfrak{h}^{-1}(\tilde{G_2}, \wedge_1) = (\tilde{H_2}, \wedge_1),$$

$\therefore \mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ is $FHS\theta Cts$.

Proposition 3.3. The statements hold but the converse is not.

(i) Each $FHS\theta Cts$ is a $FHS\theta SCts$.

(ii) Each $FHS\theta Cts$ is a $FHSCts$.

(iii) Each $FHS\theta SCts$ is a $FHSSCts$.

(iv) Each $FHSCts$ is a $FHS\mathcal{P}Cts$.

(v) Each $FHSSCts$ is a $FHS\theta\mathcal{P}Cts$.

(vi) Each $FHS\mathcal{P}Cts$ is a $FHS\theta\mathcal{P}Cts$.

Proof.

Consider the map $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$

(i) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHS\theta Cts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta os$ in $\mathfrak{M}$. Since all $FHS\theta os$ are $FHS\theta Sos$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta Sos$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHS\theta SCts$.

(ii) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHS\theta Cts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta os$ in $\mathfrak{M}$. Since all $FHS\theta os$ are $FHSos$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHSos$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHSCts$.

(iii) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHS\theta SCts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta Sos$ in $\mathfrak{M}$. Since all $FHS\theta Sos$ are $FHSSos$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHSSos$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHSSCts$.

(iv) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHSCts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHSos$ in $\mathfrak{M}$. Since all $FHSos$ are $FHS\mathcal{P}os$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\mathcal{P}os$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHS\mathcal{P}Cts$.

(v) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHSSCts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHSSos$ in $\mathfrak{M}$. Since all $FHSSos$ are $FHS\theta\mathcal{P}os$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta\mathcal{P}os$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHS\theta\mathcal{P}Cts$.

(vi) Let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. As $\mathfrak{h}$ is $FHS\mathcal{P}Cts$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\mathcal{P}os$ in $\mathfrak{M}$. Since all $FHS\mathcal{P}os$ are $FHS\theta\mathcal{P}os$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta\mathcal{P}os$ in $\mathfrak{M}$. Thus $\mathfrak{h}$ is a $FHS\theta\mathcal{P}Cts$.

Example 3.4. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H}_1, \wedge_1)$, $(\tilde{H}_2, \wedge_2)$, $(\tilde{H}_3, \wedge_2)$, $(\tilde{H}_4, \wedge_3)$, $(\tilde{H}_5, \wedge_3)$, $(\tilde{H}_6, \wedge_3)$, $(\tilde{H}_7, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H}_1, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_2, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_3, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_4, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_5, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_6, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_7, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H}_1, \wedge_1), (\tilde{H}_2, \wedge_2), (\tilde{H}_3, \wedge_2), (\tilde{H}_4, \wedge_3), (\tilde{H}_5, \wedge_3), (\tilde{H}_6, \wedge_3), (\tilde{H}_7, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G}_1, \wedge_3)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_6}, \wedge_3)$$

$\mathfrak{h}$ is $FHS\theta\mathcal{S}Cts$ but not $FHS\theta Cts$ because $(\tilde{G_1}, \wedge_3)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_6}, \wedge_3)$ is a $FHS\theta Sos$ but not $FHS\theta os$.

Example 3.5. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1)$, $(\tilde{H_2}, \wedge_2)$, $(\tilde{H_3}, \wedge_2)$, $(\tilde{H_4}, \wedge_3)$, $(\tilde{H_5}, \wedge_3)$, $(\tilde{H_6}, \wedge_3)$, $(\tilde{H_7}, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_3}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_4}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_5}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_6}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_7}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_2), (\tilde{H_3}, \wedge_2), (\tilde{H_4}, \wedge_3), (\tilde{H_5}, \wedge_3), (\tilde{H_6}, \wedge_3), (\tilde{H_7}, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G_1}, \wedge_1)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_1) = \begin{Bmatrix} \langle(c_2, d_1), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.8}\}\rangle, \\ \langle(c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\}\rangle, \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \rightarrow (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_1) = (\tilde{H_1}, \wedge_1)$$

$\mathfrak{h}$ is $FHSCts$ but not $FHS\theta Cts$ because $(\tilde{G_1}, \wedge_1)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_1) = (\tilde{H_1}, \wedge_1)$ is a $FHSos$ but not $FHS\theta os$.

Example 3.6. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1)$, $(\tilde{H_2}, \wedge_2)$, $(\tilde{H_3}, \wedge_2)$, $(\tilde{H_4}, \wedge_3)$, $(\tilde{H_5}, \wedge_3)$, $(\tilde{H_6}, \wedge_3)$, $(\tilde{H_7}, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle(a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_2) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_3}, \wedge_2) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_4}, \wedge_3) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle(a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_5}, \wedge_3) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle(a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_6}, \wedge_3) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle(a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_7}, \wedge_3) = \begin{Bmatrix} \langle(a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle(a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle(a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_2), (\tilde{H_3}, \wedge_2), (\tilde{H_4}, \wedge_3), (\tilde{H_5}, \wedge_3), (\tilde{H_6}, \wedge_3), (\tilde{H_7}, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G_1}, \wedge_3)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \rightarrow (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_4}, \wedge_3)$$

$\mathfrak{h}$ is $FHS\mathcal{S}Cts$ but not $FHS\theta\mathcal{S}Cts$ because $(\tilde{G_1}, \wedge_3)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_4}, \wedge_3)$ is a $FHSSos$ but not $FHS\theta Sos$.

Example 3.7. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1)$, $(\tilde{H_2}, \wedge_2)$, $(\tilde{H_3}, \wedge_2)$, $(\tilde{H_4}, \wedge_3)$, $(\tilde{H_5}, \wedge_3)$, $(\tilde{H_6}, \wedge_3)$, $(\tilde{H_7}, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_3}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_4}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_5}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_6}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_7}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_8}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H}_1, \wedge_1), (\tilde{H}_2, \wedge_2), (\tilde{H}_3, \wedge_2), (\tilde{H}_4, \wedge_3), (\tilde{H}_5, \wedge_3), (\tilde{H}_6, \wedge_3), (\tilde{H}_7, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G}_1, \wedge_3)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G}_1, \wedge_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.3}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G}_1, \wedge_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G}_1, \wedge_3) = (\tilde{H}_8, \wedge_3)$$

$\mathfrak{h}$ is $FHS\mathcal{P}Cts$ but not $FHSCts$ because $(\tilde{G}_1, \wedge_3)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G}_1, \wedge_3) = (\tilde{H}_8, \wedge_3)$ is a $FHS\mathcal{P}os$ but not $FHSos$.

Example 3.8. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q_1' \times Q_2'$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q_1' = \{c_1, c_2, c_3\}, Q_2' = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H}_1, \wedge_1)$, $(\tilde{H}_2, \wedge_2)$, $(\tilde{H}_3, \wedge_2)$, $(\tilde{H}_4, \wedge_3)$, $(\tilde{H}_5, \wedge_3)$, $(\tilde{H}_6, \wedge_3)$, $(\tilde{H}_7, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H}_1, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_2, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_3, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_4, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_5, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_6, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_7, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H}_8, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_2), (\tilde{H_3}, \wedge_2), (\tilde{H_4}, \wedge_3), (\tilde{H_5}, \wedge_3), (\tilde{H_6}, \wedge_3), (\tilde{H_7}, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G_1}, \wedge_1)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.3}, \frac{\mathfrak{n}_2}{0.2}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.3}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_3)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$
$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_8}, \wedge_3)$$

$\mathfrak{h}$ is $FHS\theta\mathcal{P}Cts$ but not $FHSSCts$ because $(\tilde{G_1}, \wedge_3)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_8}, \wedge_3)$ is a $FHS\theta\mathcal{P}os$ but not $FHSSos$.

Example 3.9. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q_1' \times Q_2'$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q_1' = \{c_1, c_2, c_3\}, Q_2' = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1)$, $(\tilde{H_2}, \wedge_2)$, $(\tilde{H_3}, \wedge_2)$, $(\tilde{H_4}, \wedge_3)$, $(\tilde{H_5}, \wedge_3)$, $(\tilde{H_6}, \wedge_3)$, $(\tilde{H_7}, \wedge_3)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_3}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_4}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_5}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_6}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_7}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_8}, \wedge_3) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.4}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\}\rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_2), (\tilde{H_3}, \wedge_2), (\tilde{H_4}, \wedge_3), (\tilde{H_5}, \wedge_3), (\tilde{H_6}, \wedge_3), (\tilde{H_7}, \wedge_3)\}$ is $FHSts$.

Let the $FHSs$ $(\tilde{G_1}, \wedge_3)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_3) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.2}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.3}\}\rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_3)\}$ is $FHSts$.

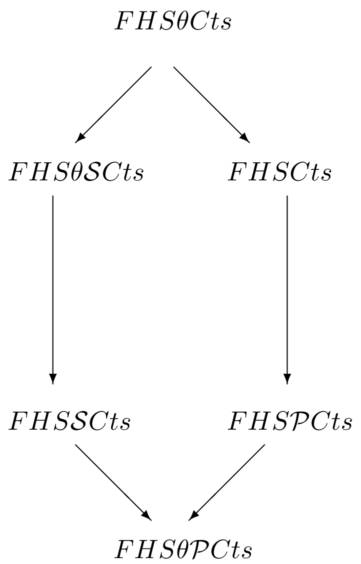
Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_8}, \wedge_3)$$

$\mathfrak{h}$ is $FHS\theta\mathcal{P}Cts$ but not $FHS\mathcal{P}Cts$ because $(\tilde{G_1}, \wedge_3)$ is $FHSos$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_3) = (\tilde{H_8}, \wedge_3)$ is a $FHS\theta\mathcal{P}os$ but not $FHS\mathcal{P}os$.

Remark 3.10. From the results discussed above, the following diagram is obtained.



Theorem 3.11. *A map $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ is $FHS\theta Cts$ iff the inverse image of each $FHScs$ in $\mathfrak{N}$ is $FHS\theta cs$ in $\mathfrak{M}$.*

Proof. Let $(\tilde{G}, \wedge)$ be a $FHScs$ in $\mathfrak{N}$. This implies that $(\tilde{G}, \wedge)^c$ is $FHSos$ in $\mathfrak{N}$. Since $\mathfrak{h}$ is $FHS\theta Cts$, $\mathfrak{h}^{-1}((\tilde{G}, \wedge)^c)$ is $FHS\theta os$ in $\mathfrak{M}$. Since $\mathfrak{h}^{-1}((\tilde{G}, \wedge)^c) = (\mathfrak{h}^{-1}(\tilde{G}, \wedge))^c$, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is a $FHS\theta cs$ in $\mathfrak{M}$.

Conversely, let $(\tilde{G}, \wedge)$ be a $FHSos$ in $\mathfrak{N}$. Then $(\tilde{G}, \wedge)^c$ is a $FHScs$ in $\mathfrak{N}$. By hypothesis, $\mathfrak{h}^{-1}((\tilde{G}, \wedge)^c)$ is $FHS\theta cs$ in $\mathfrak{M}$. Since,$\mathfrak{h}^{-1}((\tilde{G}, \wedge)^c) = (\mathfrak{h}^{-1}(\tilde{G}, \wedge))^c$, $(\mathfrak{h}^{-1}(\tilde{G}, \wedge))^c$ is $FHS\theta cs$ in $\mathfrak{M}$. Therefore, $(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$ is a $FHS\theta os$ in $\mathfrak{M}$. Hence, $\mathfrak{h}$ is $FHS\theta Cts$.

Theorem 3.12. *Let $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ be a $FHS\theta Cts$ map and $\mathfrak{g} : (\mathfrak{N}, M, \sigma) \to (P, N, \rho)$ be a $FHSCts$, then $\mathfrak{g} \circ \mathfrak{h} : (\mathfrak{M}, L, \tau) \to (P, N, \rho)$ is a $FHS\theta Cts$.*

Proof. Let $(\tilde{K}, \wedge)$ be a $FHSos$ in P. Then $\mathfrak{g}^{-1}(\tilde{K}, \wedge)$ is a $FHSos$ in $\mathfrak{N}$, by hypothesis. Since $\mathfrak{h}$ is a $FHS\theta Cts$ map $\mathfrak{h}^{-1}(\mathfrak{g}^{-1}(\tilde{K}, \wedge))$ is a $FHS\theta os$ in $\mathfrak{M}$. Hence $\mathfrak{g} \circ \mathfrak{h}$ is a $FHS\theta Cts$ map.

Theorem 3.13. *Let $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ be a $FHS\theta Cts$ map. Then the following conditions are hold:*

(i) $\mathfrak{h}(FHS\theta cl(\tilde{H}, \wedge)) \leq FHScl(\mathfrak{h}(\tilde{H}, \wedge))$, *for all* $FHScs(\tilde{H}, \wedge)$ *in* $\mathfrak{M}$.

(ii) $FHS\theta cl(\mathfrak{h}^{-1}(\tilde{G}, \wedge)) \leq \mathfrak{h}^{-1}(FHScl(\tilde{G}, \wedge))$, *for all* $FHScs(\tilde{G}, \wedge)$ *in* $\mathfrak{N}$.

Proof. (i) As $FHS\theta cl(\mathfrak{h}(\tilde{H}, \wedge))$ is a $FHS\theta cs$ in $\mathfrak{N}$ and $\mathfrak{h}$ is $FHS\theta Cts$, we have $\mathfrak{h}^{-1}(FHS\theta cl(\mathfrak{h}(\tilde{H}, \wedge)))$ is a $FHS\theta cs$ in $\mathfrak{M}$. Now, as $(\tilde{H}, \wedge) \leq \mathfrak{h}^{-1}(FHScl(\mathfrak{h}(\tilde{H}, \wedge))), FHScl(\tilde{H}, \wedge) \leq \mathfrak{h}^{-1}(FHScl(\mathfrak{h}(\tilde{H}, \wedge)))$. Therefore, $\mathfrak{h}(FHS\theta cl(\tilde{H}, \wedge)) \leq FHScl(\mathfrak{h}(\tilde{H}, \wedge))$.

(ii) By replacing $(\tilde{H}, \wedge)$ with $(\tilde{G}, \wedge)$ in (i), we get $\mathfrak{h}(FHS\theta cl(\mathfrak{h}^{-1}(\tilde{G}, \wedge))) \leq FHScl(\mathfrak{h}(\mathfrak{h}^{-1}(\tilde{G}, \wedge))) \leq FHScl(\tilde{G}, \wedge)$. Hence, $FHS\theta cl(\mathfrak{h}^{-1}(\tilde{G}, \wedge)) \leq \mathfrak{h}^{-1}(FHScl(\tilde{G}, \wedge))$.

Remark 3.14. If $\mathfrak{h}$ is $FHS\theta Cts$ then,

(i) $\mathfrak{h}(FHS\theta cl(\tilde{H}, \wedge))$ need not be equal to $FHScl(\mathfrak{h}(\tilde{H}, \wedge))$ where $(\tilde{H}, \wedge) \in \mathfrak{M}$.

(ii) $FHS\theta cl(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$ need not be equal to $\mathfrak{h}^{-1}(FHScl(\tilde{G}, \wedge))$ where $(\tilde{G}, \wedge) \in \mathfrak{N}$.

Example 3.15. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_1), (\tilde{H_3}, \wedge_2)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\} \rangle \end{Bmatrix}$$

$$(\tilde{H_3}, \wedge_2) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.6}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle, \\ \langle (a_1, b_2), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1), (\tilde{H_2}, \wedge_1)\}$ is $FHSts$.

Let the $FHSs$'s $(\tilde{G_1}, \wedge_1), (\tilde{G_2}, \wedge_1), (\tilde{G_3}, \wedge_2)$ over the universe $\mathfrak{N}$ be

$$(\tilde{G_1}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.3}, \frac{\mathfrak{n}_2}{0.2}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{G_2}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{G_3}, \wedge_2) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.8}\} \rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\} \rangle, \\ \langle (c_2, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G}_1, \wedge_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G}_1, \wedge_1) = (\tilde{H}_1, \wedge_1), \mathfrak{h}^{-1}(\tilde{G}_2, \wedge_1) = (\tilde{H}_2, \wedge_1), \mathfrak{h}^{-1}(\tilde{G}_3, \wedge_2) = (\tilde{H}_3, \wedge_2)$$

Then $\mathfrak{h}$ is $FHS\theta Cts$.

(i) $\mathfrak{h}(FHS\theta cl(\tilde{H}_3, \wedge_2)) = (\tilde{G}_1, \wedge_1)^c$, but $FHScl(\mathfrak{h}(\tilde{H}_3, \wedge_2)) = (\tilde{G}_1, \wedge_1)^c$.
Hence $\mathfrak{h}(FHS\theta cl(\tilde{H}_3, \wedge_2)) = FHScl(\mathfrak{h}(\tilde{H}_3, \wedge_2))$.

(ii) $FHS\theta cl(\mathfrak{h}^{-1}(\tilde{G}_3, \wedge_2)) = (\tilde{H}_1, \wedge_1)^c$, but $\mathfrak{h}^{-1}(FHScl(\tilde{G}_3, \wedge_2)) == (\tilde{H}_1, \wedge_1)^c$.
Hence $FHScl(\mathfrak{h}^{-1}(\tilde{G}_3, \wedge_2)) = \mathfrak{h}^{-1}(FHScl(\tilde{G}_3, \wedge_2))$.

Theorem 3.16. *$\mathfrak{h}$ is $FHS\theta Cts$ iff $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge)) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$ for all $FHScs$ $(\tilde{G}, \wedge)$ in $\mathfrak{N}$.*

Proof. Let $\mathfrak{h}$ be a $FHS\theta Cts$ and $(\tilde{G}, \wedge) \in \mathfrak{N}$. $FHSint(\tilde{G}, \wedge)$ is $FHSos$ in $\mathfrak{N}$ and hence, $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge))$ is a $FHS\theta os$ in $\mathfrak{M}$. Therefore, $FHS\theta int(\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge))) = \mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge))$. Also, $FHSint(\tilde{G}, \wedge) \leq (\tilde{G}, \wedge)$ implies that $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge)) \leq \mathfrak{h}^{-1}(\tilde{G}, \wedge)$. Therefore, $FHS\theta int(\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge))) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$. That is, $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge)) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$.

Conversely, let $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge)) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$ for all subset $(\tilde{G}, \wedge)$ of $\mathfrak{N}$. If $(\tilde{G}, \wedge)$ is $FHSos$ in $\mathfrak{N}$, then $FHSint(\tilde{G}, \wedge) = (\tilde{G}, \wedge)$. By assumption, $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge)) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$. Thus $(\mathfrak{h}^{-1}(\tilde{G}, \wedge) \leq FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$. But $FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge)) = \mathfrak{h}^{-1}(\tilde{G}, \wedge)$. Therefore, $FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge)) = \mathfrak{h}^{-1}(\tilde{G}, \wedge)$. That is, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is $FHS\theta os$ in $\mathfrak{M}$, for all $FHSos(\tilde{G}, \wedge)$ in $\mathfrak{N}$. Therefore, $\mathfrak{h}$ is $FHS\theta Cts$ on $\mathfrak{M}$.

Remark 3.17. If $\mathfrak{h}$ is $FHS\theta Cts$, then $FHS\theta int(\mathfrak{h}^{-1}(\tilde{G}, \wedge))$ need not be equal to $\mathfrak{h}^{-1}(FHSint(\tilde{G}, \wedge))$ where $(\tilde{G}, \wedge) \in \mathfrak{N}$.

Example 3.18. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q_1' \times Q_2'$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q_1' = \{c_1, c_2, c_3\}, Q_2' = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H}_1, \wedge_1)$ and $(\tilde{H}_2, \wedge_1)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H}_1, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.7}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$$(\tilde{H}_2, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\} \rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.3}, \frac{\mathfrak{m}_2}{0.5}\} \rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H}_1, \wedge_1), (\tilde{H}_2, \wedge_1)\}$ is $FHSts$.

Let the $FHSs's$ $(\tilde{G}_1, \wedge_1)$ and $(\tilde{G}_2, \wedge_1)$ over the universe $\mathfrak{N}$ be defined as

$$(\tilde{G_1}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.7}\}\rangle \end{Bmatrix}$$

$$(\tilde{G_2}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.3}, \frac{\mathfrak{n}_2}{0.2}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.5}, \frac{\mathfrak{n}_2}{0.3}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_2, d_1), \nu(a_1, b_2) = (c_1, d_2), \nu(a_2, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_1) = (\tilde{H_1}, \wedge_1), \mathfrak{h}^{-1}(\tilde{G_2}, \wedge_1) = (\tilde{H_2}, \wedge_1)$$

Then $\mathfrak{h}$ is $FHS\theta Cts$. then $FHS\theta int(\mathfrak{h}^{-1}(\tilde{G_2}, \wedge_1)) = (\tilde{H_2}, \wedge_1)$

$\mathfrak{h}^{-1}(FHSint(\tilde{G_2}, \wedge_1)) = \tilde{0}_{(\mathfrak{M},Q)}$ where $(\tilde{G_2}, \wedge_1) \in \mathfrak{N}$.

4 Fuzzy Hypersoft θ Irresolute Maps

Fuzzy hypersoft θ irresolute maps are introduced and their relevent properties are discussed in this section .

Definition 4.1. A map $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ is called a $FHS\theta$ irresolute (in short, $FHS\theta Irr$) map if $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is a $FHS\theta os$ in $(\mathfrak{M}, L, \tau)$ for every $FHS\theta os$ $(\tilde{G}, \wedge)$ of $(\mathfrak{N}, M, \sigma)$.

Theorem 4.2. Let $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ be a $FHS\theta Irr$ map. Then $\mathfrak{h}$ is a $FHS\theta Cts$ map. But not conversely.

Proof. Let $\mathfrak{h}$ be a $FHS\theta Irr$ map. Let $(\tilde{G}, \wedge)$ be any $FHSos$ on $\mathfrak{N}$. Since every $FHSos$ is a $FHS\theta os$, $(\tilde{G}, \wedge)$ in $\mathfrak{N}$. By hypothesis, $\mathfrak{h}^{-1}(\tilde{G}, \wedge)$ is a $FHS\theta os$ in $\mathfrak{M}$. Hence, $\mathfrak{h}$ is a $FHS\theta Cts$ map.

Example 4.3. Let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and $\mathfrak{N} = \{\mathfrak{n}_1, \mathfrak{n}_2\}$ be the FHS initial universes and the attributes be $L = Q_1 \times Q_2$ and $M = Q'_1 \times Q'_2$ respectively. The attributes are given as:

$$Q_1 = \{a_1, a_2, a_3\}, Q_2 = \{b_1, b_2\}$$
$$Q'_1 = \{c_1, c_2, c_3\}, Q'_2 = \{d_1, d_2\}.$$

Let $(\mathfrak{M}, L), (\mathfrak{N}, M)$ be the classes of FHS sets. Let the $FHSs$'s $(\tilde{H_1}, \wedge_1)$, $(\tilde{H_2}, \wedge_1)$ over the universe $\mathfrak{M}$ be

$$(\tilde{H_1}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.2}, \frac{\mathfrak{m}_2}{0.3}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.4}\}\rangle \end{Bmatrix}$$

$$(\tilde{H_2}, \wedge_1) = \begin{Bmatrix} \langle (a_1, b_1), \{\frac{\mathfrak{m}_1}{0.8}, \frac{\mathfrak{m}_2}{0.7}\}\rangle, \\ \langle (a_2, b_1), \{\frac{\mathfrak{m}_1}{0.5}, \frac{\mathfrak{m}_2}{0.6}\}\rangle \end{Bmatrix}$$

$\tau = \{\tilde{0}_{(\mathfrak{M},Q)}, \tilde{1}_{(\mathfrak{M},Q)}, (\tilde{H_1}, \wedge_1)\}$ is $FHSts$.

Let the $FHSs's$ $(\tilde{G_1}, \wedge_1), (\tilde{G_2}, \wedge_1)$ over the universe $\mathfrak{N}$ be

$$(\tilde{G_1}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.3}, \frac{\mathfrak{n}_2}{0.2}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.4}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$$(\tilde{G_2}, \wedge_1) = \begin{Bmatrix} \langle (c_2, d_1), \{\frac{\mathfrak{n}_1}{0.7}, \frac{\mathfrak{n}_2}{0.8}\}\rangle, \\ \langle (c_1, d_2), \{\frac{\mathfrak{n}_1}{0.6}, \frac{\mathfrak{n}_2}{0.5}\}\rangle \end{Bmatrix}$$

$\sigma = \{\tilde{0}_{(\mathfrak{N},Q)}, \tilde{1}_{(\mathfrak{N},Q)}, (\tilde{G_1}, \wedge_1), (\tilde{G_2}, \wedge_1)\}$ is $FHSts$.

Let $\mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ be a FHS mapping as follows:

$$\omega(\mathfrak{m}_1) = \mathfrak{n}_2, \omega(\mathfrak{m}_2) = \mathfrak{n}_1,$$
$$\nu(a_1, b_1) = (c_2, d_1), \nu(a_2, b_1) = (c_1, d_2), \nu(a_1, b_2) = (c_2, d_2)$$

$$\mathfrak{h}^{-1}(\tilde{G_1}, \wedge_1) = (\tilde{H_1}, \wedge_1), \mathfrak{h}^{-1}(\tilde{G_2}, \wedge_1) = (\tilde{H_2}, \wedge_1),$$

$\therefore \mathfrak{h} = (\omega, \nu) : (\mathfrak{M}, L) \to (\mathfrak{N}, M)$ is $FHS\theta Cts$ but not $FHS\theta Irr$, because the set $(\tilde{G_2}, \wedge_1)$ is a $FHS\theta os$ in $\mathfrak{N}$ but $\mathfrak{h}^{-1}(\tilde{G_2}, \wedge_1) = (\tilde{H_2}, \wedge_1)$ is not $FHS\theta os$ in $\mathfrak{M}$

Theorem 4.4. *Let $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ and $\mathfrak{g} : (\mathfrak{N}, M, \sigma) \to (P, N, \rho)$ be $FHSIrr$ maps, then $\mathfrak{g} \circ \mathfrak{h} : (\mathfrak{M}, L, \tau) \to (P, N, \rho)$ is a $FHSIrr$ map.*

Proof. Let $(\tilde{K}, \wedge)$ be a $FHS\theta os$ in P. Then $\mathfrak{g}^{-1}(\tilde{K}, \wedge)$ is a $FHS\theta os$ in $\mathfrak{N}$. Since $\mathfrak{h}$ is a $FHS\theta Irr$ map, $\mathfrak{h}^{-1}(\mathfrak{g}^{-1}(\tilde{K}, \wedge))$ is a $FHS\theta os$ in $\mathfrak{M}$. Hence $\mathfrak{g} \circ \mathfrak{h}$ $FHS\theta Irr$ map.

Theorem 4.5. Let $\mathfrak{h} : (\mathfrak{M}, L, \tau) \to (\mathfrak{N}, M, \sigma)$ be a $FHS\theta Irr$ map and $\mathfrak{g} : (\mathfrak{N}, M, \sigma) \to (P, N, \rho)$ is a $FHS\theta Cts$ map, then $\mathfrak{g} \circ \mathfrak{h} : (\mathfrak{M}, L, \tau) \to (P, N, \rho)$ is a $FHS\theta Cts$ map.

Proof. Let $(\tilde{K}, \wedge)$ be a $FHSos$ in P. Then $\mathfrak{g}^{-1}(\tilde{K}, \wedge)$ is a $FHS\theta os$ in $\mathfrak{N}$. Since, $\mathfrak{h}$ is a $FHS\theta Irr$ map, $\mathfrak{h}^{-1}(\mathfrak{g}^{-1}(\tilde{K}, \wedge))$ is a $FHS\theta os$ in $\mathfrak{M}$. Hence, $\mathfrak{g} \circ \mathfrak{h}$ is a $FHS\theta Cts$ map.

5 Conclusions

In this paper, $FHS\theta Cts$, $FHS\theta SCts$ and $FHS\theta \mathcal{P}Cts$ maps are defined using $FHS\theta os$ and their properties are analysed with examples. Then $FHSCts$ maps are compared with $FHS\theta Cts$ maps. In addition, these maps are extended to $FHS\theta Irr$ maps and its relevant properties are discussed. In future, these findings can be extended to $FHS\theta$ open mapping, $FHS\theta$ closed mapping and FHS homeomorphic functions.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

AMS (2000) subject classification: 03E72, 54A10, 54A40, 54C05, 54C10.

References

[1] M. Abbas, G. Murtaza and F. Smarandache, *Basic operations on hypersoft sets and hypersoft point*, Neutrosophic Sets and Systems, **35**, (2020), 407-421.

[2] M. Ahsan, M. Saeed and A.U. Rahman, *A Theoretical and Analytical Approach for Fundamental Framework of Composite Mappings on Fuzzy Hypersoft Classes*, Neutrosophic Sets and Systems, **45**, (2021), 268-285.

[3] D. Ajay and J. Joseline Charisma, *Neutrosophic hypersoft topological spaces*, Neutrosophic Sets and Systems, **40**, (2021), 178-194.

[4] D. Ajay, J. Joseline Charisma, N. Boonsatit, P. Hammachukiattikul and G. Rajchakit *Neutrosophic semiopen hypersoft sets with an application to MAGDM under the COVID-19 scenario*, Hindawi Journal of Mathematics, **2021**, (2021), 1-16.

[5] M. Caldas, M. Ganster, D. N. Georgiou, S. Jafari and T. Noiri, *θ-semiopen sets and separation axioms in topological spaces*, Carpathian Journal of Mathematics, 24 **(1)** (2008), 13-22.

[6] M. Caldas, S. Jafari and M. M. Kov$a$r, *Some properties of θ-open sets*, Divulg. Mat., 12**(2)** (2004), 161-169.

[7] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[8] J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Internat. J. Math. Math. Sci., **19**, (1996), 303–310.

[9] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl., **37**, (1999), 19-31.

[10] Saeed, M., Ahsan, M., Siddique, M.K., Ahmad M.R., *A Study of The Fundamentals of Hypersoft Set Theory* , International Journal of Scientific Engineering Research, **11** (1), (2020).

[11] M. Saeed, A. U. Rahman, M. Ahsan, F. Smarandache, *An inclusive study on fundamentals of hypersoft set, Theory and Applications of Hypersoft Set*, Pons Publishing House, Brussels, (2021), 1–23.

[12] S. Saha, *Fuzzy δ-continuous mappings*, Journal of Mathematical Analysis and Applications, **126** (1987), 130-142.

[13] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl., **61**, (2011), 1786-1799.

[14] F. Smarandache, *Extension of soft set to hypersoft set, and then to plithogenic hypersoft set*, Neutrosophic Sets and Systems, **22**, (2018), 168-170.

[15] N. V. Velicko, *H-Closed Topological Spaces*, Amer. Math. Soc. Transl., **78** (No 2), (1968),103-118.

[16] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (3), (1965), 338–353.