The Applications of Runge-Kutta Numerical Methods to Numerical Solutions of Several Neutrosophic Problems

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Abstract

In This paper, we develop the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we present generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we apply our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

Keywords: Neutrosophic problem; Neutrosophic Runge-Kutta; Numerical stability; Numerical method

1. Introduction

Numerical analysis is a broad branch of applied mathematics concerned with finding numerical solutions and numerical approximations for many mathematical problems and models related to algebraic equations or differential equations alike, through devising some algorithms that are easy to apply and program on the computer [1-3]. Numerical analysis is also concerned with finding good estimates of the error resulting from the approximation process, and with the issue of studying the stability of the resulting numerical solutions

Neutrosophic logic is a revolutionary logic introduced by Smarandache [6] as a new generalization of fuzzy logic that considers the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena. It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even in computer science [10-13]. The Runge-Kutta method is one of the reference methods in numerical analysis that has been dealt with and developed by many researchers around the world. Different ranks of this method have been used to find numerical solutions and approximate errors for various problems in applied mathematics [4-9]. The applications of neutrosophic methods in numerical analysis have been studied by many authors see [14-18]. Where, we can see neutrosophic modelled problems with their approximate solutions and absolute errors were presented numerically [19-25].

This has motivated us to develop the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we present generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we apply our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

2. Main Discussion

The neutrosophic Runge-Kutta method of rank 5:

$$y_{n+1} + z_{n+1}I = y_n + z_nI + \frac{h + dI}{48 + 48I}(8(k_1 + k_1I) + 20(k_2 + k_2I) + 12(k_3 + k_3I) + 6(k_4 + k_4I) + (k_5 + k_5I) + (k_6 + k_6I))$$

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Where:

$$k_1 + k_1 I = f(x_n + t_n I, y_n + z_n I)$$

$$k_2 + k_2 I = f(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I))$$

$$k_3 + k_3 I = f(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_2 + k_2 I))$$

$$k_4 + k_4 I = f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_3 + k_3 I))$$

$$k_5 + k_5 I = f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_4 + k_4 I))$$

$$k_6 + k_6 I = f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_5 + k_5 I))$$

The neutrosophic Runge-Kutta method of rank 6:

$$y_{n+1} + z_{n+1} I = y_n + z_n I + \frac{h + dI}{96 + 96I}(16(k_1 + k_1 I) + 40(k_2 + k_2 I) + 24(k_3 + k_3 I) + 12(k_4 + k_4 I) + 2(k_5 + k_5 I) + (k_6 + k_6 I) + (k_7 + k_7 I))$$

Where:

$$k_1 + k_1 I = f(x_n + t_n I, y_n + z_n I)$$

$$k_2 + k_2 I = f(x_n + t_n I + \frac{h}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I))$$

$$k_3 + k_3 I = f(x_n + t_n I + \frac{h}{2}, y_n + z_n I_n + \frac{1}{2}(h + dI)(k_2 + k_2 I))$$

$$k_4 + k_4 I = f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_3 + k_3 I))$$

$$k_5 + k_5 I = f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_4 + k_4 I))$$

$$k_6 + k_6 I = f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_5 + k_5 I))$$

$$k_7 + k_7 I = f(x_n + t_n I + h, y_n + z_n I + (h + dI)(k_6 + k_6 I))$$

The neutrosophic Runge-Kutta method of rank 7:

$$y_{n+1} + z_{n+1} I = y_n + z_n I + z_n I + \frac{h + dI}{192}(32(k_1 + k_1 I) + 80(k_2 + k_2 I) + 48(k_3 + k_3 I) + 24(k_4 + k_4 I) + 4(k_5 + k_5 I) + 2(k_6 + k_6 I) + (k_7 + k_7 I) + (k_8 + k_8 I))$$

Where:

$$k_1 + k_1 I = f(x_n + t_n I, y_n + z_n I)$$

$$k_2 + k_2 I = f(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_1 + k_1 I))$$

$$k_3 + k_3 I = f(x_n + t_n I + \frac{h + dI}{2}, y_n + z_n I + \frac{1}{2}(h + dI)(k_2 + k_2 I))$$

$$k_4 + k_4 I = f(x_n + t_n I_n + h + dI, y_n + z_n I + (h + dI)(k_3 + k_3 I))$$

$$k_5 + k_5 I = f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_4 + k_4 I))$$

$$k_6 + k_6 I = f(x_n + t_n I_n + h + dI, y_n + z_n I + (h + dI)(k_5 + k_5 I))$$

$$k_7 + k_7 I = f(x_n + t_n I + h + dI, y_n + z_n I + (h + dI)(k_6 + k_6 I))$$

$$k_8 + k_8 I = f(x_n + t_n I_n + h + dI, y_n + z_n I + (h + dI)(k_7 + k_7 I))$$

The stability analysis:

Consider the following initial values:

$$(y + zI)' = f(x + tI, y + zI), (y + zI)(x_0 + t_0 I) = y_0 + z_0 I, x + tI \in [a + cI, b + sI] \quad (1)$$

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The general formula:

$$y_{n+1} + z_{n+1}I = y_n + z_n I + (h + dI)\phi(x_n + t_n I_n, y_n + z_n I, h + dI) \quad (2)$$

The general formula o of phase R :

$$(y + zI)(x_{n+1} + t_{n+1}I) - (y_{n+1} + z_{n+1}I) = O((h + dI)^{p+1}) \quad (3)$$

we get the following:

$$(y + zI)(x_{n+1} + t_{n+1}I) = E(\lambda(h + dI))(y_n + z_n I) \quad (4)$$

we get :

$$(y + zI)' = \lambda(y + zI), (y + zI)(x_0 + t_0 I) = (y + zI)_0 \quad (5)$$

Thus:

$$E(\overline{(h + dI)}) = 1 + \overline{(h + dI)} + \frac{1}{2!}\overline{(h + dI)}^2 + \ldots\ldots\ldots + \frac{1}{p!}\overline{(h + dI)}^p + O(\overline{(h + dI)}^{p+1}) \quad (6)$$

RK		stability period
5	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!}$	$(-3.100183 + 2.1I, 0)$
6	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!} + \frac{\overline{(h + dI)}^6}{6!}$	$(-3.55642 - 1.6I, 0)$
7	$1 + \overline{(h + dI)} + \frac{\overline{(h + dI)}^2}{2!} + \frac{\overline{(h + dI)}^3}{3!} + \frac{\overline{(h + dI)}^4}{4!} + \frac{\overline{(h + dI)}^5}{5!} + \frac{\overline{(h + dI)}^6}{6!} + \frac{\overline{(h + dI)}^7}{7!}$	$(-3.235 + 1.067I, 0)$

Direct applications

Example (1)

Solve the following system of neutrosophic differential equations:

$$(y + zI)_1' = (y + zI)_2, (y + zI)_1(0) = 1 + I$$

$$(y + zI)_2' = -1001 - 1001I, \quad (y + zI)_2 - 1000I \ (y + zI)_1, (y + zI)_2(I) = -1 - I$$

We will take the value of the step length $h = 0.001 + 0.001I$.

Example (2)

Solve the following system of differential equations:

$$(y + zI)_1' = (600 + 300I)(y + zI)_1{}^2 \left((y + zI)_2 - (y + zI)_1{}^3 \right), (y + zI)_1(I) = 0.1 + I$$

$$(y + zI)_2' = (-200 - 100I)\left((y + zI)_2 - (y + zI)_1{}^3\right) + 2(1 - (y + zI)_2), (y + zI)_2(I) = -0.1 - I$$

Table 1: the results of solving the problem in the first example.

ERK method of the fifth rank error values		Truth values		ERK method-Kutta of the fifth rank values	
y_1 error values	y_2 error values	y_1 Values	y_2 Values	y_1 Values	y_2 Values
I	I	1+I	-1-I	1+I	-1-I
3.2355e-14+I	3.2213e-14+I	0.928+0.928 I	-0.928-0.928I	0.928+0.928I	-0.928-0.928I
6.31e-14+I	6.31e-14+I	0.93311+I	-(0.93311+I)	0.9447+I	-(0.9447+I)
9.3452e-14+I	9.3452e-14+I	0.9332+I	-(0.9332+I)	0.9332+I	-(0.9332+I)
1.22173e-11+I	1.22173e-11+I	0.96643+I	-(0.96643+I)	0.96643+I	-(0.96643+I)
1.10342e-11+I	1.10342e-11+I	0.9115+I	-(0.9115+I)	0.9115+I	-(0.9115+I)

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1.330913e-11+I	1.330913e-11+I	0.658807+I	-(0.658807+I)	0.658807+I	-(0.658807+I)
2.11295e-11+I	2.11295e-11+I	0.9443861+I	0.9443861+I	0.9443861+I	0.9443861+I
2.3387e-11+I	2.3387e-11+I	0.981293+I	-(0.981293+I)	0.981293+I	-(0.981293+I)
2.208876e-11+I	2.208876e-11+I	0.9088+I	_(0.9088+I)	0.9088+I	-(0.9088+I)
3.11045e-11+I	3.11045e-11+I	0.77602+I	-(0.77602+I)	0.77602+I	-(0.77602+I)

Table 2: results of solving the problem in the second example.

ERK method of the fifth rank error values		**Truth values**		**ERK method of the fifth rank values**	
y_1**error values**	y_2 **error values**	y_1 **Values**	y_2 **Values**	y_1**Values**	y_2 **Values**
I	I	0.1+I	-0.1-I	0.1+I	-0.1-I
5.46645e-9+I	1.212e-7+I	0.092237+I	-0.073498-I	0. 092234+I	-0.073498-I
1.221376e-8+I	4.2437e-7+I	0.0911369+I	-0.0379445-I	0.0911366+I	-0.0379444-I
1.0987e-8+I	5.7787e-7+I	0.094696+I	-0.046064-I	0.094693+I	-0.049621-I
2.311209e-8+I	6.52014e-7+I	0.092064+I	-0.022398-I	0.092060+I	-0.0333913-I
2.3329e-9+I	6.66783e-8+I	0.095833+I	-0.231998+I	0.095830+I	-0.011690133-I
2.1092e-9+I	6.11038e-8+I	0.094155+I	-0.0100078-I	0.094151+I	-0.0017110-I
2.51375e-9+I	6.1167e-05+I	0.097964+I	-0.01437-I	0.097961+I	-0.01433-I
2.256248e-9+I	6.2334e-05+I	0.03326+I	-0.01125-I	0.03323+I	-0.01121-I
2.3143e-9+I	5.673108e-05+I	0.08873+I	-0.008514-I	0.08870+I	-0.008510-I
2.100123e-9+I	5.4431e-05+I	0.0897664+I	-0.0023638-I	0.0897661+I	-0.0023634-I

Neutrosophic IRK Method of the fifth rank:

Depending on the neutrosophic explicit Runge-Kutta method of fifth rank, the neutrosophic implicit Runge-Kutta formula of fifth rank is in the following form:

$$y_n + z_n I = y_{n+1} + z_{n+1} I + \frac{h + dI}{48 + 48I}(8(k_1 + k_1 I) + 20(k_2 + k_2 I) + 12(k_3 + k_3 I) + 6(k_4 + k_4 I) + (k_5 + k_5 I) + (k_6 + k_6 I))$$

Where:

$$k_1 + k_1 I = f(x_{n+1} + t_{n+1} I, y_{n+1} + z_{n+1} I)$$

$$k_2 + k_2 I = f(x_{n+1} + t_{n+1} I - \frac{h + dI}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_1 + k_1 I))$$

$$k_3 + k_3 I = f(x_{n+1} + t_{n+1} I - \frac{h + dI}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_2 + k_2 I))$$

$$k_4 + k_4 I = f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_3 + k_3 I))$$

$$k_5 + k_5 I = f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_4 + k_4 I))$$

$$k_6 + k_6 I = f(x_{n+1} + t_{n+1} I - h - dI, y_{n+1} + z_{n+1} I - (h + dI)(k_5 + k_5 I)).$$

Neutrosophic IRK Method of the sixth rank:

Depending on the neutrosophic explicit Runge-Kutta method of the sixth rank, the neutrosophic implicit Runge-Kutta formula of the sixth rank is in the following form:

$$y_n + z_n I = y_{n+1} + z_{n+1} I - + \frac{h + dI}{96 + 96I}(16(k_1 + k_1 I) + 40(k_2 + k_2 I) + 24(k_3 + k_3 I) + 12(k_4 + k_4 I) + 2(k_5 + k_5 I) + (k_6 + k_6 I) + (k_7 + k_7 I))$$

Where:

$$k_1 + k_1 I = f(x_{n+1} + t_{n+1} I, y_{n+1} + z_{n+1} I)$$

$$k_2 + k_2 I = f(x_{n+1} + t_{n+1} I - \frac{(h + dI)}{2}, y_{n+1} + z_{n+1} I - \frac{1}{2}(h + dI)(k_1 + k_1 I))$$

$$k_3 + k_3 I = f(x_{n+1} + t_{n+1} I - \frac{(h + dI)}{2}, y_{n+1} + z_{n+1} I_n - \frac{1}{2}(h + dI)(k_2 + k_2 I))$$

$$k_4 + k_4 I = f(x_{n+1} + t_{n+1} I - (h + dI), y_{n+1} + z_{n+1} I - (h + dI)(k_3 + k_3 I))$$

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$$k_5 + k_5 I = f(x_{n+1} + t_{n+1}I - (h + dI), y_{n+1} + z_{n+1}I - (h + dI)(k_4 + k_4 I))$$
$$k_6 + k_6 I = f(x_{n+1} + t_{n+1}I - (h + dI), y_{n+1} + z_{n+1}I - (h + dI)(k_5 + k_5 I))$$
$$k_7 + k_7 I = f(x_{n+1} + t_{n+1}I - (h + dI), y_{n+1} + z_{n+1}I - (h + dI)(k_6 + k_6 I))$$

Stability of Higher-order NRK methods:

Consider the general formula of the neutrosophic Runge-Kutta method from the phase:

$$y_{n+1} + z_{n+1}I = y_n + (h + dI)\phi(x_{n+1} + t_{n+1}I, y_{n+1} + z_{n+1}I, h + dI) \quad (7)$$

The neutrosophic implicit Runge-Kutta formula (7) of phase R is said to be of rank P if:

$$(y + zI)(x_{n+1} + t_{n+1}I) - y_{n+1} - z_{n+1}I = O((h + dI)^{p+1})$$

Using formula (7) on the equation of differences we get

$$(y + zI)_{n+1} = E(\lambda h)(y + zI)_n \quad (8)$$

Using the test question:

$$(y + zI)' = \lambda(y + zI), (y + zI)((x + tI)_0) = (y + zI)_0 \quad (9)$$

We get:

$$E(\overline{h + dI}) = 1 + I + \overline{h + dI} + \frac{1}{2!}\overline{(h + dI)}^2 + \ldots\ldots\ldots + \frac{1}{(p+1)!}\overline{(h + dI)}^{p+1} + O(\overline{(h + dI)}^{p+1}) \quad (10)$$

Where $\overline{h + dI} = \lambda(h + dI)$ are polynomials of degree R in $\overline{h + dI}$

Therefore, the absolute stability intervals of the implicit Runge-Kutta methods are of higher rank, as in the following table. Considering that these periods were found using the following diagram:

RK	r_2	stability period
5	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!}$	(-3.123+I,I)
6	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!} + \frac{\overline{h + dI}^7}{7!}$	(-3.223+I,I)
7	$1 + \overline{h + dI} + \frac{\overline{h + dI}^2}{2!} + \frac{\overline{h + dI}^3}{3!} + \frac{\overline{h + dI}^4}{4!} + \frac{\overline{h + dI}^5}{5!} + \frac{\overline{h + dI}^6}{6!} + \frac{\overline{h + dI}^7}{7!} + \frac{\overline{h + dI}^8}{8!}$	(-4.112809+I,I)

Direct Applications:

Example 3

Solve the system of equations in the first example using the neutrosophic IRK method of the fifth rank.

We will take the step length value h=0.002+0.001I.

Example 4

Solve the system of equations in the second example using the neutrosophic IRK method of the fifth rank.

We will take the step length value h=0.001+ 0.001I.

Table 3: results of neutrosophic IRK of fifth order for the first example.

ERK method of the fifth rank error values		Truth values		ERK method of the fifth rank values	
y_1 error values	y_2 error values	y_1 Values	y_2 Values	y_1 Values	y_2 Values
I	I	I	-I	1+I	-1-I
3.110572e-15+I	3.11057e-15+I	0.996+I	-(0.996+I)	0.996+I	-(0.996+I)
6.7886e-15+I	6.7882e-15+I	0.995+I	-(0.995+I)	0.995+I	-(0.995+I)
1.009551e-15+I	1.009531e-15+I	0.994+I	-(0.994+I)	0.994+I	-(0.994+I)
1.32219e-14+I	1.32216e-14+I	0.992+I	-(0.992+I)	0.992+I	-(0.992+I)

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1.649304e-14+I	1.649301e-14+I	0.991+I	-(0.991+I)	0.991+I	-(0.991+I)
2.08629e-14+I	2.08625e-14+I	0.98831+I	-(0.98831+I)	0.98831+I	-(0.98831+I)
2.3198e-14+I	2.3194e-14+I	0.9842+I	-(0.9842+I)	0.9842+I	-(0.9842+I)
2.6497e-14+I	2.6492e-14+I	0.9833+I	-(0.9833+I)	0.9833+I	-(0.9833+I)
3.06558e-14+I	2.6492e-14+I	0.9821+I	-(0.9821+I)	0.9821+I	-(0.9821+I)
3.32185e-14+I	3.32180e-14+I	0.9811+I	-(0.9811+I)	0.9811+I	-(0.9811+I)

Table 4: results of neutrosophic IRK of fifth order for the second example.

ERK of the fifth rank error values		**Truth values**		**ERK of the fifth rank values**	
y_1 **error values**	y_2 **error values**	y_1 **Values**	y_2 **Values**	y_1 **Values**	y_2 **Values**
I	I	0.1+**I**	-0.1-I	0.1+I	-0.1-I
2.2231e-07+I	8.98476e-06+I	0.09221+I	-0.0797-I	0.09211+I	-0.079715-**I**
9.113634e-07+I	2.5845e-05+I	0.09864+I	-0.06415-I	0.09851+I	-0.063142-**I**
1.32565e-06+I	3.589395e-05+I	0.0986675+I	-0.049564-I	0.098101+I	-0.049602-**I**
1.773654e-06+I	4.968te-05+I	0.092316+I	-0.033798-I	0.09793+I	-0.03284-**I**
1.4895e-06+I	4.10496e-05+I	0.0989886+I	-0.023369-I	0.098234	-0.029302-**I**
1.5856375e-07+I	368485e-06+I	0.09847364 +I	-0.0211988-I	0.091341+I	-0.0221784-**I**
1.1124e-07+I	3.104e-06+I	0.097111+I	-0.017769-I	0.0966+I	-0.01422313-**I**
1.00285e-07+I	2.8406e-06+I	0.097332+I	-0.010015-I	0.09665+I	-0.01100173-**I**
1.393e-07+I	2.10038e-06+I	0.0969984+I	-0.0056315-I	0.0965543+I	-0.005567-**I**
1.0440285e-07+I	1.938694e-06+I	0.0964453+I	-0.0011286-I	0.097754+I	-0.00111101-**I**

Table 5: results of neutrosophic IRK of sixth order for the second example.

ERK of the sixth rank error values		**Truth values**		**ERK of the sixth rank values**	
y_1 **error values**	y_2 **error values**	y_1 **Values**	y_1 **error values**	y_2 **error values**	y_1 **Values**
I	I	0.1+**I**	I	I	0.1+**I**
2.2231e-07+I	8.98476e-06+I	0.09221+0.0001I	2.2231e-07+I	8.98476e-06+0.0001I	0.09221+0.00002I
9.113634e-07+I	2.5845e-05+I	0.09864+0.0002I	9.113634e-07+I	2.5845e-05+0.00001I	0.09864+0.00002I
1.32565e-06+I	3.589395e-05+I	0.0986675+ 0.0003I	1.32565e-06+I	3.589395e-05+0.00002I	0.0986675+0.00004I
1.773654e-06+I	4.968te-05+I	0.092316+0. 0004I	1.773654e-06+I	4.968te-05+0.000002I	0.092316+0.00004I
1.4895e-06+I	4.10496e-05+I	0.0989886+ 0.0005I	1.4895e-06+I	4.10496e-05+0.00003I	0.0989886+0.00004I
1.5856375e-07+I	368485e-06+I	0.09847364 +0.0006I	1.5856375e-07+I	368485e-06+0.00004I	0.09847364+0.00000112I
1.1124e-07+I	3.104e-06+I	0.097111+0. 0007I	1.1124e-07+I	3.104e-06+0.000031I	0.097111+0.00000112I
1.00285e-07+I	2.8406e-06+I	0.097332+0. 0008I	1.00285e-07+I	2.8406e-06+0.000032I	0.097332+0.00000112I
1.393e-07+I	2.10038e-06+I	0.0969984+ 0.0009I	1.393e-07+I	2.10038e-06+0.00000112I	0.0969984+0.00000112I
1.0440285e-07+I	1.938694e-06+I	0.0964453+ 0.001I	1.0440285e-07+I	1.938694e-06+0.000004472I	0.0964453+0.00000112I

3. Conclusion

In This paper, we developed the Runge-Kutta numerical method to be applied on neutrosophic problems of high orders, where we presented generalized neutrosophic versions of Runge-Kutta methods of rank five, six and seven to use them in finding numerical solutions for some neutrosophic differential problems. In addition, we applied our generalized methods to some solid problems with many illustrated examples and numerical tables for comparing the results and the absolute errors.

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