Neutrosophic sets in IUP-algebras: a new exploration

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Abstract

The notions of neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals of IUP-algebras are introduced, and their basic properties are investigated. Conditions for neutrosophic sets to be neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals of IUP-algebras are provided. Relations between neutrosophic IUP-subalgebras (resp., neutrosophic IUP-ideals, neutrosophic IUP-filters, neutrosophic strong IUP-ideals) and their level subsets are considered.

Keywords: IUP-algebra; neutrosophic IUP-subalgebra; neutrosophic IUP-ideal; neutrosophic IUP-filter; neutrosophic strong IUP-ideal

1 Introduction

Zadeh[30] commenced the concept of fuzzy sets (FSs) in 1965, an important concept. After that, Atanasov[4] introduced the notion of intuitionistic fuzzy sets (IFSs) in 1986 as a generalization of FSs. Afterwards, Smaradanche suggested the notion of neutrosophic sets (NSs) in 2004, a generalization of IFSs. Since the NSs were discovered, many researchers have been interested in them and have researched this notion extensively. On the generalizations of NSs and their application to numerous logical algebras, such as in 2014, Alblowi et al.[2] introduced the new concepts of NSs. Broumi et al.[8] introduced the notion of rough NSs. Salama et al.[19] introduced the notion of the characteristic function of an NS. Salama and Smarandache[18] introduced the notion of neutrosophic crisp sets. In 2015, Broumi and Smarandache[6] introduced the notion of interval neutrosophic rough set. Hussain and Shabir[12] introduced the notion of algebraic structures of neutrosophic soft sets. Broumi and Smarandache[7] introduced the notion of soft interval-valued neutrosophic rough sets. In 2016, Smarandache[21] introduced the notion of operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. Khan et al.[15] introduced an NS approach for characterising left almost semigroups. In 2017, Song et al.[23] introduced the notion of interval NSs applied to ideals in BCK/BCI-algebras. Zhang et al.[31] introduced the notion of neutrosophic regular filters and fuzzy regular filters in pseudo-BCI algebras. Alias et al.[3] introduced the notion of rough neutrosophic multisets. In 2018, Borzooei et al.[5] introduced the notion of positive implicative BMBJ neutrosophic ideals in BCK-algebras. In 2019, Songsaeng and Iampan[24] applied NS theory to UP-algebras. Saha and Broumi[17] introduced the new operators on interval-valued NSs.

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Hashim et al.[11] introduced the notion of interval neutrosophic vague sets. In 2020, Songsaeng and Iampan[27] introduced several key concepts in the study of UP-algebras, including special neutrosophic UP-subalgebras, special neutrosophic near UP-filters, special neutrosophic UP-filters, special neutrosophic UP-ideals, and special neutrosophic strong UP-ideals. Their work marked a significant advancement in applying neutrosophic sets to UP-algebraic structures. In subsequent research, they extended this framework by applying neutrosophic cubic sets to UP-algebras,[26] further enriching the theory. Additionally, Songsaeng and Iampan[25] investigated the image and inverse image of neutrosophic cubic sets within UP-algebras. They defined these images under any function in a non-empty set and explored their properties in the context of neutrosophic cubic UP-subalgebras, neutrosophic cubic near UP-filters, neutrosophic cubic UP-filters, neutrosophic cubic UP-ideals, and neutrosophic cubic strong UP-ideals under certain UP-homomorphisms. This comprehensive study deepened the understanding of how neutrosophic structures interact with UP-algebras, providing a foundation for further exploration and application in algebraic theory. In 2021, James and Mathew[14] introduced the notion of lattice-valued NSs. Songsaeng et al.[28] introduced the concepts of neutrosophic implicative, comparative, and shift UP-filters in UP-algebras, expanding the theoretical foundation of UP-algebras and offering new avenues for exploration in neutrosophic logic and algebraic systems. In 2022, Hadi and Al-Swidi[10] introduced the notion of neutrosophic axial sets. In 2023, Al-Hijjawi and Alkhazaleh[1] introduced the notion of the possibility neutrosophic hypersoft sets.

In 2022, Iampan et al.[13] introduced the concept of IUP-algebras, a novel algebraic structure that defined four primary subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. Their work not only presented the fundamental properties of these subsets but also opened new pathways for mathematical research and applications. The introduction of IUP-algebras has since become a focal point in algebraic studies, inspiring extensive research that investigates its principles and expands its theoretical boundaries. Building on this momentum, Chanmanee et al.[9] in 2023 proposed the direct product of infinite families of IUP-algebras. Their research introduced the concept of weak direct products and presented key results regarding (anti-)IUP-homomorphisms in this context. These contributions significantly enhanced the structural understanding of IUP-algebras and established foundational tools for further exploration of the algebra's properties. In 2024, Kuntama et al.[16] advanced the field by integrating FS theory into IUP-algebras. They introduced fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals, meticulously analyzing the properties and interactions of these subsets. This research expanded the applicability of IUP-algebras, offering new perspectives and mathematical tools that bridge algebraic structures with fuzzy logic. Further broadening this theoretical framework, Suayngam et al.[29] introduced the notion of intuitionistic fuzzy IUP-algebras in 2024. Their work combined IFS theory with IUP-algebras, leading to the development of intuitionistic fuzzy IUP-subalgebras, ideals, filters, and strong ideals. This innovative approach enriched the study of IUP-algebras, presenting new hybrid structures that have the potential to inspire a wide range of applications and future research.

From reviewing the literature, it can be seen that many researchers have studied the study of NSs and are being studied continuously. Since IUP-algebras were released and published in 2022 and are an interesting new algebraic system, our researchers are interested in applying the concept of NSs to IUP-algebras. Therefore, we will study NSs and apply this notion to a subset of IUP-algebras, that is, IUP-subalgebras, IUP-filters, IUP-ideals and strong IUP-ideals, and research their properties and relationships. We will study the relationship between their NSs and level subsets. We have divided this article's content into four sections. Section 1 will describe related research and the inspiration for this article. Section 2 introduces the definition of IUP-algebras, providing examples and essential properties. We will also review the definitions of IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals and show their relationship. Section 3 reviews the definitions of NSs, introduces the notions of neutrosophic IUP-subalgebras, neutrosophic IUP-filters, neutrosophic IUP-ideals, and neutrosophic strong IUP-ideals, and gives examples. Afterwards, we will find the critical properties of the four concepts and show their generalizations. This section's main result is to show the relationship between characteristic functions, level subsets, and their NSs. Section 4 summarizes the results of the research and recommends further studies and extensions of this research.

2 Preliminaries

Before we dive into our research, it's essential to revisit the core concepts of IUP-algebras. Understanding their fundamental properties and key definitions will provide a solid foundation for the following discussions and insights.

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Definition 2.1. [13] An algebra $X = (X; \cdot, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:

$$(\forall x \in X)(0 \cdot x = x) \quad \text{(IUP-1)}$$
$$(\forall x \in X)(x \cdot x = 0) \quad \text{(IUP-2)}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot (x \cdot z) = y \cdot z) \quad \text{(IUP-3)}$$

For simplicity, we will refer to X as the IUP-algebra $X = (X; \cdot, 0)$ unless stated otherwise.

Example 2.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	3	0	5	1	2	4
2	5	2	0	4	1	3
3	1	3	4	0	5	2
4	4	5	3	2	0	1
5	2	4	1	5	3	0

then $X = (X, \cdot, 0)$ is an IUP-algebra.

Example 2.3. [13] Let $(G, \bullet, e)$ be a group where every element is its own inverse. In this case, $(G, \bullet, e)$ naturally forms an IUP-algebra.

Example 2.4. [13] Let X be a set and $\mathcal{P}(X)$ means the power set of X. It follows from Example 2.3 that $(\mathcal{P}(X), \triangle, \emptyset)$ is an IUP-algebra where the binary operation $\triangle$ is defined as the symmetric difference of any two sets.

Example 2.5. [13] Let $(G, \bullet, e)$ be a group with identity element e. Define a binary operation $\bullet$ on G by:

$$(\forall x, y \in G)(x \bullet y = yx^{-1}) \quad (2.1)$$

Then $(G, \bullet, e)$ is an IUP-algebra.

Proposition 2.6. [13] *In an IUP-algebra $X = (X; \cdot, 0)$, the following assertions are valid.*

$$(\forall x, y \in X)((x \cdot 0) \cdot (x \cdot y) = y) \quad (2.2)$$
$$(\forall x \in X)((x \cdot 0) \cdot (x \cdot 0) = 0) \quad (2.3)$$
$$(\forall x, y \in X)((x \cdot y) \cdot 0 = y \cdot x) \quad (2.4)$$
$$(\forall x \in X)((x \cdot 0) \cdot 0 = x) \quad (2.5)$$
$$(\forall x, y \in X)(x \cdot ((x \cdot 0) \cdot y) = y) \quad (2.6)$$
$$(\forall x, y \in X)(((x \cdot 0) \cdot y) \cdot x = y \cdot 0) \quad (2.7)$$
$$(\forall x, y, z \in X)(x \cdot y = x \cdot z \Leftrightarrow y = z) \quad (2.8)$$
$$(\forall x, y \in X)(x \cdot y = 0 \Leftrightarrow x = y) \quad (2.9)$$
$$(\forall x \in X)(x \cdot 0 = 0 \Leftrightarrow x = 0) \quad (2.10)$$
$$(\forall x, y, z \in X)(y \cdot x = z \cdot x \Leftrightarrow y = z) \quad (2.11)$$
$$(\forall x, y \in X)(x \cdot y = y \Rightarrow x = 0) \quad (2.12)$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot 0 = (z \cdot y) \cdot (z \cdot x)) \quad (2.13)$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (z \cdot x) \cdot (z \cdot y) = 0) \quad (2.14)$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (x \cdot z) \cdot (y \cdot z) = 0) \quad (2.15)$$

the right and the left cancellation laws hold (2.16)

Within IUP-algebras, four fundamental subsets stand out: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets form a critical framework that deepens our understanding and facilitates the application of IUP-algebras across different mathematical contexts.

Definition 2.7. [13] A nonempty subset S of X is called

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(i) an *IUP-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in S)(x \cdot y \in S) \tag{2.17}$$

(ii) an *IUP-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S \tag{2.18}$$
$$(\forall x, y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S) \tag{2.19}$$

(iii) an *IUP-ideal* of X if it satisfies the condition (2.18) and the following condition:

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S) \tag{2.20}$$

(iv) a *strong IUP-ideal* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S) \tag{2.21}$$

According to,[13] IUP-filters represent a unifying concept encompassing both IUP-ideals and IUP-subalgebras. These two subsets, IUP-ideals and IUP-subalgebras, are generalizations of strong IUP-ideals. Particularly, in an IUP-algebra X, strong IUP-ideals are equivalent to the entire algebra X itself. This hierarchical relationship among these subsets is visually represented in Figure 1, illustrating the structure of special subsets within IUP-algebras.

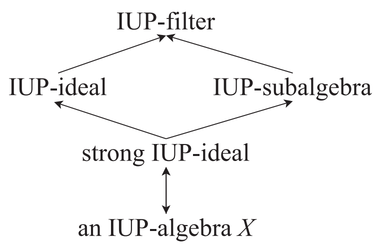

Figure 1: Special subsets of IUP-algebras

3 Main results

Before exploring the definition of NSs, it's crucial to revisit the foundational concepts that support them. This background will offer essential context and deepen our understanding of NSs, setting the stage for a more insightful discussion.

Definition 3.1. [20] A *neutrosophic set* (briefly, NS) in a nonempty set X is an object $\mathcal{A}$ having the form

$$\mathcal{A} = \{(x, \mathcal{A}_T(x), \mathcal{A}_I(x), \mathcal{A}_F(x)) \mid x \in X\} \tag{3.1}$$

where $\mathcal{A}_T : X \to [0,1]$ is a truth membership function, $\mathcal{A}_I : X \to [0,1]$ is an indeterminate member function, and $\mathcal{A}_F : X \to [0,1]$ is a false membership function.

To streamline notation, we represent an NS as $\mathcal{A} = (X, \mathcal{A}_T, \mathcal{A}_I, \mathcal{A}_F)$, where $\mathcal{A}$ is defined as

$$\{(x, \mathcal{A}_T(x), \mathcal{A}_I(x), \mathcal{A}_F(x)) \mid x \in X\}.$$

Definition 3.2. Let f be an FS in X. The FS $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in X$ is called the *complement* of f in X.

Definition 3.3. Let $\mathcal{A}$ be an NS in a nonempty set X. The NS $\overline{\mathcal{A}} = (X, \overline{\mathcal{A}}_T, \overline{\mathcal{A}}_I, \overline{\mathcal{A}}_F)$ is called the *complement* of $\mathcal{A}$ in X.

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We expand the concept of NSs to IUP-algebras by introducing four novel categories: neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals. This innovative application broadens the theoretical scope of IUP-algebras and enhances their practical utility, paving the way for fresh insights and applications in the field.

Definition 3.4. An NS $\mathcal{A}$ in X is called a *neutrosophic IUP-subalgbra* of X if it satisfies the following properties:

$$(\forall x, y \in X)(\mathcal{A}_T(x \cdot y) \geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\}) \tag{3.2}$$
$$(\forall x, y \in X)(\mathcal{A}_I(x \cdot y) \leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\}) \tag{3.3}$$
$$(\forall x, y \in X)(\mathcal{A}_F(x \cdot y) \geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}) \tag{3.4}$$

Definition 3.5. An NS $\mathcal{A}$ in X is called a *neutrosophic IUP-ideal* of X if it satisfies the following properties:

$$(\forall x \in X)(\mathcal{A}_T(0) \geq \mathcal{A}_T(x)) \tag{3.5}$$
$$(\forall x \in X)(\mathcal{A}_I(0) \leq \mathcal{A}_I(x)) \tag{3.6}$$
$$(\forall x \in X)(\mathcal{A}_F(0) \geq \mathcal{A}_F(x)) \tag{3.7}$$
$$(\forall x, y, z \in X)(\mathcal{A}_T(x \cdot z) \geq \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}) \tag{3.8}$$
$$(\forall x, y, z \in X)(\mathcal{A}_I(x \cdot z) \leq \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}) \tag{3.9}$$
$$(\forall x, y, z \in X)(\mathcal{A}_F(x \cdot z) \geq \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}) \tag{3.10}$$

Definition 3.6. An NS $\mathcal{A}$ in X is called a *neutrosophic IUP-filter* of X if it satisfies (3.5), (3.6), (3.7), and the following properties:

$$(\forall x, y \in X)(\mathcal{A}_T(y) \geq \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}) \tag{3.11}$$
$$(\forall x, y \in X)(\mathcal{A}_I(y) \leq \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}) \tag{3.12}$$
$$(\forall x, y \in X)(\mathcal{A}_F(y) \geq \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}) \tag{3.13}$$

Definition 3.7. An NS $\mathcal{A}$ in X is called a *neutrosophic strong IUP-ideal* of X if it satisfies the following properties:

$$(\forall x, y \in X)(\mathcal{A}_T(x \cdot y) \geq \mathcal{A}_T(y)) \tag{3.14}$$
$$(\forall x, y \in X)(\mathcal{A}_I(x \cdot y) \leq \mathcal{A}_I(y)) \tag{3.15}$$
$$(\forall x, y \in X)(\mathcal{A}_F(x \cdot y) \geq \mathcal{A}_F(y)) \tag{3.16}$$

Lemma 3.8. *Every neutrosophic IUP-subalgebra of X satisfies* (3.5)*,* (3.6)*, and* (3.7)*.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned}
\mathcal{A}_T(0) &= \mathcal{A}_T(x \cdot x) && \text{(by (IUP-2))}\\
&\geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(x)\} && \text{(by (3.2))}\\
&= \mathcal{A}_T(x),\\
\mathcal{A}_I(0) &= \mathcal{A}_I(x \cdot x) && \text{(by (IUP-2))}\\
&\leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(x)\} && \text{(by (3.3))}\\
&= \mathcal{A}_I(x),\\
\mathcal{A}_F(0) &= \mathcal{A}_F(x \cdot x) && \text{(by (IUP-2))}\\
&\geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(x)\} && \text{(by (3.2))}\\
&= \mathcal{A}_F(x).
\end{aligned}$$

Hence, it satisfies (3.5), (3.6), and (3.7). □

Theorem 3.9. *Every neutrosophic strong IUP-ideal of X satisfies* (3.5)*,* (3.6)*, and* (3.7)*.*

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Proof. Assume that neutrosophic strong IUP-ideal of X. Let $x \in X$. Thus,

$$\begin{aligned}
\mathcal{A}_T(0) &= \mathcal{A}_T(x \cdot x) && \text{(by (IUP-2))} \\
&\geq \mathcal{A}_T(x), && \text{(by (3.14))} \\
\mathcal{A}_I(0) &= \mathcal{A}_I(x \cdot x) && \text{(by (IUP-2))} \\
&\leq \mathcal{A}_I(x), && \text{(by (3.15))} \\
\mathcal{A}_F(0) &= \mathcal{A}_F(x \cdot x) && \text{(by (IUP-2))} \\
&\geq \mathcal{A}_F(x). && \text{(by (3.16))}
\end{aligned}$$

Hence, It satisfies (3.5), (3.6), and (3.7). □

Theorem 3.10. *A neutrosophic strong IUP-ideal and constant NS coincide.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned}
\mathcal{A}_T(x) &= \mathcal{A}_T((x \cdot 0) \cdot 0) && \text{(by (2.5))} \\
&\geq \mathcal{A}_T(0), && \text{(by (3.14))} \\
\mathcal{A}_I(x) &= \mathcal{A}_I((x \cdot 0) \cdot 0) && \text{(by (2.5))} \\
&\leq \mathcal{A}_I(0), && \text{(by (3.15))} \\
\mathcal{A}_F(x) &= \mathcal{A}_F((x \cdot 0) \cdot 0) && \text{(by (2.5))} \\
&\geq \mathcal{A}_F(0). && \text{(by (3.16))}
\end{aligned}$$

Hence, $\mathcal{A}$ is a constant of X.

Conversely, it is obvious that every constant NF is a neutrosophic strong IUP-ideal. □

Theorem 3.11. *Every neutrosophic strong IUP-ideal of X is a neutrosophic IUP-subalgebra of X.*

Proof. It is straightforward by Theorem 3.10. □

Example 3.12. Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	1	3	2
2	3	4	0	2	5	1
3	2	5	3	0	1	4
4	4	2	1	5	0	3
5	1	3	5	4	2	0

Then X is an IUP-algebra. We define $\mathcal{A}$ on X as follows:

$$\mathcal{A}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.1 & 0.6 & 0.6 & 0.1 & 0.1 \end{pmatrix}$$

$$\mathcal{A}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 1 & 0.9 & 0.9 & 1 & 1 \end{pmatrix}$$

$$\mathcal{A}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.2 & 0.4 & 0.4 & 0.2 & 0.2 \end{pmatrix}$$

Then $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. Since $\mathcal{A}_T(2 \cdot 0) = \mathcal{A}_T(3) = 0.6 \ngeq 0.8 = \mathcal{A}_T(0)$, $\mathcal{A}_I(2 \cdot 0) = \mathcal{A}_I(3) = 0.9 \nleq 0.6 = \mathcal{A}_I(0)$, and $\mathcal{A}_F(4 \cdot 0) = \mathcal{A}_F(4) = 0.2 \ngeq 0.5 = \mathcal{A}_F(0)$. Hence, $\mathcal{A}$ is not a neutrosophic strong IUP-ideal of X.

Theorem 3.13. *Every neutrosophic strong IUP-ideal of X is a neutrosophic IUP-ideal of X.*

Proof. It is straightforward by Theorem 3.10. □

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Example 3.14. Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	5	3	0	2	1	4
3	4	2	5	0	3	1
4	3	5	1	4	0	2
5	2	4	3	1	5	0

Then X is an IUP-algebra. We define $\mathcal{A}$ on X as follows:

$$\mathcal{A}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.1 & 0.1 & 0.6 & 0.6 & 0.1 \end{pmatrix}$$

$$\mathcal{A}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.3 & 0.9 & 0.9 & 0.5 & 0.5 & 0.9 \end{pmatrix}$$

$$\mathcal{A}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.9 & 0.2 & 0.2 & 0.3 & 0.3 & 0.2 \end{pmatrix}$$

Then $\mathcal{A}$ is a neutrosophic IUP-ideal of X. Since $\mathcal{A}_T(5 \cdot 0) = \mathcal{A}_T(2) = 0.1 \ngeq 0.7 = \mathcal{A}_T(0)$, $\mathcal{A}_I(1 \cdot 0) = \mathcal{A}_I(1) = 0.9 \nleq 0.3 = \mathcal{A}_I(0)$, and $\mathcal{A}_F(1 \cdot 3) = \mathcal{A}_F(5) = 0.2 \ngeq 0.3 = \mathcal{A}_F(3)$. Hence, $\mathcal{A}$ is not a neutrosophic strong IUP-ideal of X.

Theorem 3.15. *Every neutrosophic IUP-ideal of X is a neutrosophic IUP-filter of X.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-ideal of X. By the assumption, it satisfies (3.5), (3.6), and (3.7). Let $x, y \in X$.

$$\begin{aligned} \mathcal{A}_T(y) &= \mathcal{A}_T(0 \cdot y) && \text{(by (IUP-1))} \\ &\geq \min\{\mathcal{A}_T(0 \cdot (x \cdot y)), \mathcal{A}_T(x)\} && \text{(by (3.8))} \\ &= \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}, && \text{(by (IUP-1))} \\ \mathcal{A}_I(y) &= \mathcal{A}_I(0 \cdot y) && \text{(by (IUP-1))} \\ &\leq \max\{\mathcal{A}_I(0 \cdot (x \cdot y)), \mathcal{A}_I(x)\} && \text{(by (3.9))} \\ &= \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}, && \text{(by (IUP-1))} \\ \mathcal{A}_F(y) &= \mathcal{A}_F(0 \cdot y) && \text{(by (IUP-1))} \\ &\geq \min\{\mathcal{A}_F(0 \cdot (x \cdot y)), \mathcal{A}_F(x)\} && \text{(by (3.10))} \\ &= \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}. && \text{(by (IUP-1))} \end{aligned}$$

Hence, $\mathcal{A}$ is a neutrosophic IUP-filter of X. □

Example 3.16. Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	2	4	0	5	1	3
3	3	2	1	0	5	4
4	4	3	5	1	0	2
5	1	5	3	4	2	0

Then X is an IUP-algebra. We define $\mathcal{A}$ on X as follows:

$$\mathcal{A}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.1 & 0.4 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

$$\mathcal{A}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.8 & 0.5 & 0.8 & 0.8 & 0.8 \end{pmatrix}$$

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$$\mathcal{A}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.2 & 0.3 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Then $\mathcal{A}$ is a neutrosophic IUP-filter of X. Since $\mathcal{A}_T(1 \cdot 5) = \mathcal{A}_T(1) = 0.1 \ngeq 0.4 = \min\{0.4, 0.4\} = \min\{\mathcal{A}_T(2), \mathcal{A}_T(2)\} = \min\{\mathcal{A}_T(1 \cdot 3), \mathcal{A}_T(2)\} = \min\{\mathcal{A}_T(1 \cdot (2 \cdot 5)), \mathcal{A}_T(2)\}$, $\mathcal{A}_I(1 \cdot 4) = \mathcal{A}_I(3) = 0.8 \nleq 0.5 = \max\{0.2, 0.5\} = \max\{\mathcal{A}_I(0), \mathcal{A}_I(2)\} = \max\{\mathcal{A}_I(1 \cdot 1), \mathcal{A}_I(2)\} = \max\{\mathcal{A}_I(1 \cdot (2 \cdot 4)), \mathcal{A}_I(2)\}$, and $\mathcal{A}_F(5 \cdot 3) = \mathcal{A}_F(4) = 0.2 \ngeq 0.3 = \min\{0.5, 0.3\} = \min\{\mathcal{A}_F(0), \mathcal{A}_F(2)\} = \min\{\mathcal{A}_F(5 \cdot 5), \mathcal{A}_F(2)\} = \min\{\mathcal{A}_F(5 \cdot (2 \cdot 3)), \mathcal{A}_F(2)\}$. Hence, $\mathcal{A}$ is not a neutrosophic IUP-ideal of X.

Theorem 3.17. *Every neutrosophic IUP-subalgebra of X is a neutrosophic IUP-filter of X.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. By Lemma 3.8, it satisfies (3.5), (3.6), and (3.7). Let $x, y \in X$.

$$\begin{aligned}
\mathcal{A}_T(y) &= \mathcal{A}_T(0 \cdot y) && \text{(by (IUP-1))} \\
&= \mathcal{A}_T((x \cdot 0) \cdot (x \cdot y)) && \text{(by (IUP-3))} \\
&\geq \min\{\mathcal{A}_T(x \cdot 0), \mathcal{A}_T(x \cdot y)\} && \text{(by (3.2))} \\
&\geq \min\{\min\{\mathcal{A}_T(x), \mathcal{A}_T(0)\}, \mathcal{A}_T(x \cdot y)\} && \text{(by (3.2))} \\
&= \min\{\mathcal{A}_T(x), \mathcal{A}_T(x \cdot y)\}, && \text{(by (3.5))} \\
\mathcal{A}_I(y) &= \mathcal{A}_I(0 \cdot y) && \text{(by (IUP-1))} \\
&= \mathcal{A}_I((x \cdot 0) \cdot (x \cdot y)) && \text{(by (IUP-3))} \\
&\leq \max\{\mathcal{A}_I(x \cdot 0), \mathcal{A}_I(x \cdot y)\} && \text{(by (3.3))} \\
&\leq \max\{\max\{\mathcal{A}_I(x), \mathcal{A}_I(0)\}, \mathcal{A}_I(x \cdot y)\} && \text{(by (3.3))} \\
&= \max\{\mathcal{A}_I(x), \mathcal{A}_I(x \cdot y)\}, && \text{(by (3.6))} \\
\mathcal{A}_F(y) &= \mathcal{A}_F(0 \cdot y) && \text{(by (IUP-1))} \\
&= \mathcal{A}_F((x \cdot 0) \cdot (x \cdot y)) && \text{(by (IUP-3))} \\
&\geq \min\{\mathcal{A}_F(x \cdot 0), \mathcal{A}_F(x \cdot y)\} && \text{(by (3.4))} \\
&\geq \min\{\min\{\mathcal{A}_F(x), \mathcal{A}_F(0)\}, \mathcal{A}_F(x \cdot y)\} && \text{(by (3.4))} \\
&= \min\{\mathcal{A}_F(x), \mathcal{A}_F(x \cdot y)\}. && \text{(by (3.7))}
\end{aligned}$$

Hence, $\mathcal{A}$ is a neutrosophic IUP-filter of X. □

Example 3.18. [13] Let $\mathbb{R}^*$ be the set of all nonzero real numbers. Define a binary operation $\cdot$ on $\mathbb{R}^*$ by:

$$(\forall x, y \in \mathbb{R}^*)(x \cdot y = \frac{y}{x}).$$

Thus, $(\mathbb{R}^*, \cdot, 1)$ is an IUP-algebra.

Example 3.19. From Example 3.18, let $P = \{x \in \mathbb{R}^* \mid x \geq 1\}$. Then $1 \in P$. Next, let $x, y, z \in \mathbb{R}^*$ be such that $x \cdot (y \cdot z) \geq 1$ and $y \geq 1$. Then $\frac{z}{yx} \geq 1$. Thus, $x \cdot z = \frac{z}{x} = (\frac{z}{yx})y \geq 1$, that is, $x \cdot z \in P$. Hence, P is an IUP-ideal of $\mathbb{R}^*$. Then P is an IUP-filter of $\mathbb{R}^*$. From Theorem 3.26 and 3.27, that is, $\mathcal{A}^{G}[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ are neutrosophic IUP-ideal and neutrosophic IUP-filter of $\mathbb{R}^*$. Implies that $\mathcal{A}$ are neutrosophic IUP-ideal and neutrosophic IUP-filter. Since $1, 3 \in s$ but $3 \cdot 1 = \frac{1}{3} \in P$, we have P is not an IUP-subalgebra of $\mathbb{R}^*$. From Theorem 3.25, that is, $\mathcal{A}^{G}[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is not a neutrosophic IUP-subalgebra. Implies that $\mathcal{A}$ is not a neutrosophic IUP-subalgebra.

Example 3.20. Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	5	2	3	1	0	4
5	4	3	1	2	5	0

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Then X is an IUP-algebra. We define $\mathcal{A}$ on X as follows:

$$\mathcal{A}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.2 & 0.2 & 0.6 & 0.2 & 0.2 \end{pmatrix}$$

$$\mathcal{A}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0.3 & 0.3 & 0.1 & 0.3 & 0.3 \end{pmatrix}$$

$$\mathcal{A}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.3 & 0.3 & 0.5 & 0.3 & 0.3 \end{pmatrix}$$

Then $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. Since $\mathcal{A}_T(1 \cdot 2) = \mathcal{A}_T(4) = 0.2 \ngeq 0.6 = \min\{0.6, 0.6\} = \min\{\mathcal{A}_T(3), \mathcal{A}_T(3)\} = \min\{\mathcal{A}_T(1\cdot(3\cdot 2)), \mathcal{A}_T(3)\}$, $\mathcal{A}_I(2\cdot 1) = \mathcal{A}_I(5) = 0.3 \nleq 0.1 = \max\{\mathcal{A}_I(3), \mathcal{A}_I(3)\} = \max\{\mathcal{A}_I(2\cdot(3\cdot 1)), \mathcal{A}_I(3)\}$, and $\mathcal{A}_F(4\cdot 1) = \mathcal{A}_F(2) = 0.3 \ngeq 0.5 = \min\{0.6, 0.5\} = \min\{\mathcal{A}_F(0), \mathcal{A}_F(3)\} = \min\{\mathcal{A}_F(4 \cdot (3 \cdot 1)), \mathcal{A}_F(3)\}$. Hence, $\mathcal{A}$ is not a neutrosophic IUP-ideal of X.

The study revealed a relationship between the four concepts: neutrosophic IUP-ideals and neutrosophic IUP-subalgebras are generalizations of neutrosophic strong IUP-ideals of IUP-algebras, where neutrosophic strong IUP-ideals of IUP-algebras can only be a constant NS. Neutrosophic IUP-filters are a generalization of neutrosophic IUP-ideals and neutrosophic IUP-subalgebras. We summarize the relationship between these four concepts, shown in Figure 2.

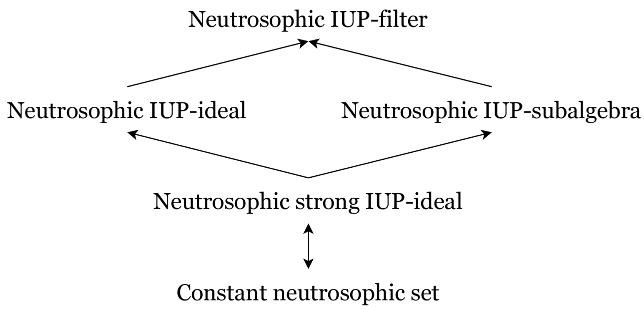

Figure 2: Neutrosophic sets in IUP-algebras

Theorem 3.21. *If $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X satisfying the following condition:*

$$(\forall x, y \in X)\left(x \cdot y \neq 0 \Rightarrow \begin{cases} \mathcal{A}_T(x) \geq \mathcal{A}_T(y) \\ \mathcal{A}_I(x) \leq \mathcal{A}_I(y) \\ \mathcal{A}_F(x) \geq \mathcal{A}_F(y) \end{cases} \right) \tag{3.17}$$

then $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X.

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X satisfying the condition (3.17). Let $x, y \in X$.

Case 1: Suppose $x \cdot y = 0$. Thus,

$$\begin{aligned} \mathcal{A}_T(x \cdot y) &= \mathcal{A}_T(0) \\ &\geq \mathcal{A}_T(y), && \text{(by (3.5))} \\ \mathcal{A}_I(x \cdot y) &= \mathcal{A}_I(0) \\ &\leq \mathcal{A}_I(y), && \text{(by (3.6))} \\ \mathcal{A}_F(x \cdot y) &= \mathcal{A}_F(0) \\ &\geq \mathcal{A}_F(y). && \text{(by (3.7))} \end{aligned}$$

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Case 2: Suppose $x \cdot y \neq 0$. Thus,

$$\begin{aligned}
\mathcal{A}_T(x \cdot y) &\geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\} && \text{(by (3.2))} \\
&= \mathcal{A}_T(y), \\
\mathcal{A}_I(x \cdot y) &\leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\} && \text{(by (3.6))} \\
&= \mathcal{A}_I(y), \\
\mathcal{A}_F(x \cdot y) &\geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\} && \text{(by (3.7))} \\
&= \mathcal{A}_F(y).
\end{aligned}$$

Hence, $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X. □

Theorem 3.22. *If $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X satisfying the following condition:*

$$\mathcal{A}_T = \mathcal{A}_I = \mathcal{A}_F \tag{3.18}$$

then $\mathcal{A}$ is a neutrosophic IUP-filter of X.

Proof. It is straightforward by Theorem 3.10. □

Theorem 3.23. *If $\mathcal{A}$ is a neutrosophic IUP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X) \begin{pmatrix} \mathcal{A}_T(y \cdot (x \cdot z)) = \mathcal{A}_T(x \cdot (y \cdot z)) \\ \mathcal{A}_I(y \cdot (x \cdot z)) = \mathcal{A}_I(x \cdot (y \cdot z)) \\ \mathcal{A}_F(y \cdot (x \cdot z)) = \mathcal{A}_F(x \cdot (y \cdot z)) \end{pmatrix} \tag{3.19}$$

then $\mathcal{A}$ is a neutrosophic IUP-ideal of X.

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-filter of X satisfying the condition (3.19). By the assumption, it satisfies (3.5), (3.6), and (3.7). Let $x, y \in X$. Thus,

$$\begin{aligned}
\mathcal{A}_T(x \cdot z) &\geq \min\{\mathcal{A}_T(y \cdot (x \cdot z)), \mathcal{A}_T(y)\} && \text{(by (3.11))} \\
&= \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}, \\
\mathcal{A}_I(x \cdot z) &\leq \max\{\mathcal{A}_I(y \cdot (x \cdot z)), \mathcal{A}_I(y)\} && \text{(by (3.12))} \\
&= \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}, \\
\mathcal{A}_F(x \cdot z) &\geq \min\{\mathcal{A}_F(y \cdot (x \cdot z)), \mathcal{A}_F(y)\} && \text{(by (3.13))} \\
&= \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}.
\end{aligned}$$

Hence, $\mathcal{A}$ is a neutrosophic IUP-ideal of X. □

For any fixed numbers $\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^- \in [0, 1]$ such that $\alpha^+ > \alpha^-, \beta^+ > \beta^-, \gamma^+ > \gamma^-$ and a nonempty subset G of X, an NS $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}] = (X, \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}], \mathcal{A}_I^G[^{\beta^-}_{\beta^+}], \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}])$ in X, where $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}], \mathcal{A}_I^G[^{\beta^-}_{\beta^+}]$, and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}]$ are function on X which are given as follows:

$$\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}] = \begin{cases} \alpha^+ & \text{if } x \in G \\ \alpha^- & \text{otherwise} \end{cases}$$

$$\mathcal{A}_I^G[^{\beta^-}_{\beta^+}] = \begin{cases} \beta^- & \text{if } x \in G \\ \beta^+ & \text{otherwise} \end{cases}$$

$$\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}] = \begin{cases} \gamma^+ & \text{if } x \in G \\ \gamma^- & \text{otherwise} \end{cases}$$

Lemma 3.24. *Let G be a nonempty subset of X. Then the constant 0 of X is in G if and only if the characteristic NS $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ satisfies (3.5), (3.6), and (3.7).*

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Proof. Assume that the constant 0 of X is in G. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](0) = \alpha^+$, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](0) = \beta^-$, and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](0) = \gamma^+$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](0) = \alpha^+ \geq \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)$, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](0) = \beta^- \leq \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x)$, and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](0) = \gamma^+ \geq \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x)$ for all $x \in X$, that is, $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies (3.5), (3.6), and (3.7).

Conversely, assume that $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies (3.5), (3.6), and (3.7). Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](0) \geq \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)$ for all $x \in X$. Since G is a nonempty subset of X, we let $a \in G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](0) \geq \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](a) = \alpha^+$, so $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](0) = \alpha^+$. Hence, the constant 0 of X is in G. □

Theorem 3.25. *A nonempty subset G is an IUP-subalgebra of X if and only if the characteristic NS $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic IUP-subalgebra of X.*

Proof. Assume that G is an IUP-subalgebra of X. Let $x, y \in X$. Then

Case 1 : Suppose $x, y \in G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$ and $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+ \geq \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 2 : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^-$ or $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^-$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) \geq \alpha^- = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 1' : Suppose $x, y \in G$. Then $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x) = \beta^-$ and $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^-$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^- \leq \beta^- = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 2' : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x) = \beta^+$ or $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^+$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot y) \leq \beta^+ = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 1" : Suppose $x, y \in G$. Then $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^+$ and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^+$. Since G is an IUP-subalgebra of X, We have $x \cdot y \in G$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^+ \geq \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Case 2" : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^-$ or $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^-$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y) \geq \gamma^- = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Hence, the characteristic NS $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic IUP-subalgebra of X.

Conversely, assume that the characteristic NS $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic IUP-subalgebra of X. Let $x, y \in G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$ and $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. By (3.2), we have $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) \geq \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\} = \min\{\alpha^+, \alpha^+\} = \alpha^+$. Thus $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+$, that is, $x \cdot y \in G$. Hence, G is an IUP-subalgebra of X. □

Theorem 3.26. *A nonempty subset G is an IUP-ideal of X if and only if the characteristic NS $\mathcal{A}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic IUP-ideal of X.*

Proof. Assume that G is an IUP-ideal of X. Since $0 \in G$, it follows from Lemma 3.24 that $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}]$, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}]$, and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}]$ satisfy (3.5), (3.6), and (3.7), respectively. Next, let $x, y, z \in X$.

Case 1 : Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) = \alpha^+ \geq \alpha^+ = \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 2 : Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^-$ or $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^-$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) \geq \alpha^- = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

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Case 1' : Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot z) = \beta^- \leq \beta^- = \max\{\beta^-, \beta^-\} = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 2' : Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)) = \beta^+$ or $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^+$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot z) \leq \beta^+ = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 1" : Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot z) = \gamma^+ \geq \gamma^+ = \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Case 2" : Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)) = \gamma^-$ or $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^-$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot z) \geq \gamma^- = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Hence, $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is a neutrosophic IUP-ideal of X.

Conversely, assume that the characteristic NS $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is a neutrosophic IUP-ideal of X. Since $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}]$ satisfies (3.5), it follow from Lemma 3.24 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^+$ and $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Thus, $\min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+$. By (3.8), we have $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) \geq \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+$, that is, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) = \alpha^+$. Hence, $x \cdot z \in G$, so G is an IUP-ideal. □

Theorem 3.27. *A nonempty subset G is an IUP-filter of X if and only if the characteristic NS $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is a neutrosophic IUP-filter of X.*

Proof. Assume that G is an IUP-filter of X. Since $0 \in G$, it follows from Lemma 3.24 that $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}]$, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}]$, and $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}]$ satisfy (3.5), (3.6), and (3.7), respectively. Next, let $x, y \in X$.

Case 1 : Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+ \geq \alpha^+ = \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)\}$.

Case 2 : Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^-$ or $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^-$. Thus, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) \geq \alpha^- = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)\}$.

Case 1' : Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^- \leq \beta^- = \max\{\beta^-, \beta^-\} = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot y), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x)\}$.

Case 2' : Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^+$ or $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x) = \beta^+$. Thus, $\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](y) \leq \beta^+ = \max\{\mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x \cdot y), \mathcal{A}_I^G[^{\beta^-}_{\beta^+}](x)\}$.

Case 1" : Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^+ \geq \gamma^+ = \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x)\}$.

Case 2" : Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^-$ or $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^-$. Thus, $\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](y) \geq \gamma^- = \min\{\mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y), \mathcal{A}_F^G[^{\gamma^+}_{\gamma^-}](x)\}$.

Hence, $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is a neutrosophic IUP-filter of X.

Conversely, assume that the characteristic NS $\mathcal{A}^G[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}]$ is a neutrosophic IUP-filter of X. Since $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}]$ satisfies (3.5), it follow from Lemma 3.24 that $0 \in G$. Next, let $x, y \in G$ be such that $x \cdot y \in G$ and $x \in G$. Then $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+$ and $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$. Thus, $\min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^+$. By (3.11), we have $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \min\{\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y), \mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^+$, that is, $\mathcal{A}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Hence, $y \in G$, so G is an IUP-filter of X. □

Theorem 3.28. *A nonempty subset G is a strong IUP-ideal of X if and only if the characteristic NS $\mathcal{A}^G\left[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}\right]$ is a neutrosophic strong IUP-ideal of X.*

Proof. It is straightforward by Theorem 3.10. □

Lemma 3.29. [29] *Let f be an FS in a nonempty set X. Then the following statements hold:*

$$(\forall x, y \in X)(1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}) \tag{3.20}$$

$$(\forall x, y \in X)(1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}) \tag{3.21}$$

Lemma 3.30. [29] *Let f be an FS in a nonempty set X. Then the following statements hold:*

$$(\forall x, y, z \in X)(f(z) \geq \min\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \leq \max\{\overline{f}(x), \overline{f}(y)\}) \tag{3.22}$$

$$(\forall x, y, z \in X)(f(z) \leq \max\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \geq \min\{\overline{f}(x), \overline{f}(y)\}) \tag{3.23}$$

Theorem 3.31. *An NS $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X if and only if the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy* (3.2)*, and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy* (3.3)*.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. Then

$$\mathcal{A}_T(x \cdot y) \geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\},$$
$$\mathcal{A}_I(x \cdot y) \leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\},$$
$$\mathcal{A}_F(x \cdot y) \geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}.$$

Thus,

$$\overline{\mathcal{A}}_T(x \cdot y) \leq \max\{\overline{\mathcal{A}}_T(x), \overline{\mathcal{A}}_T(y)\}, \quad \text{(by (3.22))}$$
$$\overline{\mathcal{A}}_I(x \cdot y) \geq \min\{\overline{\mathcal{A}}_I(x), \overline{\mathcal{A}}_I(y)\}, \quad \text{(by (3.23))}$$
$$\overline{\mathcal{A}}_F(x \cdot y) \leq \max\{\overline{\mathcal{A}}_F(x), \overline{\mathcal{A}}_F(y)\}. \quad \text{(by (3.22))}$$

Hence, the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.2), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.3).

Conversely, assume that the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.2), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.3). Then $\mathcal{A}_T$ and $\mathcal{A}_F$ satisfy (3.2), and $\mathcal{A}_I$ satisfy (3.3). Thus, $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. □

Theorem 3.32. *An NS $\mathcal{A}$ is a neutrosophic IUP-ideal of X if and only if the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy* (3.5) *and* (3.8)*, and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy* (3.6) *and* (3.9)*.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-ideal of X. Then

$$\mathcal{A}_T(0) \geq \mathcal{A}_T(x),$$
$$\mathcal{A}_I(0) \leq \mathcal{A}_I(x),$$
$$\mathcal{A}_F(0) \geq \mathcal{A}_F(x),$$
$$\mathcal{A}_T(x \cdot z) \geq \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\},$$
$$\mathcal{A}_I(x \cdot z) \leq \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\},$$
$$\mathcal{A}_F(x \cdot z) \geq \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}.$$

Thus,

$$\overline{\mathcal{A}}_T(0) \leq \overline{\mathcal{A}}_T(x),$$
$$\overline{\mathcal{A}}_I(0) \geq \overline{\mathcal{A}}_I(x),$$
$$\overline{\mathcal{A}}_F(0) \leq \overline{\mathcal{A}}_F(x),$$
$$\overline{\mathcal{A}}_T(x \cdot z) \leq \max\{\overline{\mathcal{A}}_T(x \cdot (y \cdot z)), \overline{\mathcal{A}}_T(y)\},$$
$$\overline{\mathcal{A}}_I(x \cdot z) \geq \min\{\overline{\mathcal{A}}_I(x \cdot (y \cdot z)), \overline{\mathcal{A}}_I(y)\},$$
$$\overline{\mathcal{A}}_F(x \cdot z) \leq \max\{\overline{\mathcal{A}}_F(x \cdot (y \cdot z)), \overline{\mathcal{A}}_F(y)\}.$$

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Hence, the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.5) and (3.8), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.6) and (3.9).

Conversely, assume that the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.5) and (3.8), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.6) and (3.9). Then $\mathcal{A}_T$ and $\mathcal{A}_F$ satisfy (3.5) and (3.8), and $\mathcal{A}_I$ satisfy (3.6) and (3.9). Hence, $\mathcal{A}$ is a neutrosophic IUP-ideal of X. □

Theorem 3.33. *An NS $\mathcal{A}$ is a neutrosophic IUP-filter of X if and only if the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy* (3.5) *and* (3.11)*, and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy* (3.6) *and* (3.12).

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-ideal of X. Then

$$\begin{aligned}
\mathcal{A}_T(0) &\geq \mathcal{A}_T(x), \\
\mathcal{A}_I(0) &\leq \mathcal{A}_I(x), \\
\mathcal{A}_F(0) &\geq \mathcal{A}_F(x), \\
\mathcal{A}_T(y) &\geq \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}, \\
\mathcal{A}_I(y) &\leq \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}, \\
\mathcal{A}_F(y) &\geq \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\mathcal{A}}_T(0) &\leq \overline{\mathcal{A}}_T(x), \\
\overline{\mathcal{A}}_I(0) &\geq \overline{\mathcal{A}}_I(x), \\
\overline{\mathcal{A}}_F(0) &\leq \overline{\mathcal{A}}_F(x), \\
\overline{\mathcal{A}}_T(y) &\leq \max\{\overline{\mathcal{A}}_T(x \cdot y), \overline{\mathcal{A}}_T(x)\}, \\
\overline{\mathcal{A}}_I(y) &\geq \min\{\overline{\mathcal{A}}_I(x \cdot y), \overline{\mathcal{A}}_I(x)\}, \\
\overline{\mathcal{A}}_F(y) &\leq \max\{\overline{\mathcal{A}}_F(x \cdot y), \overline{\mathcal{A}}_F(x)\}.
\end{aligned}$$

Hence, the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.5) and (3.11), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.6) and (3.12).

Conversely, assume that the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy (3.5) and (3.11), and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy (3.6) and (3.12). Then $\mathcal{A}_T$ and $\mathcal{A}_F$ satisfy (3.5) and (3.11), and $\mathcal{A}_I$ satisfy (3.6) and (3.12). Hence, $\mathcal{A}$ is a neutrosophic IUP-filter of X. □

Theorem 3.34. *An NS $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X if and only if the FSs $\mathcal{A}_T$, $\overline{\mathcal{A}}_I$, and $\mathcal{A}_F$ satisfy* (3.14)*, and the FSs $\overline{\mathcal{A}}_T$, $\mathcal{A}_I$, and $\overline{\mathcal{A}}_F$ satisfy* (3.15).

Proof. It is straightforward by Theorem 3.10. □

The following four theorems are derived directly by applying Theorems 3.31, 3.32, 3.33, and 3.34, respectively.

Theorem 3.35. *An NS $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X if and only if NS $*\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_F)$, $\square\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_F, \mathcal{A}_F)$, $\Diamond\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \mathcal{A}_F)$, $\triangle\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, $\blacksquare\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_T)$, $\blacklozenge\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, and $\blacktriangle\mathcal{A} = (\mathcal{A}_F, \overline{\mathcal{A}}_F, \mathcal{A}_F)$ are neutrosophic IUP-subalgebras of X.*

Theorem 3.36. *An NS $\mathcal{A}$ is a neutrosophic IUP-ideal of X if and only if NS $*\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_F)$, $\square\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_F, \mathcal{A}_F)$, $\Diamond\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \mathcal{A}_F)$, $\triangle\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, $\blacksquare\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_T)$, $\blacklozenge\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, and $\blacktriangle\mathcal{A} = (\mathcal{A}_F, \overline{\mathcal{A}}_F, \mathcal{A}_F)$ are neutrosophic IUP-ideals of X.*

Theorem 3.37. *An NS $\mathcal{A}$ is a neutrosophic IUP-filter of X if and only if NS $*\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_F)$, $\square\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_F, \mathcal{A}_F)$, $\Diamond\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \mathcal{A}_F)$, $\triangle\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, $\blacksquare\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_T)$, $\blacklozenge\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, and $\blacktriangle\mathcal{A} = (\mathcal{A}_F, \overline{\mathcal{A}}_F, \mathcal{A}_F)$ are neutrosophic IUP-filters of X.*

Theorem 3.38. *An NS $\mathcal{A}$ is a neutrosophic strong IUP-ideal of X if and only if NS $*\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_F)$, $\square\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_F, \mathcal{A}_F)$, $\Diamond\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \mathcal{A}_F)$, $\triangle\mathcal{A} = (\mathcal{A}_T, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, $\blacksquare\mathcal{A} = (\mathcal{A}_T, \overline{\mathcal{A}}_T, \mathcal{A}_T)$, $\blacklozenge\mathcal{A} = (\overline{\mathcal{A}}_I, \mathcal{A}_I, \overline{\mathcal{A}}_I)$, and $\blacktriangle\mathcal{A} = (\mathcal{A}_F, \overline{\mathcal{A}}_F, \mathcal{A}_F)$ are neutrosophic strong IUP-ideals of X.*

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Definition 3.39. [22] Let f be an FS in a nonempty set X. For any $t \in [0,1]$, the sets

$$U(f;t) = \{x \in X \mid f(x) \geq t\}, \tag{3.24}$$
$$L(f;t) = \{x \in X \mid f(x) \leq t\}, \tag{3.25}$$
$$E(f;t) = \{x \in X \mid f(x) = t\} \tag{3.26}$$

are called an upper t-level subset and a lower t-level subset of f, respectively. The sets

$$U^{+}(f;t) = \{x \in X \mid f(x) > t\}, \tag{3.27}$$
$$L^{-}(f;t) = \{x \in X \mid f(x) < t\} \tag{3.28}$$

are called an upper t-strong level subset and a lower t-strong level subset of f, respectively.

Before delving into the theorems that explore the connection between level subsets and their associated NSs, it is crucial to understand the foundational concepts. Level subsets are pivotal in defining NSs by outlining how membership degrees are distributed. The subsequent theorem articulates this relationship, offering valuable insights into the underlying structure of NSs.

Theorem 3.40. *An NS $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. Let $\alpha \in [0,1]$ be such that $U(\mathcal{A}_T;\alpha) \neq \emptyset$. Let $x, y \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(x) \geq \alpha$ and $\mathcal{A}_T(y) \geq \alpha$. Thus, $\min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\} \geq \alpha$. By (3.2), we have $\mathcal{A}_T(x \cdot y) \geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\} \geq \alpha$, that is, $\mathcal{A}_T(x \cdot y) \geq \alpha$. Thus, $x \cdot y \in U(\mathcal{A}_T;\alpha)$. Hence, $U(\mathcal{A}_T;\alpha)$ is an IUP-subalgebra of X.

Let $\beta \in [0,1]$ be such that $L(\mathcal{A}_I;\beta) \neq \emptyset$. Let $x, y \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(x) \leq \beta$ and $\mathcal{A}_I(y) \leq \beta$. Thus, $\max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\} \leq \beta$. By (3.3), we have $\mathcal{A}_I(x \cdot y) \leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\} \leq \beta$, that is, $\mathcal{A}_I(x \cdot y) \leq \beta$. Thus, $x \cdot y \in L(\mathcal{A}_I;\beta)$. Hence, $L(\mathcal{A}_I;\beta)$ is an IUP-subalgebra of X.

Let $\gamma \in [0,1]$ be such that $U(\mathcal{A}_F;\gamma) \neq \emptyset$. Let $x, y \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(x) \geq \gamma$ and $\mathcal{A}_F(y) \geq \gamma$. Thus, $\min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\} \geq \gamma$. By (3.4), we have $\mathcal{A}_F(x \cdot y) \geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\} \geq \gamma$, that is, $\mathcal{A}_F(x \cdot y) \geq \gamma$. Thus, $x \cdot y \in U(\mathcal{A}_F;\gamma)$. Hence, $U(\mathcal{A}_F;\gamma)$ is an IUP-subalgebra of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Let $\alpha = \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\}$. Then $\mathcal{A}_T(x) \geq \alpha$ and $\mathcal{A}_T(y) \geq \alpha$. Thus, $x, y \in U(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_T;\alpha)$ is an IUP-subalgebra of X. By (2.17), we have $x \cdot y \in U(\mathcal{A}_T;\alpha)$. Thus, $\mathcal{A}_T(x \cdot y) \geq \alpha = \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\}$.

Let $x, y \in X$. Let $\beta = \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\}$. Then $\mathcal{A}_I(x) \leq \beta$ and $\mathcal{A}_I(y) \leq \beta$. Thus, $x, y \in L(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{A}_I;\beta)$ is an IUP-subalgebra of X. By (2.17), we have $x \cdot y \in L(\mathcal{A}_I;\beta)$. Thus, $\mathcal{A}_I(x \cdot y) \leq \beta = \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\}$.

Let $x, y \in X$. Let $\gamma = \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}$. Then $\mathcal{A}_F(x) \geq \gamma$ and $\mathcal{A}_F(y) \geq \gamma$. Thus, $x, y \in U(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_F;\gamma)$ is an IUP-subalgebra of X. By (2.17), we have $x \cdot y \in U(\mathcal{A}_F;\gamma)$. Thus, $\mathcal{A}_F(x \cdot y) \geq \gamma = \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}$.

Hence, $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. □

Theorem 3.41. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{A}$ in X is a neutrosophic IUP-ideal of X. Let $\alpha \in [0,1]$ be such that $U(\mathcal{A}_T;\alpha) \neq \emptyset$. Let $a \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(a) \geq \alpha$. By (3.5), we have $\mathcal{A}_T(0) \geq \mathcal{A}_T(a) \geq \alpha$. Thus, $0 \in U(\mathcal{A}_T;\alpha)$. Let $x, y, z \in lom$ be such that $x \cdot (y \cdot z) \in U(\mathcal{A}_T;\alpha)$ and $y \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(x \cdot (y \cdot z)) \geq \alpha$ and $\mathcal{A}_T(y) \geq \alpha$. Thus, $\min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\} \geq \alpha$. By (3.8), we have $\mathcal{A}_T(x \cdot z) \geq \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\} \geq \alpha$. Thus, $x \cdot z \in U(\mathcal{A}_T;\alpha)$. Hence, $U(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X.

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Let $\beta \in [0,1]$ be such that $L(\mathcal{A}_I;\beta) \neq \emptyset$. Let $b \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(b) \leq \beta$. By (3.6), we have $\mathcal{A}_I(0) \leq \mathcal{A}_I(b) \leq \beta$. Thus, $0 \in L(\mathcal{A}_I;\beta)$. Let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in L(\mathcal{A}_I;\beta)$ and $y \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(x \cdot (y \cdot z)) \leq \beta$ and $\mathcal{A}_I(y) \leq \beta$. Thus, $\max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\} \leq \beta$. By (3.9), we have $\mathcal{A}_I(x \cdot z) \leq \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\} \leq \beta$. Thus, $x \cdot z \in L(\mathcal{A}_I;\beta)$. Hence, $L(\mathcal{A}_I;\beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U(\mathcal{A}_F;\gamma) \neq \emptyset$. Let $c \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(c) \geq \gamma$. By (3.7), we have $\mathcal{A}_F(0) \geq \mathcal{A}_F(c) \geq \gamma$. Thus, $0 \in U(\mathcal{A}_F;\gamma)$. Let $x,y,z \in lom$ be such that $x \cdot (y \cdot z) \in U(\mathcal{A}_F;\gamma)$ and $y \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(x \cdot (y \cdot z)) \geq \gamma$ and $\mathcal{A}_F(y) \geq \gamma$. Thus, $\min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\} \geq \gamma$. By (3.10), we have $\mathcal{A}_F(x \cdot z) \geq \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\} \geq \gamma$. Thus, $x \cdot z \in U(\mathcal{A}_F;\gamma)$. Hence, $U(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are either empty or IUP-ideals of X. Let $x \in X$. Let $\alpha = \mathcal{A}_T(x)$. Then $\mathcal{A}_T(x) \geq \alpha$. Thus, $x \in U(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(0) \geq \alpha = \mathcal{A}_T(x)$. Let $x,y,z \in X$. Let $\alpha = \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}$. Then $\mathcal{A}_T(x \cdot (y \cdot z)) \geq \alpha$ and $\mathcal{A}_T(y) \geq \alpha$. Thus, $x \cdot (y \cdot z), y \in U(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U(\mathcal{A}_T;\alpha)$. Thus, $\mathcal{A}_T(x \cdot z) \geq \alpha = \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}$.

Let $x \in X$. Let $\beta = \mathcal{A}_I(x)$. Then $\mathcal{A}_I(x) \leq \beta$. Thus, $x \in L(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{A}_I;\beta)$ is an IUP-ideal of X. By (2.18), we have $0 \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(0) \leq \beta = \mathcal{A}_I(x)$. Let $x,y,z \in X$. Let $\beta = \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}$. Then $\mathcal{A}_I(x \cdot (y \cdot z)) \leq \beta$ and $\mathcal{A}_I(y) \leq \beta$. Thus, $x \cdot (y \cdot z), y \in L(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_I;\beta)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in L(\mathcal{A}_I;\beta)$. Thus, $\mathcal{A}_I(x \cdot z) \leq \beta = \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}$.

Let $x \in X$. Let $\gamma = \mathcal{A}_F(x)$. Then $\mathcal{A}_F(x) \geq \gamma$. Thus, $x \in U(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X. By (2.18), we have $0 \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(0) \geq \gamma = \mathcal{A}_F(x)$. Let $x,y,z \in X$. Let $\gamma = \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}$. Then $\mathcal{A}_F(x \cdot (y \cdot z)) \geq \gamma$ and $\mathcal{A}_F(y) \geq \gamma$. Thus, $x \cdot (y \cdot z), y \in U(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U(\mathcal{A}_F;\gamma)$. Thus, $\mathcal{A}_F(x \cdot z) \geq \gamma = \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}$. Hence, $\mathcal{A}$ is a neutrosophic IUP-ideal of X. □

Theorem 3.42. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-filter of X if and only if for all $\alpha,\beta,\gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{A}$ in X is a neutrosophic IUP-filter of X. Let $\alpha \in [0,1]$ be such that $U(\mathcal{A}_T;\alpha) \neq \emptyset$. Let $a \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(a) \geq \alpha$. By (3.5), we have $\mathcal{A}_T(0) \geq \mathcal{A}_T(a) \geq \alpha$. Thus, $0 \in U(\mathcal{A}_T;\alpha)$. Let $x,y \in X$ be such that $x \cdot y \in U(\mathcal{A}_T;\alpha)$ and $x \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(x \cdot y) \geq \alpha$ and $\mathcal{A}_T(x) \geq \alpha$. Thus, $\min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\} \geq \alpha$. By (3.11), we have $\mathcal{A}_T(y) \geq \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\} \geq \alpha$. Thus, $y \in U(\mathcal{A}_T;\alpha)$. Hence, $U(\mathcal{A}_T;\alpha)$ is an IUP-filter of X.

Let $\beta \in [0,1]$ be such that $L(\mathcal{A}_I;\beta) \neq \emptyset$. Let $b \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(b) \leq \beta$. By (3.6), we have $\mathcal{A}_I(0) \leq \mathcal{A}_I(b) \leq \beta$. Thus, $0 \in L(\mathcal{A}_I;\beta)$. Let $x,y \in X$ be such that $x \cdot y \in L(\mathcal{A}_I;\beta)$ and $x \in L(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(x \cdot y) \leq \beta$ and $\mathcal{A}_I(x) \leq \beta$. Thus, $\max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\} \leq \beta$. By (3.12), we have $\mathcal{A}_I(y) \leq \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\} \leq \beta$. Thus, $y \in L(\mathcal{A}_I;\beta)$. Hence, $L(\mathcal{A}_I;\beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U(\mathcal{A}_F;\gamma) \neq \emptyset$. Let $c \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(c) \geq \gamma$. By (3.7), we have $\mathcal{A}_F(0) \geq \mathcal{A}_F(c) \geq \gamma$. Thus, $0 \in U(\mathcal{A}_F;\gamma)$. Let $x,y \in X$ be such that $x \cdot y \in U(\mathcal{A}_F;\gamma)$ and $x \in U(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(x \cdot y) \geq \gamma$ and $\mathcal{A}_F(x) \geq \gamma$. Thus, $\min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\} \geq \gamma$. By (3.13), we have $\mathcal{A}_F(y) \geq \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\} \geq \gamma$. Thus, $y \in U(\mathcal{A}_F;\gamma)$. Hence, $U(\mathcal{A}_F;\gamma)$ is an IUP-filter of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [0,1]$, the sets $U(\mathcal{A}_T;\alpha)$, $L(\mathcal{A}_I;\beta)$, and $U(\mathcal{A}_F;\gamma)$ are empty or IUP-filters of X. Let $x \in X$. Let $\alpha = \mathcal{A}_T(x)$. Then $\mathcal{A}_T(x) \geq \alpha$. Thus, $x \in U(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_T;\alpha)$ is an IUP-filter of X. By (2.18), we have $0 \in U(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(0) \geq \alpha = \mathcal{A}_T(x)$. Let $x,y \in X$. Let $\alpha = \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}$. Then $\mathcal{A}_T(x \cdot y) \geq \alpha$ and $\mathcal{A}_T(x) \geq \alpha$. Thus, $x \cdot y, x \in U(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_T;\alpha)$ is an IUP-filter of X. By (2.19), we have $y \in U(\mathcal{A}_T;\alpha)$. Thus, $\mathcal{A}_T(y) \geq \alpha = \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}$.

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Let $x \in X$. Let $\beta = \mathcal{A}_I(x)$. Then $\mathcal{A}_I(x) \leq \beta$. Thus, $x \in L(\mathcal{A}_I; \beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{A}_I; \beta)$ is an IUP-filter of X. By (2.18), we have $0 \in L(\mathcal{A}_I; \beta)$. Then $\mathcal{A}_I(0) \leq \beta = \mathcal{A}_I(x)$. Let $x, y \in X$. Let $\beta = \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}$. Then $\mathcal{A}_I(x \cdot y) \leq \beta$ and $\mathcal{A}_I(x) \leq \beta$. Thus, $x \cdot y, x \in L(\mathcal{A}_I; \beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{A}_I; \beta)$ is an IUP-filter of X. By (2.19), we have $y \in L(\mathcal{A}_I; \beta)$. Thus, $\mathcal{A}_I(y) \leq \beta = \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}$.

Let $x \in X$. Let $\gamma = \mathcal{A}_F(x)$. Then $\mathcal{A}_F(x) \geq \gamma$. Thus, $x \in U(\mathcal{A}_F; \gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_F; \gamma)$ is an IUP-filter of X. By (2.18), we have $0 \in U(\mathcal{A}_F; \gamma)$. Then $\mathcal{A}_F(0) \geq \gamma = \mathcal{A}_F(x)$. Let $x, y \in X$. Let $\gamma = \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}$. Then $\mathcal{A}_F(x \cdot y) \geq \gamma$ and $\mathcal{A}_F(x) \geq \gamma$. Thus, $x \cdot y, x \in U(\mathcal{A}_F; \gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{A}_F; \gamma)$ is an IUP-filter of X. By (2.19), we have $y \in U(\mathcal{A}_F; \gamma)$. Thus, $\mathcal{A}_F(y) \geq \gamma = \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}$. Hence, $\mathcal{A}$ is a neutrosophic IUP-filter of X. □

Theorem 3.43. *An NS $\mathcal{A}$ in X is a neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the sets $U(\mathcal{A}_T; \alpha)$, $L(\mathcal{A}_I; \beta)$, and $U(\mathcal{A}_F; \gamma)$ are either empty or strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 3.10. □

Theorem 3.44. *An NS $\mathcal{A}$ in X is a neutrosophic strong IUP-ideal of X if and only if the sets $E(\mathcal{A}_T; \mathcal{A}_T(0))$, $E(\mathcal{A}_I; \mathcal{A}_I(0))$, and $E(\mathcal{A}_F; \mathcal{A}_F(0))$ are strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 3.10. □

Theorem 3.45. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the sets $U^+(\mathcal{A}_T; \alpha)$, $L^-(\mathcal{A}_I; \beta)$, and $U^+(\mathcal{A}_F; \gamma)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{A}$ in X is a neutrosophic IUP-subalgebra of X. Let $\alpha \in [0, 1]$ be such that $U^+(\mathcal{A}_T; \alpha) \neq \emptyset$. Let $x, y \in U^+(\mathcal{A}_T; \alpha)$. Then $\mathcal{A}_T(x) > \alpha$ and $\mathcal{A}_T(y) > \alpha$. Thus, $\min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\} > \alpha$. By (3.2), we have $\mathcal{A}_T(x \cdot y) \geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\} > \alpha$. Thus, $x \cdot y \in U^+(\mathcal{A}_T; \alpha)$. Hence, $U^+(\mathcal{A}_T; \alpha)$ is an IUP-subalgebra of X.

Let $\beta \in [0, 1]$ be such that $L^-(\mathcal{A}_I; \beta) \neq \emptyset$. Let $x, y \in L^-(\mathcal{A}_I; \beta)$. Then $\mathcal{A}_I(x) < \beta$ and $\mathcal{A}_I(y) < \beta$. Thus, $\max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\} < \beta$. By (3.3), we have $\mathcal{A}_I(x \cdot y) \leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\} < \beta$. Thus, $x \cdot y \in L^-(\mathcal{A}_I; \beta)$. Hence, $L^-(\mathcal{A}_I; \beta)$ is an IUP-subalgebra of X.

Let $\gamma \in [0, 1]$ be such that $U^+(\mathcal{A}_F; \gamma) \neq \emptyset$. Let $x, y \in U^+(\mathcal{A}_F; \gamma)$. Then $\mathcal{A}_F(x) > \gamma$ and $\mathcal{A}_F(y) > \gamma$. Thus, $\min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\} > \gamma$. By (3.2), we have $\mathcal{A}_F(x \cdot y) \geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\} > \gamma$. Thus, $x \cdot y \in U^+(\mathcal{A}_F; \gamma)$. Hence, $U^+(\mathcal{A}_F; \gamma)$ is an IUP-subalgebra of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0, 1]$, the sets $U^+(\mathcal{A}_T; \alpha)$, $L^-(\mathcal{A}_I; \beta)$, and $U^+(\mathcal{A}_F; \gamma)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Assume that $\mathcal{A}_T(x \cdot y) < \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\}$. Let $\alpha = \mathcal{A}_T(x \cdot y)$. Then $\mathcal{A}_T(x) > \alpha$ and $\mathcal{A}_T(y) > \alpha$. Thus, $x, y \in U^+(\mathcal{A}_T; \alpha)$. By the assumption, we have $U^+(\mathcal{A}_T; \alpha)$ is an IUP-subalgebra. By (2.17), we have $x \cdot y \in U^+(\mathcal{A}_T; \alpha)$. So $\mathcal{A}_T(x \cdot y) > \alpha = \mathcal{A}_T(x \cdot y)$, which is a contradiction. Thus, $\mathcal{A}_T(x \cdot y) \geq \min\{\mathcal{A}_T(x), \mathcal{A}_T(y)\}$.

Let $x, y \in X$. Assume that $\mathcal{A}_I(x \cdot y) > \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\}$. Let $\beta = \mathcal{A}_I(x \cdot y)$. Then $\mathcal{A}_I(x) < \beta$ and $\mathcal{A}_I(y) < \beta$. Thus, $x, y \in L^-(\mathcal{A}_I; \beta)$. By the assumption, we have $L^-(\mathcal{A}_I; \beta)$ is an IUP-subalgebra. By (2.17), we have $x \cdot y \in L^-(\mathcal{A}_I; \beta)$. So $\mathcal{A}_I(x \cdot y) < \beta = \mathcal{A}_I(x \cdot y)$, which is a contradiction. Thus, $\mathcal{A}_I(x \cdot y) \leq \max\{\mathcal{A}_I(x), \mathcal{A}_I(y)\}$.

Let $x, y \in X$. Assume that $\mathcal{A}_F(x \cdot y) < \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}$. Let $\gamma = \mathcal{A}_F(x \cdot y)$. Then $\mathcal{A}_F(x) > \gamma$ and $\mathcal{A}_F(y) > \gamma$. Thus, $x, y \in U^+(\mathcal{A}_F; \gamma)$. By the assumption, we have $U^+(\mathcal{A}_F; \gamma)$ is an IUP-subalgebra. By (2.17), we have $x \cdot y \in U^+(\mathcal{A}_F; \gamma)$. So $\mathcal{A}_F(x \cdot y) > \gamma = \mathcal{A}_F(x \cdot y)$, which is a contradiction. Thus, $\mathcal{A}_F(x \cdot y) \geq \min\{\mathcal{A}_F(x), \mathcal{A}_F(y)\}$.

Hence, $\mathcal{A}$ is a neutrosophic IUP-subalgebra of X. □

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Theorem 3.46. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{A}_T;\alpha)$, $L^-(\mathcal{A}_I;\beta)$, and $U^+(\mathcal{A}_F;\gamma)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{A}$ in X is a neutrosophic IUP-ideal of X. Let $\alpha \in [0,1]$ be such that $U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. Let $a \in U^+(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(a) > \alpha$. By (3.5), we have $\mathcal{A}_T(0) \geq \mathcal{A}_T(a) > \alpha$. Thus, $0 \in U^+(\mathcal{A}_T;\alpha)$. Let $x, y, z \in U^+(\mathcal{A}_T;\alpha)$ be such that $x \cdot (y \cdot z), y \in U^+(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(x \cdot (y \cdot z)) > \alpha$ and $\mathcal{A}_T(y) > \alpha$. Thus, $\min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\} > \alpha$. By (3.8). we have $\mathcal{A}_T(x \cdot z) \geq \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\} > \alpha$. Thus, $x \cdot z \in U^+(\mathcal{A}_T;\alpha)$. Hence, $U^+(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X.

Let $\beta \in [0,1]$ be such that $L^-(\mathcal{A}_I;\beta) \neq \emptyset$. Let $b \in L^-(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(b) < \beta$. By (3.6), we have $\mathcal{A}_I(0) \leq \mathcal{A}_I(b) < \beta$. Thus, $0 \in L^-(\mathcal{A}_I;\beta)$. Let $x, y, z \in L^-(\mathcal{A}_I;\beta)$ be such that $x \cdot (y \cdot z), y \in L^-(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(x \cdot (y \cdot z)) < \beta$ and $\mathcal{A}_T(y) < \beta$. Thus, $\max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\} < \beta$. By (3.9). we have $\mathcal{A}_I(x \cdot z) \leq \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\} > \beta$. Thus, $x \cdot z \in L^-(\mathcal{A}_I;\beta)$. Hence, $L^-(\mathcal{A}_I;\beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. Let $c \in U^+(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(c) > \gamma$. By (3.7), we have $\mathcal{A}_F(0) \geq \mathcal{A}_F(c) > \gamma$. Thus, $0 \in U^+(\mathcal{A}_F;\gamma)$. Let $x, y, z \in U^+(\mathcal{A}_F;\gamma)$ be such that $x \cdot (y \cdot z), y \in U^+(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(x \cdot (y \cdot z)) > \gamma$ and $\mathcal{A}_F(y) > \gamma$. Thus, $\min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\} > \gamma$. By (3.10). we have $\mathcal{A}_F(x \cdot z) \geq \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\} > \gamma$. Thus, $x \cdot z \in U^+(\mathcal{A}_F;\gamma)$. Hence, $U^+(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{A}_T;\alpha)$, $L^-(\mathcal{A}_I;\beta)$, and $U^+(\mathcal{A}_F;\gamma)$ are either empty or IUP-ideals of X. Let $x \in X$. Assume that $\mathcal{A}_T(0) < \mathcal{A}_T(x)$. Let $\alpha = \mathcal{A}_T(0)$. Then $x \in U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{A}_T;\alpha)$. So $\mathcal{A}_T(0) > \alpha = \mathcal{A}_T(0)$, which is a contradiction. Thus, $\mathcal{A}_T(0) \geq \mathcal{A}_T(x)$. Let $x, y, z \in X$. Assume that $\mathcal{A}_T(x \cdot z) < \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}$. Let $\alpha = \mathcal{A}_T(x \cdot z)$. Then $x \cdot (y \cdot z), y \in U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U^+(\mathcal{A}_T;\alpha)$. So $\mathcal{A}_T(x \cdot z) > \alpha = \mathcal{A}_T(x \cdot z)$, which is a contradiction. Thus, $\mathcal{A}_T(x \cdot z) \geq \min\{\mathcal{A}_T(x \cdot (y \cdot z)), \mathcal{A}_T(y)\}$.

Let $x \in X$. Assume that $\mathcal{A}_I(0) > \mathcal{A}_I(x)$. Let $\beta = \mathcal{A}_I(0)$. Then $x \in L^-(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{A}_I;\beta)$ is an IUP-ideal of X. By (2.18), we have $0 \in L^-(\mathcal{A}_I;\beta)$. So $\mathcal{A}_I(0) < \beta = \mathcal{A}_I(0)$, which is a contradiction. Thus, $\mathcal{A}_I(0) \leq \mathcal{A}_I(x)$. Let $x, y, z \in X$. Assume that $\mathcal{A}_I(x \cdot z) > \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}$. Let $\beta = \mathcal{A}_I(x \cdot z)$. Then $x \cdot (y \cdot z), y \in L^-(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{A}_I;\beta)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in L^-(\mathcal{A}_I;\beta)$. So $\mathcal{A}_I(x \cdot z) < \beta = \mathcal{A}_I(x \cdot z)$, which is a contradiction. Thus, $\mathcal{A}_I(x \cdot z) \leq \max\{\mathcal{A}_I(x \cdot (y \cdot z)), \mathcal{A}_I(y)\}$.

Let $x \in X$. Assume that $\mathcal{A}_F(0) < \mathcal{A}_F(x)$. Let $\gamma = \mathcal{A}_F(0)$. Then $x \in U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{A}_F;\gamma)$. So $\mathcal{A}_F(0) > \gamma = \mathcal{A}_F(0)$, which is a contradiction. Thus, $\mathcal{A}_F(0) \geq \mathcal{A}_F(x)$. Let $x, y, z \in X$. Assume that $\mathcal{A}_F(x \cdot z) < \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}$. Let $\gamma = \mathcal{A}_F(x \cdot z)$. Then $x \cdot (y \cdot z), y \in U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X. By (2.20), we have $x \cdot z \in U^+(\mathcal{A}_F;\gamma)$. So $\mathcal{A}_F(x \cdot z) > \gamma = \mathcal{A}_F(x \cdot z)$, which is a contradiction. Thus, $\mathcal{A}_F(x \cdot z) \geq \min\{\mathcal{A}_F(x \cdot (y \cdot z)), \mathcal{A}_F(y)\}$.

Hence, $\mathcal{A}$ is a neutrosophic IUP-ideal of X. □

Theorem 3.47. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{A}_T;\alpha)$, $L^-(\mathcal{A}_I;\beta)$, and $U^+(\mathcal{A}_F;\gamma)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{A}$ in X is a neutrosophic IUP-filter of X. Let $\alpha \in [0,1]$ be such that $U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. Let $a \in U^+(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(a) > \alpha$. By (3.5), we have $\mathcal{A}_T(0) \geq \mathcal{A}_T(a) > \alpha$. Thus, $0 \in U^+(\mathcal{A}_T;\alpha)$. Let $x, y \in U^+(\mathcal{A}_T;\alpha)$ be such that $x \cdot y, x \in U^+(\mathcal{A}_T;\alpha)$. Then $\mathcal{A}_T(x \cdot y) > \alpha$ and $\mathcal{A}_T(x) > \alpha$. Thus, $\min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\} > \alpha$. By (3.11). we have $\mathcal{A}_T(y) \geq \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\} > \alpha$. Thus, $y \in U^+(\mathcal{A}_T;\alpha)$. Hence, $U^+(\mathcal{A}_T;\alpha)$ is an IUP-filter of X.

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Let $\beta \in [0,1]$ be such that $L^-(\mathcal{A}_I;\beta) \neq \emptyset$. Let $b \in L^-(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(b) < \beta$. By (3.6), we have $\mathcal{A}_I(0) \leq \mathcal{A}_I(b) < \beta$. Thus, $0 \in L^-(\mathcal{A}_I;\beta)$. Let $x,y \in L^-(\mathcal{A}_I;\beta)$ be such that $x \cdot y, x \in L^-(\mathcal{A}_I;\beta)$. Then $\mathcal{A}_I(x \cdot y) < \beta$ and $\mathcal{A}_T(x) < \beta$. Thus, $\max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\} < \beta$. By (3.12). we have $\mathcal{A}_I(y) \leq \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\} > \beta$. Thus, $y \in L^-(\mathcal{A}_I;\beta)$. Hence, $L^-(\mathcal{A}_I;\beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. Let $c \in U^+(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(c) > \gamma$. By (3.7), we have $\mathcal{A}_F(0) \geq \mathcal{A}_F(c) > \gamma$. Thus, $0 \in U^+(\mathcal{A}_F;\gamma)$. Let $x,y \in U^+(\mathcal{A}_F;\gamma)$ be such that $x \cdot y, x \in U^+(\mathcal{A}_F;\gamma)$. Then $\mathcal{A}_F(x \cdot y) > \gamma$ and $\mathcal{A}_F(x) > \gamma$. Thus, $\min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\} > \gamma$. By (3.13). we have $\mathcal{A}_F(y) \geq \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\} > \gamma$. Thus, $y \in U^+(\mathcal{A}_F;\gamma)$. Hence, $U^+(\mathcal{A}_F;\gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{A}_T;\alpha)$, $L^-(\mathcal{A}_I;\beta)$, and $U^+(\mathcal{A}_F;\gamma)$ are either empty or IUP-filters of X. Let $x \in X$. Assume that $\mathcal{A}_T(0) < \mathcal{A}_T(x)$. Let $\alpha = \mathcal{A}_T(0)$. Then $x \in U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_T;\alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{A}_T;\alpha)$. So $\mathcal{A}_T(0) > \alpha = \mathcal{A}_T(0)$, which is a contradiction. Thus, $\mathcal{A}_T(0) \geq \mathcal{A}_T(x)$. Let $x,y \in X$. Assume that $\mathcal{A}_T(y) < \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}$. Let $\alpha = \mathcal{A}_T(y)$. Then $x \cdot y, x \in U^+(\mathcal{A}_T;\alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_T;\alpha)$ is an IUP-filter of X. By (2.19), we have $y \in U^+(\mathcal{A}_T;\alpha)$. So $\mathcal{A}_T(y) > \alpha = \mathcal{A}_T(y)$, which is a contradiction. Thus, $\mathcal{A}_T(y) \geq \min\{\mathcal{A}_T(x \cdot y), \mathcal{A}_T(x)\}$.

Let $x \in X$. Assume that $\mathcal{A}_I(0) > \mathcal{A}_I(x)$. Let $\beta = \mathcal{A}_I(0)$. Then $x \in L^-(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{A}_I;\beta)$ is an IUP-filter of X. By (2.18), we have $0 \in L^-(\mathcal{A}_I;\beta)$. So $\mathcal{A}_I(0) < \beta = \mathcal{A}_I(0)$, which is a contradiction. Thus, $\mathcal{A}_I(0) \leq \mathcal{A}_I(x)$. Let $x,y \in X$. Assume that $\mathcal{A}_I(y) > \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}$. Let $\beta = \mathcal{A}_I(y)$. Then $x \cdot y, x \in L^-(\mathcal{A}_I;\beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{A}_I;\beta)$ is an IUP-filter of X. By (2.19), we have $y \in L^-(\mathcal{A}_I;\beta)$. So $\mathcal{A}_I(y) < \beta = \mathcal{A}_I(y)$, which is a contradiction. Thus, $\mathcal{A}_I(y) \leq \max\{\mathcal{A}_I(x \cdot y), \mathcal{A}_I(x)\}$.

Let $x \in X$. Assume that $\mathcal{A}_F(0) < \mathcal{A}_F(x)$. Let $\gamma = \mathcal{A}_F(0)$. Then $x \in U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_F;\gamma)$ is an IUP-filter of X. By (2.18), we have $0 \in U^+(\mathcal{A}_F;\gamma)$. So $\mathcal{A}_F(0) > \gamma = \mathcal{A}_F(0)$, which is a contradiction. Thus, $\mathcal{A}_F(0) \geq \mathcal{A}_F(x)$. Let $x,y \in X$. Assume that $\mathcal{A}_F(y) < \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}$. Let $\gamma = \mathcal{A}_F(y)$. Then $x \cdot y, x \in U^+(\mathcal{A}_F;\gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{A}_F;\gamma)$ is an IUP-filter of X. By (2.19), we have $y \in U^+(\mathcal{A}_F;\gamma)$. So $\mathcal{A}_F(y) > \gamma = \mathcal{A}_F(y)$, which is a contradiction. Thus, $\mathcal{A}_F(y) \geq \min\{\mathcal{A}_F(x \cdot y), \mathcal{A}_F(x)\}$.

Hence, $\mathcal{A}$ is a neutrosophic IUP-filter of X. □

Theorem 3.48. *An NS $\mathcal{A}$ in X is a neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{A}_T;\alpha)$, $L^-(\mathcal{A}_I;\beta)$, and $U^+(\mathcal{A}_F;\gamma)$ are either empty or strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 3.10. □

Definition 3.49. Let $\mathcal{A}$ be an NS in X. For any $\alpha, \beta, \gamma \in [0,1]$, the sets

$$ULU_{\mathcal{A}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{A}_T \geq \alpha, \mathcal{A}_I \leq \beta, \mathcal{A}_F \geq \gamma\}, \tag{3.29}$$

$$LUL_{\mathcal{A}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{A}_T \leq \alpha, \mathcal{A}_I \geq \beta, \mathcal{A}_F \leq \gamma\}, \tag{3.30}$$

$$E_{\mathcal{A}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{A}_T = \alpha, \mathcal{A}_I = \beta, \mathcal{A}_F = \gamma\} \tag{3.31}$$

are called a ULU-(α,β,γ)-level subset, an LUL-(α,β,γ)-level subset, and an E-(α,β,γ)-level subset of $\mathcal{A}$, respectively.

The following five corollaries are derived directly by applying Theorems 3.40, 3.41, 3.42, 3.43, and 3.44, respectively.

Corollary 3.50. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{A}}(\alpha,\beta,\gamma)$ is either empty or an IUP-subalgebra of X.*

Corollary 3.51. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{A}}(\alpha,\beta,\gamma)$ is either empty or an IUP-ideal of X.*

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Corollary 3.52. *An NS $\mathcal{A}$ in X is a neutrosophic IUP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the set $ULU_{\mathcal{A}}(\alpha, \beta, \gamma)$ is either empty or an IUP-filter of X.*

Corollary 3.53. *An NS $\mathcal{A}$ in X is a neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the set $ULU_{\mathcal{A}}(\alpha, \beta, \gamma)$ is either empty or a strong IUP-ideal of X.*

Corollary 3.54. *An NS $\mathcal{A}$ in X is a neutrosophic strong IUP-ideal of X if and only if the set $E_{\mathcal{A}}(\mathcal{A}_T(0), \mathcal{A}_I(0), \mathcal{A}_F(0))$ is a strong IUP-ideal of X, that is, $E(\mathcal{A}_T, \mathcal{A}_T(0)) = X$, $E(\mathcal{A}_I, \mathcal{A}_I(0)) = X$, and $E(\mathcal{A}_F, \mathcal{A}_F(0)) = X$.*

4 Conclusion and future direction

In this paper, we have introduced and explored several new concepts within the realm of IUP-algebras: neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals. We have examined their fundamental properties and analyzed the intricate relationships between these neutrosophic structures and their level subsets, shedding light on their unique characteristics and interactions.

In our upcoming research, we plan to extend these findings to explore various types of NSs within IUP-algebras. We will also look at how soft set theory and cubic set theory can be used with neutrosophic IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals. This will help us find new dimensions and possible insights in these complex algebraic structures.

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