Some Types of Nth-Locally Compactness Spaces

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Abstract

This work focuses on n^{th}-locally compact spaces, which are topologies with locally compactness properties. Furthermore, the properties of these spaces will be studied in terms of locally compact spaces. Theoretical conclusions have been given and proven, and well-known theorems for locally compact spaces have been extended to n^{th}-topologies. An instance case is offered to back up the findings.

Keywords: Compact spaces; n^{th}-locally compact spaces; Metacompact space; Bitopological spaces

1 Introduction

One of the most important aspects of set theoretic topology is the development and study of the connections between various types of topological spaces located between countably paracompact spaces. In this context, the class of locally compact spaces is important since it naturally lies between the other classes. Where the locally compact space known as every a in Υ=(Υ, ϖ) has a neighborhood which is itself contained in a compact space [2] and A n^{th}-topologecal space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is called n^{th}-locally metacompact space if every a in Υ has a neighborhood which is itself contained in a n^{th}-compact space. One distinguishing property of these kinds of spaces is that they share numerous fundamental separation axioms, such as normality and collection wise Hausdorff. This makes them extremely important, both theoretically and practically, whether we consider problems from sn^{th}ctly topological or other disciplines of mathematics. A set Υ is called a n^th-topological space if each point in Υ has a fundamental system of almost open neighborhoods. The concept of a n^th-topological group, which is the n^th-topological version of a topological group, was also discussed in earlier works. Throughout this paper, all spaces under consideration are assumed to be nonempty and Υ_0 spaces (i.e., the intersection of any two open neighborhoods of a point a is itself an open neighborhood of a, for all points a in the space).

2 Literature review

The concept of a locally compact in topological space (Υ, ϖ) was presented by [7]. Recent research [2], [3], [6] has delved deeper into these areas. This paper explores the concept of n^{th}-locally compact and n^{th}-locally metacompact in n^{th}-topological spaces and presents associated conclusions. In the next section, we introduce

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the concept of n^{th}-locally compactness in n^{th}-topological spaces, discuss its features, and apply it to other spaces. We examine well-known definitions that will be applied in the sequel. In section two, we explore n^{th}-locally metacompactness in n^{th}-topological spaces and demonstrate several features of these spaces. The terms ϖ_u,ϖ_{dis},ϖ_{cof}, and ϖ_{coc} represent the ordinary, discrete, co-finite, and co-countable topologies, respectively.The concept of bitopological spaces can be represented as Υ = $(\Upsilon, \varpi_1, \varpi_2)$ where ϖ_1,ϖ_2 are two topologies on Υ. This is connected to prior research on bitopological spaces and tri-topological spaces. [4] explained bi-Hausdorff, bi-regular, and bi-normal spaces using a set of standard results known as Tietze extintion. [5] and [1] conducted additional research on bitopological spaces. A tri-open cover of Υ = $(\Upsilon, \varpi_1, \varpi_2)$ is defined as $Q = Q_1 \cup Q_1, Q \cap Q_1 = \Gamma$. [2], [3]explained the grow, nearly expand, and feebly expand abilities of bitopological space, similarly on tri-topological spaces. The purpose of this study is to present and investigate a new type of n^{th}-compact space: the n^{th}-locally compact space. n^{th}-topological spaces are sets containing three topologies, represented as Υ = $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$, where $\varpi_{i's}$ are topologies on Υ for all i=1,2,..,n. n^{th}topological spaces have variations that correspond to well-known topological space features.

We list some of them here for completeness:
• A n^{th}-topological space $(X, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-Hausdorff if two distinct points a and b have disjoint W_1 ϖ_1 and W_2 ϖ_2 with a W_1 and y W_2.
The concept of n^{th}-topological spaces has been explored in various prior studies, where a set is equipped with n^{th} topologies, each satisfying specific axioms. Researchers like [5], [1] conducted further research in the field of bitopological spaces, which involve two topologies on the same set.

3 Preliminaries

Definition 2.1 [1]

Let Υ be a non-empty set, $\varpi \subset$ P(Υ)= {A: A $\subseteq \Upsilon$}
ϖ is called topology on Υ if the following conditions are satisfied:
(i) φ, $\Upsilon \in \varpi$.
(ii) For all A, B $\in \varpi$, we have A $\cap$ B $\in \varpi$.
(iii) If $\mathrm{E} = \{\mathrm{A}_\alpha : \alpha \in \lambda, \mathrm{A}_\alpha \in \varpi\}, then \bigcup \mathrm{E} = \bigcup_{\alpha\in\lambda} \mathrm{A}_\alpha \in \varpi$.

Definition 2.2

Let Υ be a non-empty set, $\varpi_i \subset$ P(Υ) where i=1,2,...,n.
we say that $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is " n^{th}-topological space " if ϖ_i is topology on Υ, for all i=1,2,..,n.

Example 1. consider X= {m,n,o} be a non-empty set and let
ϖ_1 = {φ, Υ, {m}} $\subset$ p(Υ)
ϖ_2 = {φ , Υ, {m} , {n} , {m,n}} $\subset$ p(Υ)
ϖ_3 = {φ , Υ, {n} , {o} , {n,o}} $\subset$ p(Υ)
ϖ_i's satisfies the condition of topological space, i=1,2,3. So, $(\Upsilon, \varpi_1, \varpi_2, \varpi_3)$ is tri-topological space.
For example ϖ={$\varphi, X, \{a\}, \{b\}$} not topological space.

Definition 2.3

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space and M $\subset \Upsilon$, then:
(i) M is called ϖ_i-Open set, If M $\in \varpi_i$ for some i=1,2,3.
(ii) M is called ϖ_i-closed set, If $\mathrm{M}^c \in \varpi_i$ for some i=1,2,3.
(iii) M is called ϖ_i-clopen set, If M and M^c are both in ϖ_i for some i=1,2,3.

Definition 2.4

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Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space, $\Upsilon \neq \varphi$ and M is subset of Υ, then a $\in \Upsilon$ is called n^{th}-Limit point of M if for all u_a ϖ_i-open set such that $u_a \cap (\mathrm{M} - \{a\}) \neq \varphi$.
The set of all n^{th}-limit points is called n^{th}-derived set and it is denoted by M' ={a : a is n^{th}-limit point of M}.
Properties of n^{th}partite derived set: Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_N)$ be a n^{th}-topological space and let M,O $\subset \Upsilon$, then:
(i) $\varphi' = \varphi$.
(ii) $if M \subset O, then M' \subset O'$.
(iii) $(M \cup O)' = M' \cup O'$.
(iv) $(M \cap O)' \subset M' \cap O'$.

Proof: (i)

Suppose that $\varphi' \neq \varphi$, then there exist z $\in \varphi'$, then for all ϖ_i-open set u_z, i=1,2,...,n. We have $u_z \cap (\varphi - \{z\})$ $\neq \varphi$, but $(\varphi - \{z\})$=φ, then $u_z \cap \varphi \neq \varphi$, thus $\varphi \neq \varphi$ and that is contradiction. So, φ'=φ.

Definition 2.5

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space, $\Upsilon \neq \varphi$ and M is subset of Υ, then the n^{th}-closure set is denoted by
$\overline{\mathrm{M}}$ = M$\cup$M$'$
Properties of n^{th}partite closure set: Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space and let M,O $\subset \Upsilon$, then:
(i) $\overline{\varphi} = \varphi$ and $\overline{\Upsilon} = \Upsilon$.
(ii) $\overline{\mathrm{M} \cup \mathrm{O}} = \overline{\mathrm{M}} \cup \overline{\mathrm{O}}$ and $\overline{\mathrm{M} \cap \mathrm{O}} \subset \overline{\mathrm{M}} \cap \overline{\mathrm{O}}$.
(iii)$\overline{M}$ is ϖ_i-closed set.
(iv) $M = \overline{M}$ if and only if M is ϖ_i-closed set.
(v) a $\in \overline{M}$ if and only if for all ϖ_i-open set u_a such that a $\in u_a$ we have $u_a$$\cap$M$\neq$$\varphi$.

Proof: (ii)
$\overline{M \cup O} = (M \cup O) \cup (M \cup O)^{'}$ = $(M \cup O) \cup (M^{'} \cup O^{'})$ = $(M \cup M^{'})(O \cup O^{'})$ = $\overline{M} \cup \overline{O}$.
and $\overline{M \cap O} = (M \cap O) \cap (M \cap O)^{'} \subset (M \cap O) \cap (M^{'} \cap O^{'})$ = $(M \cup M^{'})(O \cap O^{'})$ = $\overline{M} \cap \overline{O}$.

Definition 2.6

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space , $\Upsilon \neq \varphi$ and M is subset of Υ, then a point a $\in$ M is said to be n^{th}-Interior point of M if there exist at least one neighborhood of a (N(a,ε)) such that N(a,ε) $\subseteq$ M.
The set of all n^{th}-interior point is called the n^{th}-Interior set and it is denoted by $\mathrm{M}^{\circ} \equiv INT(M) = \overline{M^C}^{C}$.
Properties of n^{th}partite interior set: Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space and let M,O $\subset \Upsilon$, then:
(i) $\varphi^{\circ} = \varphi$ and $\Upsilon^{\circ} = \Upsilon$.
(ii) $(M \cap O)^{\circ} = M^{\circ} \cap O^{\circ}$ and $M^{\circ} \cup O^{\circ} \subset (M \cup O)^{\circ}$.
(iii) M$^{\circ}$ is ϖ_i-open set.
(iv) n $\in$ M$^{\circ}$ if and only if there exist ϖ_i-open set u_n such that n $\in u_n \subset$ M.

Definition 2.7

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space, $\Upsilon \neq \varphi$ and A is subset of Υ, then the point a is said to be n^{th}-Exterior point of A if there exist at least one neighborhood of a such that N (a, ε) $\cap$ A= φ.
The set of all n^{th}-Exterior point is called n^{th}-Exterior set and it is denoted by
EX(A)=Int(A^c)= $\overline{A}^{C}$.
Properties of n^{th}partite exterior set: Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space and let A,B $\subset \Upsilon$, then:
(i) EX(φ) =Υ and EX(Υ)=φ.
(ii) if A $\subset$ B, then EX(B) $\subset$ EX(A).

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(iii) EX(A) is β_i-open set.
(iv) e $\in$ EX(A) if and only if there exsit ϖ_i-open set u_e such that e$\in u_e \subset$A^c.

Proof: (iii)

since EX(A) = $INT(A^c)$, then $EX(A) = \overline{A^{c^c}}^c$, but $A^{cc} = A$ thus, $EX(A) = \overline{A}^c$ and by definition of n^{th}-closure set we have $\overline{A}$ is ϖ_i-closed set so, the complement of ϖ_i-closed set is ϖ_i-open set, therefor EX(A) is ϖ_i-open set. As a result the interior set is also ϖ_i-open set.

Definition 2.8

Let $(\Upsilon, \varpi_1, \varpi_2, \varpi_3)$ be a n^{th}-topological space, $\Upsilon \neq \varphi$ and A is subset of Υ, then the point a is said to be n^{th}-Boundary point of A, If every neighborhood of a satisfy that N(a,ε) $\cap$ A $\neq \varphi$ and N(a, ε) $\cap A^c \neq \varphi$.
The set of all n^{th}-boundary point is called n^{th}-Boundary set, and it is denoted by
Bd(A) = $\overline{\text{A}} -$ A$^\circ$ = $\overline{\text{A}} \cap \overline{\text{A}^c}$.
Properties of n^{th}partite boundary set: Let $(\Upsilon, \varpi_1, \varpi_2, \varpi_3)$ be a n^{th}-topological space and let A,B $\subset \Upsilon$, then:
(i) Bd(φ) = Bd(X)= φ.
(ii) Bd(A) is ϖ_i-closed set.
(iii) b $\in$ Bd(A) if and only if for all ϖ_i-open set u_b such that b $\in u_b$ we have $u_b \cap$ A $\neq \varphi$ and $u_b \cap A^c \neq \varphi$.

Proof: (iii)

Let b $\in$ Bd(A) and u_b be a ϖ_i-open set such that b u_b, then b $\in$ ($\overline{\text{A}} \cap \overline{\text{A}^C}$) if and only if b $\in \overline{A}$ and b $\in \overline{A^c}$ if and only if b $\in$ (A $\cup$ A') and b $\in A^c \cup (A^c)'$ if and only if (b $\in$ A or b $\in$ A') and (b $\in A^c$ or b $\in (A^c)'$ if and only if b $\in$ A' and b $\in A^c$ if and only if $u_b \cap (A/\{b\}) \neq \varphi$ and $b \cap A^c \neq \varphi$, but $b \subset u_b$, so we have $u_b \cap$ A $\neq \varphi$ and $u_b \cap A^c \neq \varphi$.

Definition 2.9 [5]

A n^{th}-topological space (Υ, ϖ) is T_0-space ,if for all tow different element a and b in Υ, there exist open set u_a such that a $\in u_a$ and b $\notin u_a$,or there exist open set v_b such that b $\in v_b$ and a $\notin v_b$.

Definition 2.10

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi n)$ is n^{th}-T_0-space if for all two different element a and b in Υ, there exist ϖ_i-open set u_a such that a $\in u_a$ and b $\notin u_a$, or there exist ϖ_j-open set v_b such that b $\in v_b$ and a $\notin v_b$, where i $\neq$ j and i,j = 1,2,...,n.

Theorem 2.1 Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space, then the following are equivalent:
(i) Υ is n^{th}-T_0-space.
(ii) for all two different elements m and n in Υ, we have m $\notin \overline{\{n\}}$ or n $\notin \overline{\{m\}}$.
(iii) for all two different elements m and n in Υ, we have $\overline{\{m\}} \neq \overline{\{n\}}$.

Proof: (**(i) implies (ii)**)

Let a $\neq$ b, then there exist ϖ_i-open set such that a $\in u_a$ and b $\notin u - a$ or there exist ϖ_j-open set v_b such that b $\in v_b$ and a $\notin v_b$ where i=1,2,..,n. So, we have a $\in u_a$ and $u_a \cap \{b\}$ = φ or b in v_b and $v_b \cap \{a\}$ = φ. Thus, a $\notin \overline{\{b\}}$ or b $\notin \overline{\{a\}}$.
(**(ii) implies (iii)**)

Let a $\neq$ b, then if a $\notin \overline{\{b\}}$ and a $\in \overline{\{a\}}$, then we have $\overline{\{a\}} \neq \overline{\{b\}}$. Additionally, if b $\notin \overline{\{a\}}$ and b $\in \overline{\{b\}}$, then we have $\overline{\{a\}} \neq \overline{\{b\}}$.
(**(iii) implies (i)**)

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Let a $\neq$ b and by given $\overline{\{a\}} \neq \overline{\{b\}}$, but a $\in \overline{\{a\}}$ and b $\in \overline{\{b\}}$, then a$\notin X - \overline{\{a\}}$=$v_b$ which is ϖ_i open set in Υ since $\overline{\{a\}}$ is ϖ_i-closed set in Υ and b$\in X - \overline{\{a\}}$=v_b, where i=1,2,...,3. Thus, Υ is n^{th}-T_0-space.

Definition 2.11

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_1-space, if for all two different element a and b in Υ, there exist ϖ_i-open set u_a such that a $\in u_a$ and b $\notin u_a$, and there exist ϖ_j-open set v_b such that b $\in v_b$ and a $\notin v_b$, where i$\neq$j and i, j = 1,2,...,n.

Definition 2.12

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_2-space, if for all two different element a and b in Υ, there exist ϖ_i-open set u_a such that a $\in u_a$ and there exist ϖ_j-open set v_b such that b $\in v_b$ and $u_a \cap v_b = \varphi$, where i$\neq$j and i, j=1,2,...,n.

Definition 2.13

A topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-$T_{2\frac{1}{2}}$-space, if for all two different element a and b in Υ, there exist ϖ_i-closed set A_a and B_b such that a $\in A_a$, b $\in B_b$ and $A_a \cap B_b = \varphi$, i=1,2,...,n.

Definition 2.14

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-regular space, if $\forall a\notin$ A and A is ϖ_i-closed set, there exist ϖ_i-open set u_a and ϖ_j-open set v_A such that a $\in u_a$, A $\subset v_A$ and $u_a \cap v_A$ =φ, where i$\neq$j, i, j=1,2,...,n.

Theorem 2.2 A space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-regular space if and only if for all a $\in u_a$,where u_a is ϖ_i-open set, there exist ϖ_i-open set w_a such that a$\in w_a \subset \overline{w_a} \subset u_a$.

Proof:
$(\rightarrow)$
Let a $\in u_a$, then a $\notin {u_a}^c$,but ${u_a}^c$ is ϖ_i-closed set, then we can say that ${u_a}^c$=A. So, by definition of n^{th}-regular space, there exist ϖ_i-open set w_a and v_A such that a $\in w_a$, A$\subset v_A$ and $w_a \cap v_A$ =φ, but clearly $w_a \subset \overline{w_a}$.
It is enough to show $\overline{w_a} \subset u_a$, now $w_a \cap v_A$=φ, then we can say that $w_a \subset {v_A}^c$, then $\overline{w_a} \subset \overline{{v_A}^c} = {v_A}^c$. So, we have $\overline{w_a} \subset {v_A}^c$, but ${u_a}^c$= A $\subset v_A$, then ${u_a}^c \subset v_A$, then ${v_A}^c \subset u_a$, thus $\overline{w_a} \subset u_a$. We are done.
$(\leftarrow)$
Let a $\notin$ A and A is ϖ_i-open set, then a $\in A^c$ and A^c is ϖ_i-open set, then by given there exist ϖ_i-open set w_a such that a $\in w_a \subset \overline{w_a} \subset A^c$. Now we have two givens, a $\in w_a$ and A $\subset \overline{w_a}^c$, where w_a and $\overline{w_a}^c$ is ϖ_i-open sets, i=1,2,...,n ..... (i).
it is enough to show that $w_a \cap \overline{w_a}^c = \varphi$, suppose not, then there exist z such that z $\in (w_a \cap \overline{w_a}^c)$, that is implies z $\in w_a$ and z $\in \overline{w_a}^c$, then z $\in w_a$ and z $\notin w_a$ and z $\in {w_a}'$, then we have z $\in$ ($w_a \cap {w_a}^c$) that is contradiction. So, $w_a \cap \overline{w_a}^c = \varphi$ ..... (ii).
By (i) and (ii) we have, a space $(X, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-regular space.

Definition 2.15 [3]

A topological space (Υ, ϖ) is T_3-space if it is T_1-space and regular space.

Definition 2.16

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_3-space if it is n^{th}-T_1-space and n^{th}-regular space.

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Definition 2.17

A space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-normal space if for all two disjoint ϖ_i-closed set A and B, there exist ϖ_i-open sets u_A and v_B such that A $\subset u_A$, B $\subset v_B$ and $u_A \cap v_B = \varphi$.

Definition 2.18

A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_4-space if it is n^{th}-T_1-space and n^{th}-normal space.

Theorem 2.3 If $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_k-space, then it is n^{th}-T_{k-1}-space.

Example.3 If a space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_4-space, then it is n^{th}-T_3-space.

Proof: Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_4-space, then it is n^{th}-T_1-space and n^{th}-normal space, that is implies for all two disjoint ϖ_i-closed set M and O, there exist ϖ_i-open sets u_M and v_O such that M $\subset u_M$, O $\subset v_O$ and $u_M \cap v_O = \varphi$, now let b $\in$ O, then b $\in v_O$ but M $\cap$ O = φ, therefore b $\notin$ M.
Then we have $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-regular space.
Thus $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_1-space and n^{th}-regular space, hence $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-T_3-space.

Definition 2.19

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space, then a set D in $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is called n^{th}-Dense set If $\overline{D} = X$.
On another hand, if D is dense in $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$, then for all ϖ_i-open set u we have $u \cap D \neq \varphi$.

Definition 2.20

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ and $(Y, \varphi_1, \varphi_2, \varphi_3)$ are n^{th}-topological space, then the function
$f : (X, \varpi_1, \varpi_2, \varpi_3) \to (Y, \varphi_1, \varphi_2, \varphi_2)$ is called n^{th}-open function if $f(u) = v$, where u is ϖ_i-open set and v is φ_i-open set.

4 Nth-COMPACTNESS SPACES AND Nth-LOCALLY COMPACTNESS SPACES

Definition 3.1 [2] Let (Υ, ϖ) be a topological space and E $= \{\text{A}_\alpha : \alpha \in \lambda, \text{A}_\alpha \subset \text{X}\}$ is called :
(i) cover of Υ if and only if $\bigcup_{\alpha \in \lambda} \text{A}\alpha = \Upsilon$.
(ii) open cover of Υ if and only if E is cover and A_α is open set ,where $\alpha \in \lambda$.
(iii) closed cover of Υ if and only if E is cover and A_α is closed set ,where $\alpha \in \lambda$.
(iv) $C = \{\text{B}_\gamma : \gamma \in \Gamma\}$ is a subcover of E if and only if :
(i) C $\subset$ E (ii) $\bigcup_{\gamma \in \Gamma} \text{B}_\gamma = \Upsilon$
A space (Υ, ϖ) is called compact space if every open cover of Υ has a finite subcover.

Definition 3.2

Let $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ be a n^{th}-topological space and E $= \{\text{A}_\alpha : \alpha \in \lambda, \text{A}_\alpha \subset \Upsilon\}$ is called:
(i) n^{th}-cover of Υ if and only if $\bigcup_{\alpha \in \lambda} \text{A}\alpha = \Upsilon$.
(ii) n^{th}-open cover of Υ if and only if E is n^{th}-cover and A_α is ϖ_i-open set, where $\alpha \in \lambda$, i=1,2,...,n.
(iii) n^{th}-closed cover of Υ if and only if E is n^{th}-cover and A_α is ϖ_i-closed set, where $\alpha \in \lambda$, i=1,2,...,n.

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(iv) $C = \{B_\gamma : \gamma \in \Gamma\}$ is a n^{th}partite subcover of E if and only if :
(a) C $\subset$ E (b) $\bigcup_{\gamma \in \Gamma} B_\gamma = \Upsilon$.

A space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is called n^{th}partite compact space, if every n^{th}-open cover of Υ has a finite n^{th}-subcover.

Definition 3.3

If $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is a n^{th}-topological space, then ϖ is said to be n^{th}-locally compact, if each point of Υ has an n^{th}-open neighborhood whose n^{th}-closure is n^{th}-compact.
Note that : every n^{th}-compact space is n^{th}-locally compact.

Example.1 [4]

($\mathbb{R}$, ϖ_u) is not compact, where ϖ_u is the usual topology.

Proof: by contradiction, assume that $(\mathbb{R}, \varpi_u)$ is compact, so every open cover of $\mathbb{R}$ has a finite subcover, but E $= \{(-n, n) : n = 1, 2, 3, ...\}$ is open cover of Υ because $\bigcup_{n=1}^{\infty} (-n, n)$=$\mathbb{R}$ and (-n,n) is open set, so E has a finite subcover say C=$\{(-n_1, n_1), (-n_2, n_2).(-n_3, n_3), \ldots, (-n_k, n_k)\}$, then $\bigcup_{i=1}^{k} (-n_i, n_i) = \mathbb{R}$, then $(a, b) = \mathbb{R}$, where $a = \min_{i=1,\ldots,k}\{-n_i\}\ and\ b = \max_{i=1,\ldots,k}\{n_i\}$, then $\mathbb{R}$=(a,b) $\subset [a, b] \Rightarrow \mathbb{R} \subset$ [a,b] $\equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted.
$\therefore$ $(\mathbb{R}, \varpi_u)$ is not compact space.

Example.2 [7]

$(\mathbb{Q}, \varpi_{\text{cof}})$ is a compact space, where ϖ_{cof} is the co-finite topology.

Example.3

The n^{th}-topological space ($\mathbb{R}$, ϖ_{u_1}, ϖ_{u_2},..., ϖ_{u_n}) is not n^{th}-compact space, where $\varpi_{u_i's}$ ere usual topology.

Proof: by contradiction, assume that $(\mathbb{R}, \varpi_u)$ is compact, so every open cover of $\mathbb{R}$ has a finite subcover, but E $= \{(-n, n) : n = 1, 2, 3, ...\}$ is open cover of $\mathbb{R}$ because $\bigcup_{n=1}^{\infty} (-n, n)$=$\mathbb{R}$ and (-n,n) is open set, so E has a finite subcover say C=$\{(-n_1, n_1), (-n_2, n_2).(-n_3, n_3), \ldots, (-n_k, n_k)\}$, then $\bigcup_{i=1}^{k} (-n_i, n_i) = \mathbb{R}$, then $(a, b) = \mathbb{R}$ where $a = \min_{i=1,\ldots,k}\{-n_i\}\ and\ b = \max_{i=1,\ldots,k}\{n_i\}$, then $\mathbb{R}$=(a,b) $\subset [a, b] \Rightarrow \mathbb{R} \subset$ [a,b] $\equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted.
$\therefore$ $(\mathbb{R}, \varpi_u)$ is not compact space.

Example.4

The n^{th}-topological space ($\mathbb{R}$,ϖ_{u_1},ϖ_{u_2},...,ϖ_{u_n}) is not n^{th}-campact space, where $\varpi_{u_i's}$ are usual topologies.

Proof. by contradiction, assume that $(\mathbb{R}, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-compact, so
every n^{th}-open cover of $\mathbb{R}$ has a finite n^{th}-subcover, but

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E $= \{(-n,n) : n = 1,2,3,...\}$ is n^{th}-open cover of Υ because $\bigcup_{n=1}^{\infty}(-n,n)$=$\mathbb{R}$ and
(-n,n) is ϖ_{ui}-open set, i=1,2,...,n, so E has a finite n^{th}-subcover say
C=$\{(-n_1,n_1),(-n_2,n_2).(-n_3,n_3),...,(-n_k,n_k)\}$, then $\bigcup_{i=1}^{k}(-n_i,n_i) = \mathbb{R}$, then
$(a,b) = \mathbb{R}$ where $a = \min_{i=1,...,k}\{-n_i\}\, and\, b = \max_{i=1,...,k}\{n_i\}$, then $\mathbb{R}$=(a,b) $\subset [a,b]$
$\Rightarrow \mathbb{R} \subset$ [a,b] $\equiv$ bounded set so, $\mathbb{R}$ is bounded set and that is contradicted.
$\therefore (\mathbb{R},\varpi_{u_1},\varpi_{u_2},...,\varpi_{u_n})$ is not n^{th}-compact space.

Theorem 3.1 let $(\Upsilon,\varpi_1,\varpi_2,...,\varpi_n)$ be a n^{th} topological space and E$\subset\Upsilon$, then E is n^{th}-compact space if and only if E is ϖ_i-closed set and n^{th}-bounded set.

Theorem 3.2 Let A be a compact subset in a n^{th}-T_2-space Υ. Then for all $n \notin A$ there exists an open set U_n containing n such that A $\cap\, U_n = \emptyset$.

Theorem 3.3 Let M be a n^{th}-compact subset in a n^{th}-T_2-space Υ. Then for all $n \notin M$ there exists a ϖ_i-open set U_n containing n such that M $\cap\, U_n = \emptyset$, i=1,2,...,n.

5 locally Nth-metacompactness space

This section discusses the concept of locally n^{th}-metacompactness in n^{th}-topological spaces and highlights its features.

Definition 4.1

A n^{th}-topological spaces $(\Upsilon,\varpi_1,\varpi_2,...,\varpi_n)$ is called n^{th}-metacompact, if every n^{th}-open cover of the space $(\Upsilon,\varpi_1,\varpi_2,...,\varpi_n)$ has point a finite parallel refinement.

Definition 4.2 [4]

If (Υ,ϖ) is a topological space, then Υ is said to be locally metacompact, if each point of Υ has a neighborhood which is itself contained in a metacompact set.

Theorem 4.1. A n^{th}-countable locally metacompact space is n^{th}-compact.

Example 1. The n^{th}-topological space $(X,\varpi_1,\varpi_2,...,\varpi_n)$ is locally n^{th}-metacompact space, since Υ is n^{th}-compact.

Example 2. The n^{th}-topological space $(\mathbb{R},\varpi_{dis1},\varpi_{dis2},...,\varpi_{disn})$ is n^{th}-locally metacompact, where ϖ_{dis} is the discrete topology.

Theorem 4.2 If a n^{th}-topological space $(\Upsilon,\varpi_1,\varpi_2,...,\varpi_n)$ is locally n^{th}-metacompact and A is a subset of Υ which is n^{th}-closed, then it is locally n^{th}-metacompact. If moreover A is a proper subset of Υ, then A is also locally n^{th}-metacompact.

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Proof. Let $\tilde{U}$ be any n^{th}-open cover of the subspace $(A, \varpi_1, \varpi_2, ..., \varpi_n^*)$, where $\tau^* = \{U \cap A : U \in \tau\}$. Then $\tilde{U} \cup \{Z - A\}$ is a n^{th}-open cover of the locally n^{th}-metacompact space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ which has a point finite n^{th}-open parallel refinement for Υ and hence $\tilde{U}$ for A.

Corollary 4.1. Every n^{th}-metacompact space is n^{th}-locally metacompact.

Proof. Let $\tilde{U}$ be any n^{th}-open cover of the subspace $(A, \varpi_1, \varpi_2, ..., \varpi_n^*)$, where $\tau^* = \{U \cap A : U \in \tau\}$. Then $\tilde{U} \cup \{X - A\}$ is a n^{th}-open cover of the n^{th}-metacompact space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ which has a point finite n^{th}-open parallel refinement for Υ and hence $\tilde{U}$ for A.

Example 3. The n^{th}-topological space $(\mathbb{R}, \varpi_{r.r1}, \varpi_{r.r2}, ..., \varpi_{r.rn})$ is locally n^{th}-metacompact but not n^{th}-metacompact.
Where $\varpi_{\mathrm{r.r}}$ is the right-ray topology.

Theorem 4.3 A n^{th}-topological space $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is n^{th}-regular, if for each point $a \in \Upsilon$ and open set U containing a, there exists an n^{th}-open set V containing a, such that $a \in V \subset CLV \subset U$.

Theorem 4.4 Let T:$(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n) \to (Y, \varphi_1, \varphi_2, ..., \varphi_n)$ be an onto, continuous and n^{th}-open function. If $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is locally metacompact, then $(Y, \varphi_1, \varphi_2, ..., \varphi_n)$ is so.

Proof. First we show that φ is locally n^{th}-metacompact. Let $b \in Y$. Then $T^{-1}(b) \in \Upsilon$, since $(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n)$ is locally n^{th}-metacompact, then there is an n^{th}-open set U containing $T^{-1}(b)$, such that CLU is n^{th}-metacompact. Now, let T:$(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n) \to (Y, \varphi_1, \varphi_2, ..., \varphi_n)$ is n^{th}-open, then $T(U)$ is an n^{th}-open subset of Y and $b \in f(U)$. Since T:$(\Upsilon, \varpi_1, \varpi_2, ..., \varpi_n) \to (Y, \varphi_1, \varphi_2, ..., \varphi_n)$ is onto continuous, then $T(CLU)$ is n^{th}-metacompact. Thus $b \in T(U) \subset CLT(U) \subset T(CLU)$ and $T(CLU)$ is n^{th}-metacompact. So $(Y, \varphi_1, \varphi_2, ..., \varphi_n)$ is locally n^{th}-metacompact.

6 Conclusion

This study defines locally compact and locally metacompact spaces in topological and n^{th}-topological spaces respectively. Several properties of these spaces and how they interact with other topologies. Theoretically, n^{th} and tri-topological spaces were created. Future research may lead to new theorems on the finite product and mappings of n^{th} expandable spaces, feebly tripartite expandable spaces, and fuzzy n^{th}-topological spaces based on the findings.

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