On The Algebraic Classification of the 4-Cyclic Refined Neutrosophic Real Roots of Unity Group

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Abstract

This paper is dedicated to finding all 4-cyclic refined neutrosophic real solutions of the equation $X^n = 1$ which are called 4-cyclic refined real roots of unity. Also, we classify the algebraic group of these solutions as a direct product of some familiar finite cyclic groups. On the other hand, we illustrate many examples to clarify the validity of our work.

Keywords: 4-cyclic refined number; 4-cyclic neutrosophic root of unity; Abelian group; Direct product

1. Introduction

The concept of neutrosophic sets and neutrosophic structures is considered one of the most recent mathematical concepts with applications in many different scientific fields [1].

The n-cyclic refined neutrosophic structures were first defined in [9], where algebraic structures related to this definition were studied, such as rings and modules. Also, the groups and spaces generated by this mathematical systems were studied in [9-12,14].

Then in [2-4], some famous algebraic problems were studied for the n-cyclic refined neutrosophic rings, where some conjectures were put forward that discuss the classification of the group of units such as generalized Von Shtawzen's conjecture [5].

The problem of the group of units for 3-cyclic and 4-cyclic refined neutrosophic rings of integers was studied by many researchers [7-8, 13], where it was classified in [6] as a proof of first and second Von Shtawzen's conjectures.

In [15-16], the problem of finding integer n-cyclic refined neutrosophic solutions of the equation $X^n = 1$ was discussed, and many good formulas were proven.

In this work, we study the same problem for a generalized set, where we find the formulas for the solutions of the equation $X^n = 1$ in the 4-cyclic real ring, with a classification of its group structure.

2. Main Discussion

Definition:

Let $X = x_0 + \sum_{i=1}^{4} x_i I_i$ be a 4-cyclic refined neutrosophic real number with $x_i \in \mathbb{R}$, then X is called 4-cyclic refined n-th root of unity if: $X^n = 1$.

Remark:

The equation $X^n = 1$ is equivalent to:

$$\begin{cases} x_0^n = 1 \\ (\sum_{i=0}^{4} x_i)^n = 1 \\ (x_0 - x_1 + x_2 - x_3 + x_4)^n = 1 \\ (x_0 - x_2 + x_4 + i(x_1 - x_3))^n = 1 \end{cases}$$

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Discussion:

For odd values of n, we get:

$$\begin{cases} x_0 = 1 \\ \sum_{i=0}^{4} x_i = 1 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0 \quad (1) \\ x_0 - x_1 + x_2 - x_3 + x_4 = 1 \Rightarrow -x_1 + x_2 - x_3 + x_4 = 0 \quad (2) \\ x_0 - x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) \quad (3) \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \quad (4) \end{cases} ;0 \le k \le n-1$$

By using (1), (2), we get: $\begin{cases} x_2 + x_4 = 0 \\ x_1 + x_3 = 0 \end{cases}$

Thus:

$$\begin{cases} 2x_4 = \cos\left(\frac{2\pi k}{n}\right) - 1 \\ 2x_1 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

Also, $\begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases} ;0 \le k \le n-1$

$$X_k = 1 + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_2 + \left(\frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right)\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_4 \ ;0 \le k \le n-1.$$

Example:

Let $n = 3$, then:

$$X_0 = 1, X_1 = 1 + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) I_1 + \left(\frac{-1}{2}\left(\frac{-1}{2}\right) + \frac{1}{2}\right) I_2 - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) I_3 + \left(\frac{1}{2}\left(\frac{-1}{2}\right) - \frac{1}{2}\right) I_4 = 1 + \frac{\sqrt{3}}{4} I_1 + \frac{3}{4} I_2 - \frac{\sqrt{3}}{4} I_3 - \frac{3}{4} I_4.$$

$$X_2 = 1 + \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right) I_1 + \left(\frac{-1}{2}\left(\frac{-1}{2}\right) + \frac{1}{2}\right) I_2 + \left(\frac{-1}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) I_3 + \left(\frac{1}{2}\left(\frac{-1}{2}\right) - \frac{1}{2}\right) I_4 = 1 - \frac{\sqrt{3}}{4} I_1 + \frac{3}{4} I_2 + \frac{\sqrt{3}}{4} I_3 - \frac{3}{4} I_4.$$

Theorem:

The group of n-th real roots of unity in the 4-cyclic refined neutrosophic ring is isomorphic to Z_n, under the condition that n is odd.

Proof:

It is clear that the set of all n-th 4-cyclic refined neutrosophic roots of unity R_n is a cyclic group under multiplication with odd values of n and $|R_n| = n$.

Define the mapping: $f: R_n \longrightarrow C_n \ ; f(x_k) = e^{\frac{2\pi k}{n} i}$ $, 0 \le k \le n-1$.

If $X_{k_1} = X_{k_2}$, then $k_1 = k_2$, and $e^{\frac{2\pi k_1}{n} i} = e^{\frac{2\pi k_2}{n} i}$, hence $f\left(X_{k_1}\right) = f\left(X_{k_2}\right)$.

$f(X_k . X_s) = f(X_{k+s}) = e^{\frac{2\pi(k+s)}{n} i} = e^{\frac{2\pi k}{n} i} . e^{\frac{2\pi s}{n} i} = f(X_k) f(X_s)$.

And f is a bijection clearly, thus f is an isomorphism.

Even values of n:

Assume that n is even, then:

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$$x_0^n = 1 \Leftrightarrow x_0 \in \{1, -1\}$$

$$(\sum_{i=0}^{4} x_i)^n = 1 \Leftrightarrow \sum_{i=0}^{4} x_i \in \{1, -1\}$$

$$(x_0 + x_2 + x_4 - x_1 - x_3)^n = 1 \Leftrightarrow x_0 + x_2 + x_4 - x_1 - x_3 \in \{1, -1\}$$

$$[x_0 - x_2 + x_4 + i(x_1 - x_3)]^n = 1 \Leftrightarrow \begin{cases} x_0 - x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} 0 \le k \le n-1$$

Case (1):

If $x_0 = \sum_{i=0}^{4} x_i = x_0 - x_1 + x_2 - x_3 + x_4 = 1$, then:

$$X_k = 1 + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_2 - \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_4.$$

Case (2):

If $x_0 = \sum_{i=0}^{4} x_i = x_0 - x_1 + x_2 - x_3 + x_4 = -1$, then:

$$\begin{cases} -x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

And: $\begin{cases} x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) \\ x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}$

Hence:

$$X = -1 + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_2 - \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_4.$$

Case (3):

If $x_0 = 1, \sum_{i=0}^{4} x_i = x_0 + x_2 + x_4 - x_1 - x_3 = -1$, then:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -2 \\ x_2 + x_4 - x_1 - x_3 = -2 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = -2 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) - 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{3}{2} \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) \\ x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}$$

$$X = 1 + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_2 - \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - \frac{3}{2}\right) I_4.$$

Case (4):

If $x_0 = \sum_{i=0}^{4} x_i = 1, \; x_0 + x_2 + x_4 - x_1 - x_3 = -1$, then:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_4 - x_1 - x_3 = -2 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = -1 \\ x_1 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) - 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - 1 \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}$$

$$X = 1 + \left(\frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right)\right) I_2 + \left(\frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - 1\right) I_4.$$

Case (5):

If $x_0 = x_0 + x_2 + x_4 - x_1 - x_3 = 1, \sum_{i=0}^{4} x_i = -1,$, then:

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$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -2 \\ x_2 + x_4 - x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = -1 \\ x_1 + x_3 = -1 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) - 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - 1 \\ x_1 = \frac{-1}{2} + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}.$$

$$X = 1 + \left(\frac{-1}{2} + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right)\right) I_1 - \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) I_2 + \left(\frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) - 1\right) I_4.$$

Case (6):

If $x_0 = -1, \sum_{i=0}^{4} x_i = x_0 + x_2 + x_4 - x_1 - x_3 = 1$, then:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 \\ x_2 + x_4 - x_1 - x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = 2 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{3}{2} \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) \end{cases}.$$

$$X = -1 + \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_1 + \left(\frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_2 - \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + \frac{3}{2}\right) I_4.$$

Case (7):

If $x_0 = \sum_{i=0}^{4} x_i = -1,\ x_0 + x_2 + x_4 - x_1 - x_3 = 1$, then:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + x_4 - x_1 - x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = 1 \\ x_1 + x_3 = -1 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2} \end{cases}.$$

$$X = -1 + \left(\frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_1 - \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) I_2 + \left(\frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) - \frac{1}{2}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + 1\right) I_4.$$

Case (8):

If $x_0 = x_0 + x_2 + x_4 - x_1 - x_3 = -1, \sum_{i=0}^{4} x_i = 1$, then:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 \\ x_2 + x_4 - x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 + x_4 = 1 \\ x_1 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} -x_2 + x_4 = \cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 - x_3 = \sin\left(\frac{2\pi k}{n}\right) \end{cases} \Rightarrow \begin{cases} x_4 = \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + 1 \\ x_1 = \frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{-1}{2}\cos\left(\frac{2\pi k}{n}\right) \\ x_3 = \frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2} \end{cases}.$$

$$X = -1 + \left(\frac{1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_1 - \frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) I_2 + \left(\frac{-1}{2}\sin\left(\frac{2\pi k}{n}\right) + \frac{1}{2}\right) I_3 + \left(\frac{1}{2}\cos\left(\frac{2\pi k}{n}\right) + 1\right) I_4.$$

Example:

Let's try to find all 4-cyclic refined real 4-th roots of unity.

Case (1):

$$X_0 = 1$$

$$X_1 = 1 + \frac{1}{2} I_1 + \frac{1}{2} I_2 - \frac{1}{2} I_3 - \frac{1}{2} I_4$$

$$X_2 = 1 + I_2 - I_4$$

$$X_3 = 1 - \frac{1}{2} I_1 + \frac{1}{2} I_2 + \frac{1}{2} I_3 - \frac{1}{2} I_4$$

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Case (2):

$$X_0 = -1 - I_2 + I_4$$
$$X_1 = -1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{1}{2}I_4$$
$$X_2 = -1$$
$$X_3 = -1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{1}{2}I_4$$

Case (3):

$$X_0 = 1 - I_2 - I_4$$
$$X_1 = 1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{3}{2}I_4$$
$$X_2 = 1 - 2I_4$$
$$X_3 = 1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 + \frac{1}{2}I_3 - \frac{3}{2}I_4$$

Case (4):

$$X_0 = 1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4$$
$$X_1 = 1 + I_1 - I_3 - I_4$$
$$X_2 = 1 + \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{3}{2}I_4$$
$$X_3 = 1 - I_4$$

Case (5):

$$X_0 = 1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4$$
$$X_1 = 1 - I_3 - I_4$$
$$X_2 = 1 - \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{3}{2}I_4$$
$$X_3 = 1 - I_1 - I_4$$

Case (6):

$$X_0 = -1 + 2I_4$$
$$X_1 = -1 + \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{3}{2}I_4$$
$$X_2 = -1 + I_2 + I_4$$
$$X_3 = -1 - \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{3}{2}I_4$$

Case (7):

$$X_0 = -1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{3}{2}I_4$$
$$X_1 = -1 - I_3 + I_4$$
$$X_2 = -1 - \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 + \frac{1}{2}I_4$$
$$X_3 = -1 - I_1 + I_4$$

Case (8):

$$X_0 = -1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{3}{2}I_4$$
$$X_1 = -1 + I_1 + I_4$$
$$X_2 = -1 + \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{1}{2}I_4$$
$$X_3 = -1 + I_3 + I_4$$

Example:

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We will find all 4-cyclic refined neutrosophic 6-th roots of unity.

Remark that: $\left\{\frac{2\pi k}{6}\ ; 0 \le k \le 5\right\} = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

Case (1):

$$X_0 = 1$$
$$X_1 = 1 + \frac{\sqrt{3}}{4}I_1 + \frac{1}{4}I_2 - \frac{\sqrt{3}}{4}I_3 - \frac{1}{4}I_4$$
$$X_2 = 1 + \frac{\sqrt{3}}{4}I_1 + \frac{3}{4}I_2 - \frac{\sqrt{3}}{4}I_3 - \frac{3}{4}I_4$$
$$X_3 = 1 + I_2 - I_4$$
$$X_4 = 1 - \frac{\sqrt{3}}{4}I_1 + \frac{3}{4}I_2 + \frac{\sqrt{3}}{4}I_3 - \frac{3}{4}I_4$$
$$X_5 = 1 - \frac{\sqrt{3}}{4}I_1 + \frac{1}{4}I_2 + \frac{\sqrt{3}}{4}I_3 - \frac{1}{4}I_4$$

Case (2):

$$X_0 = -1 - I_2 + I_4$$
$$X_1 = -1 + \frac{\sqrt{3}}{4}I_1 - \frac{3}{4}I_2 - \frac{\sqrt{3}}{4}I_3 + \frac{3}{4}I_4$$
$$X_2 = -1 + \frac{\sqrt{3}}{4}I_1 - \frac{1}{4}I_2 - \frac{\sqrt{3}}{4}I_3 + \frac{1}{4}I_4$$
$$X_3 = -1$$
$$X_4 = -1 - \frac{\sqrt{3}}{4}I_1 - \frac{1}{4}I_2 + \frac{\sqrt{3}}{4}I_3 + \frac{1}{4}I_4$$
$$X_5 = -1 - \frac{\sqrt{3}}{4}I_1 - \frac{3}{4}I_2 + \frac{\sqrt{3}}{4}I_3 + \frac{3}{4}I_4$$

Case (3):

$$X_0 = 1 - I_2 - I_4$$
$$X_1 = 1 + \frac{\sqrt{3}}{4}I_1 - \frac{3}{4}I_2 - \frac{\sqrt{3}}{4}I_3 - \frac{5}{4}I_4$$
$$X_2 = 1 + \frac{\sqrt{3}}{4}I_1 - \frac{1}{4}I_2 + \frac{\sqrt{3}}{4}I_3 - \frac{7}{4}I_4$$
$$X_3 = 1 - 2I_4$$
$$X_4 = 1 - \frac{\sqrt{3}}{4}I_1 - \frac{1}{4}I_2 + \frac{\sqrt{3}}{4}I_3 - \frac{7}{4}I_4$$
$$X_5 = 1 - \frac{\sqrt{3}}{4}I_1 - \frac{3}{4}I_2 + \frac{\sqrt{3}}{4}I_3 - \frac{5}{4}I_4$$

Case (4):

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$$X_0 = 1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4$$
$$X_1 = 1 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{3}{4}I_4$$
$$X_2 = 1 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{5}{4}I_4$$
$$X_3 = 1 + \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{3}{2}I_4$$
$$X_4 = 1 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{5}{4}I_4$$
$$X_5 = 1 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{3}{4}I_4$$

Case (5):

$$X_0 = 1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{1}{2}I_4$$
$$X_1 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{4}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{3}{4}I_4$$
$$X_2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{4}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{5}{4}I_4$$
$$X_3 = 1 - \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{2}I_3 - \frac{3}{2}I_4$$
$$X_4 = 1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{4}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{5}{4}I_4$$
$$X_5 = 1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{4}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_3 - \frac{3}{4}I_4$$

Case (6):

$$X_0 = -1 + I_2 + I_4$$
$$X_1 = -1 - \frac{\sqrt{3}}{4}I_1 + \frac{3}{4}I_2 + \frac{\sqrt{3}}{4}I_3 + \frac{5}{4}I_4$$
$$X_2 = -1 - \frac{\sqrt{3}}{4}I_1 + \frac{1}{4}I_2 - \frac{\sqrt{3}}{4}I_3 + \frac{7}{4}I_4$$
$$X_3 = -1 + 2I_4$$
$$X_4 = -1 + \frac{\sqrt{3}}{4}I_1 + \frac{1}{4}I_2 - \frac{\sqrt{3}}{4}I_3 + \frac{7}{4}I_4$$
$$X_5 = -1 + \frac{\sqrt{3}}{4}I_1 + \frac{3}{4}I_2 - \frac{\sqrt{3}}{4}I_3 + \frac{5}{4}I_4$$

Case (7):

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$$X_0 = -1 - \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{1}{2}I_4$$

$$X_1 = -1 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{3}{4}I_4$$

$$X_2 = -1 + \left(\frac{-\sqrt{3}}{4} - \frac{1}{2}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{5}{4}I_4$$

$$X_3 = -1 - \frac{1}{2}I_1 - \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{3}{2}I_4$$

$$X_4 = -1 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{5}{4}I_4$$

$$X_5 = -1 + \left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{3}{4}I_4$$

Case (8):

$$X_0 = -1 + \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{1}{2}I_4$$

$$X_1 = -1 + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{3}{4}I_4$$

$$X_2 = -1 + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{5}{4}I_4$$

$$X_3 = -1 + \frac{1}{2}I_1 - \frac{1}{2}I_2 + \frac{1}{2}I_3 + \frac{3}{2}I_4$$

$$X_4 = -1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right)I_1 - \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{5}{4}I_4$$

$$X_5 = -1 + \left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right)I_1 + \frac{1}{4}I_2 + \left(\frac{-\sqrt{3}}{4} + \frac{1}{2}\right)I_3 + \frac{3}{4}I_4$$

Remark:

For even values of n, $|R_n| = 8n$, where (R_n, x) is the multiplicative group of n-th roots of unity.

Remark:

Since $R_4(I) \cong R \times R \times R \times \mathbb{C}$, then:

$R_n \cong Z_2 \times Z_2 \times Z_2 \times Z_n$.

3. Conclusion

In this paper we found all formulas that describe the 4-cyclic refined neutrosophic real solutions of the equation $X^n = 1$ which are called 4-cyclic refined real roots of unity. Also, we classified the algebraic group represented by these solutions as a direct product of some familiar finite abelian groups.

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