Fixed point results of Geraghty type contractions with equivalent distance

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Abstract

In this manuscript, we present the concept of $\mathscr{E}$-Geraghty contractions, demonstrating several fixed point results. Additionally, we provide an illustrative example to highlight our principal findings.

Keywords: Metric Spaces; Extended b-metric; Fixed Point; Nonlinear Contractions; Equivalent distance; Geraghty contraction

1 Introduction

Let $f : \mho \to \mho$ be a self mapping on a non empty set $\mho$. An element or a point $\vartheta' \in \mho$ is called a fixed point for f if $f\vartheta' = \vartheta'$. If d is a metric on $\mho$, then f is called contraction if there is $\omega \in [0, 1)$ such that $d(f\vartheta_1, f\vartheta_2) \leq \omega d(\vartheta_1, \vartheta_2)$, for each $\vartheta_1, \vartheta_2 \in \mho$.

The advancement of fixed point theory has been thoroughly examined by mathematicians.For instance, U. Ishtiaq et al.[2] proposed the idea of intuitionistic fuzzy double-controlled metric-like spaces, taking into account the potential scenario where the self-distance may not equal zero. In instances where the metric value is precisely zero, it is termed a "self-distance." Within this context, they effectively established multiple fixed-point results for contraction mappings. E. Karapinar et al.[3] introduced the notion of Proinov-$\mathscr{C}_b$-contraction mapping and explored its application within b-metric spaces, a notably fascinating abstract framework. Additionally, they investigated the essential conditions required to ensure the existence and uniqueness of fixed points for these mappings. In,[4] a new definition of a metric space incorporating neutrosophic numbers was introduced. This neutrosophic metric space employs the concepts of continuous triangular norms and continuous triangular conorms within the framework of intuitionistic fuzzy metric spaces. Triangular norms serve to extend the probability distribution of the triangle inequality under the conditions of metric spaces. Conversely, triangular conorms are recognized as the dual operations of triangular norms. Both triangular norms and triangular conorms play a crucial role in fuzzy operations.we are looking to integrate time fuzzy soft set and fuzzy soft set with new concepts as in the works a[6]–,[11] In,[13] the authors presented an innovative concept known as "neutrosophic fuzzy metric space," which broadens the conventional metric space framework by

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integrating the idea of neutrosophic fuzzy sets. A comprehensive examination of different structural and topological characteristics within this newly established generalization of metric space has been carried out.We can include fresh ideas, such as[7]–,[18] works using fixed point theory. Over the years, various extensions of the Banach principle have been formulated, either by altering the space involved or by adjusting the conditions of self-mapping, as referenced in sources[10]–[19] and references therein. These generalizations synthesized and broadened the previously established findings, resulting in more applicable outcomes.In addition, there are many works that have discussed broad applications, including:[24]–[30]

Henceforth, we mean by $\mho$ a nonempty set and for any self map $f : \mho \to \mho$ we mean by $\mathscr{F}_f$ the set of fixed points of f.
Let $d : \mho \times \mho \to [0, \infty)$ be a function and consider the following:

(d_1) $d(\vartheta, \vartheta') = 0$ iff $\vartheta' = \vartheta$,

(d_2) $d(\vartheta', \vartheta) = d(\vartheta, \vartheta')$,

(d_3) $d(\vartheta, \vartheta') \leq s\big[d_\gamma(\vartheta, e) + d(e, \vartheta')\big] \; \forall \, \vartheta, e, \vartheta' \in \mho$.

The function d is classified as a b-metric on the set $\mho$ if the parameter s is greater than or equal to 1. In the specific case where s equals 1, d is referred to as a metric on $\mho$. Furthermore, if d meets the conditions of d_1 and d_3 with s set to 1, it is designated as a quasi-metric on $\mho$.

In 2017 Kamran et al. introduced the notion of extended b-metric spaces in the following manner.

Definition 1.1. [?] On $\mho$, Suppose $\gamma : \mho \times \mho \to [1, \infty)$. Then, $d_\gamma : \mho \times \mho \to [0, \infty)$ is called an extended b-metric if it fulfilled the following conditions:

$(d_\gamma 1)$ $d_\gamma(\vartheta', \vartheta) = 0$ iff $\vartheta' = \vartheta$,

$(d_\gamma 2)$ $d_\gamma(\vartheta', \vartheta) = d_\gamma(\vartheta, \vartheta')$,

$(d_\gamma 3)$ $d_\gamma(\vartheta', \vartheta) \leq \gamma(\vartheta', \vartheta)\big[d_\gamma(\vartheta', e) + d_\gamma(e, \vartheta)\big], \forall \, e, \vartheta', \vartheta \in \mho$.

2 Preliminary

Bataihah and Qawasmeh[20] introduced a novel approach to explore fixed point results by creating a distance space derived from an existing one.

In this section, we will revisit the definition of the $\mathscr{E}_{A,B}$-distance. To proceed with this discussion, it is essential to consider the following class of functions, which is crucial for the advancement of this work.

Definition 2.1. [20] Let Λ be the collection of all functions $A : [0, \infty) \to [1, \infty)$ such that for each sequence (t_n) in $[0, \infty)$, $\lim_{n \to \infty} A(t_n) = 1$ iff $\lim_{n \to \infty} t_n = 0$.

The notion of equivalent distance is given as follows:

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Definition 2.2. [20]Let d be a metric on $\mho$. A function $\mathscr{E} : [0,\infty) \times \mho \times \mho \to [0,\infty)$ is said to be $[A, B]$ equivalent-distance (or shortly $\mathscr{E}_{A,B}$-distance) over $(\mho, d)$ if there are $A, B \in \Lambda$ such that for all $u, v \in \mho$ and all $t \in [0,\infty)$, we have

$$A(t)\ d(\vartheta,\theta) \leq \mathscr{E}(t,\vartheta,\theta) \leq B(t)\ d(\vartheta,\theta).$$

It is essential to recognize that every metric space produces an $\mathscr{E}_{A,B}$-distance, as illustrated in the subsequent examples.

Example 2.3. [20] Let $\mho = \mathbb{R}$, and let $d : \mho \to \mho$ be defined by $d(\vartheta,\theta) = |\vartheta - \theta|$. Then $\mathscr{E}(t,\vartheta,\theta) = 2^t|\vartheta - \theta|$. Then, $\mathscr{E}$ is $\mathscr{E}_{A,B}$-distance over $(\mho, d)$, where $A(t) = 1 + t$, and $B(t) = t + 2^t$.

Example 2.4. [20] Let $(\mho, d)$ be a metric space, and let $0 < \alpha \leq \beta \leq \gamma$ be positive real numbers. Define $\mathscr{E}(t,\vartheta,\theta) = (1+t)^\beta d(\vartheta,\theta)$ Then, $\mathscr{E}$ is $\mathscr{E}_{A,B}$-distance over $(\mho, d)$, where $A(t) = (1+t)^\alpha$, and $B(t) = (1+t)^\gamma$.

We will now provide additional examples of $\mathscr{E}$-distance functions. In the subsequent discussion, we examine the function $\mathscr{E}$ defined from $[0,\infty) \times \mho \times \mho$ to $[0,\infty)$, along with the functions A and B, which are defined from $[0,\infty)$ to $[1,\infty)$.

Example 2.5. [20] Let $(\mho, d)$ be a metric space, k be real number grater than 1, and let $\mathscr{E}(t,\vartheta,\theta) = k^t d(\vartheta,\theta)$. Then, $\mathscr{E}$ is $\mathscr{E}_{A,B}$-distance over $(\mho, d)$, where $A(t) = 1 + t$, and $B(t) = t + k^t$.

Example 2.6. [20] Let $q, \rho : \mho^2 \to [0,\infty)]$ be two equivalent metrics on $\mho$. Then
$\mathscr{E}(t,\vartheta,\theta) = q(\vartheta,\theta) + t\ \rho(\vartheta,\theta)$ is $\mathscr{E}_{A,B}$-distance on (θ, q).

Proof. Since q and ρ are equivalent, there are $\alpha, \beta \geq 0$, such that for all $\vartheta, \theta \in \mho$, we have $\alpha q(\vartheta,\theta) \leq \rho(\vartheta,\theta) \leq \beta q(\vartheta,\theta)$. So, for each $t \geq 0$, we have

$$t\alpha q(\vartheta,\theta) \leq t\rho(\vartheta,\theta) \leq t\beta q(\vartheta,\theta).$$

Thus,

$$(1 + t\alpha)q(\vartheta,\theta) \leq q(\vartheta,\theta) + t\rho(\vartheta,\theta) \leq (1 + t\beta)q(\vartheta,\theta).$$

Hence, $\mathscr{E}$ is $\mathscr{E}_{A,B}$-distance on $(\mho, q)$, where $A(t) = 1 + t\alpha$ and $B(t) = 1 + t\beta$. □

From this point forward, let A and B denote elements of Λ, and let $\mathscr{E}$ represent an $\mathscr{E}_{A,B}$-distance defined on the space $(\mho, d)$.

Additionally, in,[20] the authors established a crucial lemma that is fundamental for deriving b-metric and extended b-metric from a standard metric utilizing the $\mathscr{E}$-distance.

Lemma 2.7. [20] *Let $(\mho, d)$ be a metric space, $A, B \in \Lambda$ and $\mathscr{E}$ be $\mathscr{E}_{A,B}$-distance over $(\mho, d)$. Then, for each $\vartheta, \theta, w \in \mho$ and each $t \geq 0$, we have the following:*

1. $\mathscr{E}(t,\vartheta,\theta) = 0$ *iff* $\vartheta = \theta$,
2. $\mathscr{E}(t,\vartheta,\theta) = \mathscr{E}(t,\theta,\vartheta)$,
3. $\mathscr{E}(t,\vartheta,\theta) \leq B(t)[\mathscr{E}(t,\vartheta,w) + \mathscr{E}(t,w,\theta)]$.

Remark 2.8. [20] It follows from Lemma 2.7 that for any number $t_0 \in [0,\infty)$, $\mathscr{E}(t_0\ ,\ .\ ,\ .\ ) : \mho \times \mho \to [0,\infty)$ is a b-metric on $\mho$ with constant $s = B(t_0)$.

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One can observe that using $\mathscr{E}_{A,B}$-distance can construct an extended b-metric on $\mho$ starting by starting from a metric on $\mho$ as in the following theorem:

Theorem 2.9. [20] *Let $\mathscr{E}$ be an $E_{A.B}$-distance on $(\mho, d)$. Then $d_\gamma : \mho \times \mho \to [0,\infty)$ which defined by $d_\gamma(\vartheta,\theta) = \mathscr{E}(d(\vartheta,\theta),\vartheta,\theta)$ is an extended b-metric on $\mho$ where $\gamma : \mho \times \mho : [1,\infty)$ defined as $\gamma(\vartheta,\theta) = (B \circ d)(\vartheta,\theta) = B(d(\vartheta,\theta))$*

Proposition 2.10. [20] *Suppose (ϑ_n) is a sequence in $\mho$ and (t_n) is a sequence in $[0,\infty)$ such that $\lim\limits_{n\to\infty} B(t_n) < \infty$. Then, we have the following:*

1. *(ϑ_n) is Cauchy sequence iff $\lim\limits_{n,m,l\to\infty} \mathscr{E}(t_n,\vartheta_m,\vartheta_l) = 0$,*
2. *$\vartheta_n \to \vartheta \in \mho$ iff $\lim\limits_{n,m\to\infty} \mathscr{E}(t_n,\vartheta_m,\vartheta) = 0$.*

3 Fixed point for $\mathscr{E}$-Geraghty contractions

In this section, we introduce the contractions of Geraghty type in $\mathscr{E}$-distance spaces. First, we recall an important notion and resulut.

Definition 3.1. [23] Let S be the class of all functions $\alpha : [0,\infty) \to [0,1)$ that satisfy the following implication:

$$\alpha(t_n) \to 1 \implies t_n \to 0.$$

Geraghty in[23] proved the following fixed point result.

Theorem 3.2. [23] *Let $(\mho, d)$ be a complete metric space and $f : \mho \to \mho$. Let $\alpha \in S$ such that*

$$d(f\vartheta, f\theta) \le \alpha(d(\vartheta,\theta)d(\vartheta,\theta)\ \forall\vartheta,\ \theta \in \mho.$$

Then f has a unique fixed point.

Now, we define our contraction of Geraghty type.

Definition 3.3. Suppose there are $A, B \in \Lambda$ such that $\mathscr{E}$ is $\mathscr{E}_{A,B}$ over $(\mho, d)$. A self mapping $f : \mho \to \mho$ is said to be $\mathscr{E}$-Geraghty contraction if

$$\mathscr{E}(d(f\vartheta, f\theta), f\vartheta, f\theta) \le \alpha(d(\vartheta,\theta))\ A(d(\vartheta,\theta))\ d(\vartheta,\theta). \tag{1}$$

Lemma 3.4. *Suppose $f : \mho \to \mho$ is $\mathscr{E}$-Geraghty contraction. If $\vartheta, \theta \in \mathscr{F}_f$, then $\vartheta = \theta$.*

Lemma 3.5. *Suppose f is $\mathscr{E}$-Geraghty contraction, and $\vartheta_0 \in \mho$. Then for the Picard sequence (ϑ_n) derived by f at ϑ_0, if $\vartheta_n \ne \vartheta_{n+1}$ for each $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} d(\vartheta_n, \vartheta_{n+1}) = 0.$$

Proof. For simplicity, let $e_n = d(\vartheta_n, \vartheta_{n+1})$. Now, applying Condition 1, we get

$$\mathscr{E}(e_n, \vartheta_n, \vartheta_{n+1}) \le \alpha(e_{n-1})\ A(e_{n-1})\ e_{n-1}.$$

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By the properties of $\mathscr{E}$ and α, we get

$$e_n \leq A(e_n)\, e_n \leq \alpha(e_{n-1})\, A(e_{n-1})\, e_{n-1} < A(e_{n-1})\, e_{n-1},$$

and

$$\frac{A(e_n)\, e_n}{A(e_{n-1})\, e_{n-1}} \leq \alpha(e_{n-1}). \tag{2}$$

So, $A(e_n)\, e_n \leq A(e_{n-1})\, e_{n-1}$, and so, the sequence $(A(e_n)\, e_n : \ n \in \mathbb{N})$ is a non-increasing, and hence, there is $r \geq 0$ so that $\lim_{n\to\infty} A(e_n)\, e_n = r$. Assume that $r > 0$. Then, taking the limit whenever $n \to \infty$ in Equation 2, we get $\lim_{n\to\infty} \alpha(e_{n-1}) = 1$, which means $\lim_{n\to\infty} e_{n-1} \ = 0$, a contradiction. So, $r = 0$. Hence the result. □

Theorem 3.6. *Suppose that $(\mho, d)$ is complete and there is $\mathscr{E}_{A,B}$-distance $\mathscr{E}$ on $(\mho, d)$. Assume that $f : \mho \to \mho$ is $\mathscr{E}$-Geraghty contraction where A is continuous. Then $\mathscr{F}_f$ consists of only one element.*

Proof. Let $\vartheta_0 \in \mho$ be arbitrary, and consider the Picard sequence (ϑ_n) derived by f at ϑ_0. If there is $l \in \mathbb{N}$ such that $\vartheta_l = \vartheta_{l+1}$, then $\vartheta_l \in \mathscr{F}_f$. So suppose that $\vartheta_n \neq \vartheta_{n+1}$ for each $n \in \mathbb{N}$, . Now, our claim is to prove that (ϑ_n) is a Cauchy sequence in $(\mho, d)$. Suppose the opposite; that is (ϑ_n) is not Cauchy. Therefore, there is $\epsilon > 0$ and two sub-sequences (ϑ_{n_p}) and (ϑ_{m_p}) of (ϑ_n) such that (m_p) is selected as the minimum index for which

$$d(\vartheta_{n_p}, \vartheta_{m_p}) \geq \epsilon,\ m_p > n_p > p. \tag{3}$$

This implies that

$$d(\vartheta_{n_p}, \vartheta_{m_p-1}) < \epsilon. \tag{4}$$

Using the triangle inequality and Equations (3),(4), we get

$$\begin{aligned} \epsilon \leq d(\vartheta_{n_p}, \vartheta_{m_p}) \quad &\leq d(\vartheta_{n_p-1}, \vartheta_{m_p}) + d(\vartheta_{m_p}, \vartheta_{m_p-1}) \\ &< \epsilon + d(\vartheta_{m_p}, \vartheta_{m_p-1}). \end{aligned}$$

Taking the limit whenever $p \to \infty$ and considering Lemma 3.5, gives

$$\lim_{p\to\infty} d(\vartheta_{n_p}, \vartheta_{m_p}) = \epsilon. \tag{5}$$

Again, using the triangle inequality, we get

$$d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) - d(\vartheta_{n_p}, \vartheta_{m_p}) \leq d(\vartheta_{n_p-1}, \vartheta_{n_p}) + d(\vartheta_{m_p}, \vartheta_{m_p-1}),$$

and

$$d(\vartheta_{n_p}, \vartheta_{m_p}) - d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) \leq d(\vartheta_{n_p}, \vartheta_{n_p-1}) + d(\vartheta_{m_p-1}, \vartheta_{m_p}).$$

Therefore,

$$|d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) - d(\vartheta_{n_p}, \vartheta_{m_p})| \leq d(\vartheta_{n_p-1}, \vartheta_{n_p}) + d(\vartheta_{m_p}, \vartheta_{m_p-1}).$$

So, taking the limit as $p \to \infty$ and considering Lemma 3.5 and Equation (5), gives

$$\lim_{p\to\infty} d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) = \epsilon. \tag{6}$$

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Now, Condition 1implies that

$$\begin{aligned} A(d(\vartheta_{n_p}, \vartheta_{m_p}))\, d(\vartheta_{n_p}, \vartheta_{m_p}) &\leq \alpha(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))\, A(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))\, d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) \\ &< A(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))\, d(\vartheta_{n_p-1}, \vartheta_{m_p-1}). \end{aligned}$$

Thus,

$$\frac{A(d(\vartheta_{n_p}, \vartheta_{m_p}))}{A(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))} < \frac{d(\vartheta_{n_p-1}, \vartheta_{m_p-1})}{d(\vartheta_{n_p}, \vartheta_{m_p})}.$$

By passing k to ∞, we get

$$\lim_{p\to\infty} \frac{A(d(\vartheta_{n_p}, \vartheta_{m_p}))}{A(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))} \leq 1.$$

Also,

$$\frac{A(d(\vartheta_{n_p}, \vartheta_{m_p}))\, d(\vartheta_{n_p}, \vartheta_{m_p})}{A(d(\vartheta_{n_p-1}, \vartheta_{m_p-1}))\, d(\vartheta_{n_p-1}, \vartheta_{m_p-1})} \leq \alpha(d(\vartheta_{n_p-1}, \vartheta_{m_p-1})).$$

Therefore, by taking the limit whenever $p \to \infty$ to get $\lim_{p\to\infty} \alpha(d(\vartheta_{n_p-1}, \vartheta_{m_p-1})) = 1$. Thus, $\lim_{p\to\infty} d(\vartheta_{n_p-1}, \vartheta_{m_p-1}) = 0$, and hence $\epsilon = 0$, a contradiction. Therefore, (ϑ_n) is Cauchy, so there is some $\vartheta \in \mho$ such that (ϑ_n) converges to ϑ. Now, by Condition 1, we have

$$A(d(\vartheta_n, f\vartheta))\, d(\vartheta_n, f\vartheta) \leq \alpha(d(\vartheta_{n-1}, \vartheta))\, A(d(\vartheta_{n-1}, \vartheta))\, d(\vartheta_{n-1}, \vartheta).$$

Therefore, by passing n to ∞, we get

$$d(\vartheta, f\vartheta) \leq 0.$$

So, $\vartheta \in \mathscr{F}_f$. The uniqueness follows from Lemma 3.4. □

Example 3.7. Let $f : [0, 1] \to [0, 1]$ be defined as:

$$f\vartheta = \frac{D}{D + \vartheta^{\eta}},\ where\ \ D > 2\eta \geq 2.$$

Then $\mathscr{F}_f$ consists of only one element on $[0, 1]$.

Proof. To demonstrate this, let $\mho = [0, 1]$ and define the mapping $d : \mho \to \mho$ by $d(\vartheta, \theta) = |\vartheta - \theta|$. Consequently, $(\mho, d)$ forms a complete metric space. We define the function $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \dfrac{2\eta}{D}$, which implies that α belongs to the set S. Additionally, we define $\mathscr{E}(t, \vartheta, \theta) = 2^t d(\vartheta, \theta)$. Thus, $\mathscr{E}$ serves as an $\mathscr{E}_{A,B}$-distance on the space $(\mho, d)$, where $A(t) = 1 + t$ and $B(t) = t + 2^t$. Therefore, for all $\vartheta, \theta \in \mho$, we have the following.

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$$
\begin{aligned}
\mathscr{E}(d(f\vartheta, f\theta), f\vartheta, f\theta) &= 2^{|f\vartheta - f\theta|}|f\vartheta - f\theta| \\
&\leq 2|f\vartheta - f\theta| \\
&= 2\left|\frac{D}{D+\vartheta^\eta} - \frac{D}{D+\theta^\eta}\right| \\
&= 2\left|\frac{D\theta^\eta - D\vartheta^\eta}{(D+\theta^\eta)(D+\vartheta^\eta)}\right| \\
&\leq \frac{2}{D}\left|\vartheta^\eta - \theta^\eta\right| \\
&= \frac{2}{D}\left|\vartheta - \theta\right|\left|\vartheta^{\eta-1} + \theta\vartheta^{\eta-2} + \cdots + \vartheta\theta^{\eta-2} + \theta^{\eta-1}\right| \\
&\leq \frac{2\eta}{D}(1 + |\vartheta - \theta|)\left|\vartheta - \theta\right| \\
&= \alpha(d(\vartheta,\theta))\, A(d(\vartheta,\theta))\, d(\vartheta,\theta)
\end{aligned}
$$

Hence, $f : \mho \to \mho$ is of $\mathscr{E}$-Geraghty contraction, and so, Theorem 3.6 ensures that $\mathscr{F}_f$ has only one element. □

Corollary 3.8. *Assume that $(\mho, d)$ is a complete metric space. Let $f : \mho \to \mho$ be a function that satisfies the following condition for all $\vartheta, \theta \in \mho$ and for some constants $b \geq a > 0$:*

$$d(f\vartheta, f\theta) \leq \frac{(1 + d(\vartheta,\theta))^a}{2(1 + d(f\vartheta, f\theta))^b} d(\vartheta,\theta).$$

Under these circumstances, it can be concluded that the set $\mathscr{F}_f$ contains only a single element.

Proof. According to Example 2.4, $\mathscr{E}(t,\vartheta,\theta) = (1+t)^\beta d(\vartheta,\theta)$ is $\mathscr{E}$-distance. Also, $\alpha \in S$ where $\alpha(t) = \frac{1}{2}$. So, we have

$$d(f\vartheta, f\theta) \leq \frac{(1 + d(\vartheta,\theta))^a}{2(1 + d(f\vartheta, f\theta))^b} d(\vartheta,\theta)$$

iff

$$(1 + d(f\vartheta, f\theta))^b d(f\vartheta, f\theta) \leq \frac{1}{2}(1 + d(\vartheta,\theta))^a d(\vartheta,\theta)$$

iff

$$\mathscr{E}(d(f\vartheta, f\theta), f\vartheta, f\theta) \leq \alpha(d(\vartheta,\theta)) A(d(\vartheta,\theta)) d(\vartheta,\theta).$$

iff $f : \mho \to \mho$ is $\mathscr{E}$-Geraghty contraction. Hence by Theorem 3.6 f attains a unique fixed point. □

Corollary 3.9. *Assume that the space $(\mho, d)$ is complete. Let us consider a function $f : \mho \to \mho$ that satisfies the following condition for all $\vartheta, \theta \in \mho$ and for some constants $b \geq a > 0$:*

$$d(f\vartheta, f\theta) \leq k^{-d(f\vartheta, f\theta)} \frac{1 + d(\vartheta,\theta)}{2} d(\vartheta,\theta).$$

Under these circumstances, it can be concluded that the set $\mathscr{F}_f$ contains only a single element.

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Proof. According to Example 2.5, $\mathscr{E}(t,\vartheta,\theta)=k^t d(\vartheta,\theta)$ is $\mathscr{E}$-distance. Also, $\alpha \in S$ where $\alpha(t)=\frac{1}{2}$. So, we have

$$d(f\vartheta,f\theta)\leq k^{-d(f\vartheta,f\theta)}\frac{1+d(\vartheta,\theta)}{2}d(\vartheta,\theta).$$

iff

$$k^{d(f\vartheta,f\theta)}d(f\vartheta,f\theta)\leq\frac{1}{2}(1+d(\vartheta,\theta))d(\vartheta,\theta).$$

iff

$$\mathscr{E}(d(f\vartheta,f\theta),f\vartheta,f\theta)\leq\alpha(d(\vartheta,\theta))A(d(\vartheta,\theta))d(\vartheta,\theta).$$

iff $f:\mho\to\mho$ is $\mathscr{E}$-Geraghty contraction. Hence by Theorem 3.6 f attains a unique fixed point. □

4 Application

We will now utilize Example 3.7 to develop the subsequent application.

Let $\eta, D\in\mathbb{R}$ with the condition that $D>2\eta\geq 2$. The equation can be expressed as follows:

$$\vartheta^{\eta+1}+D(\vartheta-1)=0, \tag{7}$$

possesses a unique solution within the unit interval $[0,1]$.
To establish this, it suffices to demonstrate that the mapping defined below possesses a unique fixed point within the unit interval $[0,1]$.

$$f\vartheta=\frac{D}{D+\vartheta^{\eta}}\ ,\ where\ D>2\eta\geq 2.$$

Example 3.7 verifies that $\mathscr{F}_f$ contains a single element. Therefore, equation 7 has a unique solution.

Conclusion

We introduced the concept of $\mathscr{E}$-Geraghty contractions within the context of complete metric spaces. By employing these contractions, we explored various outcomes related to metric fixed point theory. For future studies of these results, these tools can be developed by linking them with other concepts that can be found in the following works see[31]–.[37]

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