On The Two-Fold Neutrosophic Groups and Their Algebra Properties

Ahmed Hatip[1,*]
[1]Department of Mathematics, Gaziantep University, Gaziantep, Turkey
Kollnaar5@gmail.com

Abstract

This paper is dedicated to define and study for the first time the concept of two-fold neutrosophic group by combining two-fold algebras with the classical concept of neutrosophic group. We study the elementary properties of this novel concept by many related theorems and illustrated examples.

Keywords: Two-fold algebra; Group; Neutrosophic group

1. Introduction

Two fold algebraic structures were proposed by Smarandache in [1] as a novel generalization of classical structures. In [4-8], we find many related studies that used this novel approach to generate generalized structures in both algebra and analysis. Also, two-fold structures were used in number theory and Diophantine equations theory [2-3].

In this work, we are motivated to extend previous studies on two-fold algebras to neutrosophic groups to form a new algebraic structure called two-fold neutrosophic group by combining two-fold algebras with the classical concept of neutrosophic group where we study the elementary properties of this novel concept by many related theorems and illustrated examples.

2. Main Discussion

Definition:

Let $(G,.)$ be a group, with: $t,i,f: G \to [0,1]$ such that:

$$\begin{cases} t(xy) \leq \min\big(t(x),t(y)\big) \\ i(xy) \geq \max\big(i(x),i(y)\big) \\ f(xy) \geq \max\big(f(x),f(y)\big) \end{cases} ;x,y \in G.$$

Then $(G,.,t,i,f)$ is called two-fold neutrosophic group.

Remark:

The elements of $G_N = (G,.,t,i,f)$ has the formula:

$(x)_{(t,i,f)}$;$x \in G, t = t(x), i = i(x), f = f(x)$.

Definition:

Let $A = (x)_{(t_1,i_1,f_1)}$, $B = (y)_{(t_2,i_2,f_2)} \in G_N$, we define:

$A * B = (xy)_{(t(xy),i(xy),f(xy))}$

Theorem (1):

Let $G_N = (G,.,t,i,f)$ be two-fold neutrosophic group, then:

DOI: https://doi.org/10.54216/JNFS.090103
Received: December 22, 2023 Revised: March 17, 2024 Accepted: July 12, 2024

1] $(*)$ is associative.

2] $(*)$ has an identity $(e)_{(t(e),i(e),f(e))}$

3] $A=(x)_{(t(x),i(x),f(x))}$ has an inverse $(x^{-1})_{(t(x^{-1}),i(x^{-1}),f(x^{-1}))}$.

4] If $(G,.)$ is abelian, then $(G_N,*)$ is abelian.

5] $t(x) \leq t(e), i(x) \geq i(e), f(x) \geq f(e)$ for all $x \in G$.

Definition:

Let H be a subgroup of G, then:

$H_N=\{(x)_{(t(x),i(x),f(x))}\ ; x \in H\}$ is called a two-fold neutrosophic subgroup of G_N.

Theorem (2):

Let $(G_N,*)$ be a two-fold neutrosophic group, then:

1] If H is abelian subgroup of G, then H_N is abelian two-fold neutrosophic subgroup of G_N.

2] If H is normal, then H_N is two-fold normal subgroup of G_N.

Definition:

Let H be a normal subgroup of G, and G/H be the corresponding factor group, we define:

$t', i', f': G/H \rightarrow [0,1]$ such that:

$$t'(xH)=\begin{cases} t(x) ; x \notin H \\ t(e) ; x \in H \end{cases}, i'(xH)=\begin{cases} i(x) ; x \notin H \\ i(e) ; x \in H \end{cases}, f'(xH)=\begin{cases} f(x) ; x \in H \\ f(e) ; x \notin H \end{cases}$$

Theorem (3):

Let H_N be a two-fold normal neutrosophic subgroup of G_N, then:

$G_N/H_N=\{(xH)_{(t'(xH),i'(xH),f'(xH))}\ ; x \in G\}$ is a two-fold neutrosophic group. We call it the two-fold neutrosophic factor.

Definition:

Let $g: G \rightarrow K$ be a group homomorphism, and $G_N=(G,.,t,i,f), K_N=(K,.,t_1,i_1,f_1)$be the corresponding two-fold neutrosophic groups, then: $g_N: G_N \rightarrow K_N$; $g_N\left[(x)_{(t(x),i(x),f(x))}\right]=(g(x))_{\left(t_1(g(x)),i_1(g(x)),f_1(g(x))\right)}$ is called a two-fold neutrosophic homomorphism.

If $G \cong K$, then g_N is called two-fold neutrosophic isomorphism.

Theorem (4):

Let $g_N: G_N \rightarrow K_N$ be a two-fold neutrosophic homomorphism, then:

1] $g_N\left[(e)_{(t(e),i(e),f(e))}\right]=(e_k)_{(t_1(e),i_1(e),f_1(e))}$

2] $g_N\left[(x^{-1})_{(t(x^{-1}),i(x^{-1}),f(x^{-1}))}\right]=(g(x))^{-1}{}_{(t_1((g(x))^{-1}),i_1((g(x))^{-1}),f_1((g(x))^{-1}))}$

3] If H_N is a two-fold neutrosophic subgroup of G_N, then $g_N(H_N)$ is a two-fold neutrosophic subgroup of K_N.

4] If H_N is abelian, then $g_N(H_N)$ is abelian.

5] If H_N is normal in G_N, then $g_N(H_N)$ is normal in $g_N(G_N)$.

6] $k_{er}(g_N)$ is normal subgroup of G_N.

7] $G_N/k_{er}(g_N) \cong g_N(G_N)$.

Theorem (5):

Let $g_N: G_N \rightarrow K_N$, $h_N: K_N \rightarrow S_N$ be a two-fold neutrosophic homomorphisms, then:

1] $h_N \circ g_N$ is a two-fold neutrosophic homomorphism.

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2] If h_N, g_N are isomorphisms, then $h_N \circ g_N$ is an isomorphism.

Theorem (6):

Let H_N, K_N be two-fold neutrosophic subgroups of G_N, then:

1] $H_N \cap K_N$ is a two-fold neutrosophic subgroup of G_N.

2] If H_N is normal, then $H_N K_N$ is a two-fold neutrosophic subgroup of G_N.

Example:

Take $G = (Z_7{}^*, .) = \{1,2,3,4,5,6\}$ the abelian group of integers modulo 7 under multiplication.

We define:

$t: G \to [0,1]$ such that: $t(x) = 1 \ \ ; x \in G$

$i: G \to [0,1]$ such that: $i(x) = 0 \ \ ; x \in G$

$f: G \to [0,1]$ such that: $f(x) = \frac{1}{2} \ \ ; x \in G$

$G_N = \{(1)_{\left(1,0,\frac{1}{2}\right)}, (2)_{\left(1,0,\frac{1}{2}\right)}, (3)_{\left(1,0,\frac{1}{2}\right)}, (4)_{\left(1,0,\frac{1}{2}\right)}, (5)_{\left(1,0,\frac{1}{2}\right)}, (6)_{\left(1,0,\frac{1}{2}\right)}\}$.

$H = \{1,6\}$ is a subgroup of G, then:

$H_N = \{(1)_{\left(1,0,\frac{1}{2}\right)}, (6)_{\left(1,0,\frac{1}{2}\right)}\}$ is a two-fold neutrosophic subgroup of G_N.

$G/H \cong Z_3$, with $G/H = \{H, 2H, 4H\}$, hence:

$G_N/H_N = \{(1)_{\left(1,0,\frac{1}{2}\right)}H, (2)_{\left(1,0,\frac{1}{2}\right)}H, (4)_{\left(1,0,\frac{1}{2}\right)}H\}$.

It is clear that: $|G_N| = |G| = 6$, $|G_N/H_N| = |G/H| = 3$.

Proof of theorem (1):

1] Let $A = (x)_{(t(x),i(x),f(x))}, B = (y)_{(t(y),i(y),f(y))}, C = (z)_{(t(z),i(z),f(z))} \ ; x, y, z \in G$.

$(A * B) * C = (xy)_{(t(xy),i(xy),f(xy))} * C = (xyz)_{(t(xyz),i(xyz),f(xyz))} = A * (B * C)$.

2] $A * (e)_{(t(e),i(e),f(e))} = (xe)_{(t(xe),i(xe),f(xe))} = (x)_{(t(x),i(x),f(x))} = A = (e)_{(t(e),i(e),f(e))} * A$.

3] Put $D = (x^{-1})_{(t(x^{-1}),i(x^{-1}),f(x^{-1}))}$, then:

$A * D = (e)_{(t(e),i(e),f(e))} = D * A$.

4] $(A * B) = (xy)_{(t(xy),i(xy),f(xy))} = (yx)_{(t(yx),i(yx),f(yx))} = B * A$, thus if G is abelian, then $G_N$is abelian.

5] $t(ex) = t(x) \le \min(t(x), t(e))$, thus:$t(x) \le t(e)$.

$i(ex) = i(x) \ge \max(i(x), i(e))$, thus:$i(x) \ge i(e)$.

$f(ex) = f(x) \ge \max(f(x), f(e))$, thus:$f(x) \ge f(e)$.

Proof of theorem (2):

1] Let $A = (x)_{(t(x),i(x),f(x))}, B = (y)_{(t(y),i(y),f(y))} \in H_N$, then $x, y \in H$ and H is abelian. This means that $xy = yx$ and $A * B = B * A$,thus $H_N$is abelian.

2] Assume that H is normal, and: $\begin{cases} A = (x)_{(t(x),i(x),f(x))} \in H_N \\ B = (y)_{(t(y),i(y),f(y))} \in G_N \end{cases}$ then: $y^{-1}xy \in H$ and $B^{-1}AB = (y^{-1}xy)_{(t(y^{-1}xy),i(y^{-1}xy),f(y^{-1}xy))} \in H_N$, and H_N is normal.

Proof of theorem (3):

Define $\circ: G_N/H_N \times G_N/H_N \to G_N/H_N$ such that:

$(xH)_{(t'(xH),i'(xH),f'(xH))} \circ (yH)_{\left(t'(yH),i'(yH),f'(yH)\right)} = (xyH)_{(t'(xyH),i'(xyH),f'(xyH))}$

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Since G_N is a group (factor group), then G_N/H_N will be a two-fold neutrosophic group under (○).

Proof of theorem (4):

1] Since (g) is a group homomorphism, we get:

$$\begin{cases} g(e) = e_k \\ t_1(g(e)) = t_1(e_k) \\ i_1(g(e)) = i_1(e_k) \\ f_1(g(e)) = f_1(e_k) \end{cases}$$

Hence $g_N\left[(e)_{(t(e),i(e),f(e))}\right] = (e_k)_{(t_1(e),i_1(e),f_1(e))}$.

2] Since $g(x^{-1}) = (g(x))^{-1}$ we get the proof.

3] $g_N(H_N) = (g(H))_N$, $g(H)$ is a subgroup of K, thus $(g(H))_N$ is a subgroup of K_N.

4] It holds directly from (3).

5] It holds directly from (3).

6] $k_{er}(g_N) = \left(k_{er}(g)\right)_N$, $k_{er}(g)$ is a normal subgroup of G, therefor $\left(k_{er}(g)\right)_N$ is normal in G_N.

7] $G/k_{er}(g) \cong g(G)$, thus $G_N/\left(k_{er}(g)\right)_N \cong (g(G))_N$ and $G_N/k_{er}(g_N) \cong g_N(G_N)$.

Proof of theorem (5):

The proof is similar to the classical case.

Proof of theorem (6):

It holds directly by a similar argument of the classical case in group theory.

3. Conclusion

In this paper we defined and studied for the first time the concept of two-fold neutrosophic group by combining two-fold algebras with the classical concept of neutrosophic group. We studied the elementary properties of this novel concept by many related theorems and illustrated examples.

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DOI: https://doi.org/10.54216/JNFS.090103
Received: December 22, 2023 Revised: March 17, 2024 Accepted: July 12, 2024