On the Perfect Italian Domination Numbers of Some Graph Classes

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Abstract

A function $f: V(G) \rightarrow \{0,1,2\}$ is called a Perfect Italian dominating function (PIDF) of a graph $G = (V, E)$ if $\sum_{v \in N(u)} f(v) = 2$ for every vertex $u \in V(G)$ with $f(u) = 0$. The weight of an PIDF is $w(f) = \sum_{v \in V} f(v)$. The minimum weight of all Perfect Italian dominating functions that can be conducted on a graph G is called the perfect Italian domination number of G and is denoted by $\gamma_I^p(G)$. In this paper, we study the problem on different graph classes. We determine the perfect Italian domination numbers of the circulant graphs $C_n\{1,2\}$ for $n \geq 5$ and give upper bounds for $\gamma_I^p(C_n\{1,3\})$ when $n \geq 7$. We also find this parameter for generalized Petersen graph $P(n, 2)$ when $n \geq 5$. We determine $\gamma_I^p(G)$ of strong grids $P_2 \boxtimes P_n$ and $P_3 \boxtimes P_n$ for arbitrary $n \geq 2$, then we introduce an upper bound for $\gamma_I^p(P_m \boxtimes P_n)$ when $m, n \geq 2$ are arbitraries. Finally, we determine the perfect Italian domination number of Jahangir graph $J_{s,m}$ for arbitrary $s \geq 2$ and $m \geq 3$.

Keywords: perfect Italian dominating function; perfect Italian domination number; Circulant graph; generalized Petersen graph; strong grid; Jahangir graph

1. Introduction

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v (denoted by $\deg(v)$) is the number of all vertices that are adjacent to v. Therefore, $\deg(v) = |N(v)|$. The distance $d(u, v)$ between two vertices u and v of a finite graph is the minimum length of the paths connecting them [1]. Let $Y \subseteq V$ and let F be a subset of E such that F consists of all edges of G which have endpoints in Y, then $H = (Y, F)$ is called an induced subgraph of G by Y and is denoted by $G[Y]$. A dominating vertex set of any graph $G = (V, E)$ is a subset $D \subseteq V$ satisfies that each vertex $v \in V - D$ is adjacent to at least one vertex from D. The minimum cardinality of all dominating sets of a graph G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D is a perfect dominating set if it satisfies that for every vertex $v \in V - D$; $|N(v) \cap D| = 1$. The minimum cardinality of all perfect dominating sets of G is the perfect domination number of G (denoted $\gamma_p(G)$). For more information on perfect domination in graphs see [2]. For any undefined term in this paper, [3] is recommended.

In mathematics, it is very common to conduct functions on different mathematical structures in order to achieve a certain optimization, see [4], let f be a function defined on the vertex set V of a graph G, the weight of f is the accumulated weight assigned by f to all vertices of G, i.e., $w(f) = \sum_{v \in V} f(v)$. An Italian dominating function (IDF) of a graph G is a function $f: V(G) \rightarrow \{0,1,2\}$ so that for every vertex $u \in V$, if $f(u) = 0$ then $\sum_{v \in N(u)} f(v) \geq 2$. The Italian domination number of G, denoted by $\gamma_I(G)$, is the minimum weight of all IDFs of G. This parameter was first introduced by Chellali et al. [5]. The problem was studied on trees by Henning et al. [6]. Varghese et al.

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[7] determined the Italian domination number of Sierpiński graphs. Gao et al. determined the Italian domination numbers of generalized Petersen graphs $P(n,1)$, $P(n,2)$ [8] and $P(n,3)$ [9].

Perfect Italian domination is a variation of the Italian domination problem that was introduced by Henning et al. [6]. A perfect Italian dominating function (PIDF) of a graph G is a function $f: V(G) \to \{0,1,2\}$ that satisfies the following condition, for every vertex $u \in V$, if $f(u) = 0$ then $\sum_{v \in N(u)} f(v) = 2$. The minimum weight of all PIDFs that can be conducted on a graph G is called the perfect Italian domination number of G and is denoted by$\gamma_I^p(G)$. Henning et al. [6]. found $\gamma_I^p(G)$ for some simple graphs such as paths and cycles. Varghese et al. [7] determined the perfect Italian domination number of Sierpiński graphs. Lauri et al. [10] studied the problem on planar, regular and split graphs. For more information on perfect Italian domination, see Haynes et al. [11], Banerjee et al. [12] and Pradhan et al. [13]. Varghese et al. [14] studied the relationship between the perfect Italian domination number of the Mycielskian of a graph and the perfect domination number of the graph. They also obtained the perfect Italian domination numbers of the cartesian products $P_2 \square P_n$ and $K_m \square K_n$ for any arbitrary$m, n \geq 1$.

The circulant graph $C_n(S)$ [15] with the connection set $S \subseteq \{1,2,\dots,n\}$ is a simple undirected graph which has the vertex set $V = \{v_1, v_2, \dots, v_n\}$ where subscripts are taken modulo n. Any two vertices $v_i, v_j \in V$ are adjacent if and only if $|i-j| \in S$. Therefore, the circulant graph $C_n\{1,r\}; 2 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$ is 4-regular.

The generalized Petersen graph $P(n,k)$ with $n \geq 1$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ is a 3-regular, simple and undirected graph with the vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and the edge set $E = \{u_i v_i, u_i u_{i+1}, v_i v_{i+k} \ : 1 \leq i \leq n\}$ where subscripts are taken modulo n. It is obvious that $|V(P(n,k))| = 2n$ and $|E(P(n,k))| = 3n$ For more information on the generalized Petersen graph, see [16].

We define the strong product of two paths P_m and P_n (also called a strong grid) as the graph $P_m \boxtimes P_n$ such that $V(P_m \boxtimes P_n) = V(P_m \times P_n) = \{(i,j) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices $(i,i'), (j,j')$ are adjacent if and only if:

- i is adjacent to j and $i' = j'$.
- $i = j$ and i' is adjacent to j'.
- i is adjacent to j and i' is adjacent to j'.

For more information on strong grids, see [17]. The Jahangir graph $J(s,m)$ with $s, m \geq 2$ is a simple undirected graph on $sm + 1$ vertices. It consists of a cycle C_{sm} and a central vertex v_{sm+1} that is adjacent to m vertices of C_{sm} which are $\{v_{1+is} : 1 \leq i \leq m-1\}$. i.e., the distance between every two consecutive vertices of degree 3 on C_{sm} is s. We denote the set $\{v_{1+is} : 1 \leq i \leq m-1\}$ by R. For more information on Jahangir graph, see [18].

In this paper, we determine the perfect Italian domination number of the circulant graph $C_n\{1,2\}$ for $n \geq 5$ and introduce an upper bound for the perfect Italian domination number $C_n\{1,3\}$ when $n \geq 7$. We also find this parameter for generalized Petersen graph $P(n,2)$ when $n \geq 5$. We determine $\gamma_I^p(P_2 \boxtimes P_n)$ and $\gamma_I^p(P_3 \boxtimes P_n)$ for arbitrary $n \geq 2$, then we introduce an upper bound for $\gamma_I^p(P_m \boxtimes P_n)$ when $m, n \geq 2$ are arbitraries. Finally, we determine the perfect Italian domination number of Jahangir graph $J_{s,m}$ for arbitrary $s \geq 2$ and $m \geq 3$.

Proposition 1 [6]: For any graph $G = (V,E)$; $\gamma_I^p(G) \geq \gamma_I(G)$.

Proposition 2 [6]: For any path P_n with $n \geq 2$; $\gamma_I^p(G) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proposition 3 [8]: Let $P(n,2)$ be a generalized Petersen graph with $n \geq 5$, then:

$$\gamma_I(P(n,2)) = \begin{cases} \left\lceil \frac{4n}{5} \right\rceil & if\ n \equiv 0,3,4 (mod\ 5); \\ \left\lceil \frac{4n}{5} \right\rceil + 1 & if\ n \equiv 1,2 (mod\ 5). \end{cases}$$

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Note 1: In the upcoming sections of this paper, we will use the term weight of a vertex v to represent the assigned number to v when conducting a PIDF on the studied graph. We will also use the term imperfect vertex to represent a vertex v with $f(v) = 0$ and $\sum_{u\in N(v)} f(u) \neq 2$.

Note 2: Let $f: V(G) \to \{0,1,2\}$ be a PIDF on $G = (V, E)$ and let $H = (V', E')$ be a subgraph of G, the weight of H in relation to f is the accumulated weight assigned by f to all vertices of H, i.e., $f(H) = \sum_{v\in V'} f(v)$.

Note 3: A proper PIDF $f: V(G) \to \{0,1,2\}$ separates $V(G)$ into three subsets $\{V_0, V_1, V_2\}$ so that for any $l \in \{0,2,1\}$; $V_l = \{v \in V; f(v) = l\}$.

Note 4: In all figures of this paper, we will represent a vertex v by a white circle and the assigned number to v when conducting a function on the studied graph by a small number placed inside the circle. If v is imperfect ($f(v) = 0$ and $\sum_{u\in N(v)} f(u) \neq 2$) then the circle representing v is placed inside a square.

2. Results

2.1. The perfect Italian domination number of the circulant graphs $C_n\{1, 2\}$ and $C_n\{1, 3\}$.

In this sub-section we obtain $\gamma_I^p(C_n\{1,2\})$ for $n \geq 5$. We also obtain $\gamma_I^p(C_n\{1,3\})$ when $n \equiv 0(mod\ 5)$ then we present upper bounds for $\gamma_I^p(C_n\{1,3\})$ when $n \equiv 1,2,3,4(mod\ 5)$ and $n \geq 7$.

Theorem 1. *For* $n \geq 5$, let $C_n\{1,2\}$ be a circulant graph,

$$\gamma_I^p(C_n\{1,2\}) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof: We consider the following cases for n:

Case 1. $n \equiv 0(mod\ 3)$. We begin by dividing $V(C_n\{1,2\})$ into $\frac{n}{3}$ blocks each of which consists of three vertices and we denote them as follows: $B_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}: 1 \leq i \leq \frac{n}{3}$.

Let $f: V(G) \to \{0,1,2\}$ be an arbitrary PIDF conducted on $C_n\{1,2\}$. Let us prove that $\gamma_I^p(C_n\{1,2\}) \geq \frac{n}{3}$ according to f. We assume that $\gamma_I^p(C_n\{1,2\}) \leq \frac{n}{3} - 1$. This means for at least one of $B_x \in \left\{B_i: 1 \leq i \leq \frac{n}{3}\right\}$; $f(B_x) = 0$, which means $f(v_{3x-2}) = f(v_{3x-1}) = f(v_{3x}) = 0$. We notice the following:

- Since $f(v_{3x-2}) = f(v_{3x}) = 0$ then $f(v_{3x-3}) + f(v_{3x+1}) = 2$. Otherwise v_{3x-1} is imperfect and thus $f: V(G) \to \{0,1,2\}$ is not a PIDF which is a contradiction.
- Since $f(v_{3x-1}) = f(v_{3x}) = 0$ then $f(v_{3x-4}) + f(v_{3x-3}) = 2$. Otherwise v_{3x-2} is imperfect.
- Since $f(v_{3x-2}) = f(v_{3x-1}) = 0$ then $f(v_{3x+1}) + f(v_{3x+2}) = 2$ or else v_{3x} is imperfect.

We conclude that $f(v_{3x-3}) + f(v_{3x+1}) + f(v_{3x+1}) + f(v_{3x+2}) = 4$, which means $f(B_{x-1}) + f(B_{x+1}) \geq 4$ if $f(B_x) = 0$. We notice that there are two possible configurations of f values on B_{x-1}, B_x, B_{x+1} which achieve $f(B_{x-1}) + f(B_{x+1}) = 4$ when $f(B_x) = 0$. They are:

Configuration 1: 011-000-110;

Configuration 2: 002-000-200.

Figure 1 shows that for each of these configurations:

- $f(B_{x-2}) \geq 1$ or else v_{3x-6} is imperfect.
- $f(B_{x+2}) \geq 1$ or else v_{3x+4} is imperfect.

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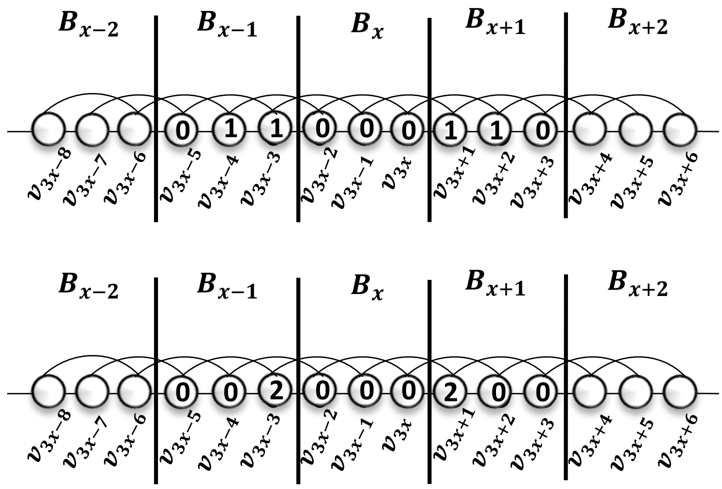

Figure 1. The weight configurations that achieve $f(B_{\mathrm{x}-1})+f(B_{\mathrm{x}+1})=4$ when $f(B_{\mathrm{x}})=0$.

Without loss of generality, we conclude that every three consecutive blocks of $C_{\mathrm{n}}\{1,2\}$ have an accumulated weight larger than 2, this means $\sum_{i=1}^{i=x-2} f(B_{\mathrm{i}})+\sum_{i=x+2}^{i=\frac{n}{3}} f(B_{\mathrm{i}}) \geq \frac{n}{3}-3$. Therefore, $\gamma_I^p(C_{\mathrm{n}}\{1,2\})=\sum_{i=1}^{i=n} f(B_{\mathrm{i}}) \geq \frac{n}{3}-3+f(B_{\mathrm{x}-1})+f(B_{\mathrm{x}})+f(B_{\mathrm{x}+1})$ which means $\gamma_I^p(C_{\mathrm{n}}\{1,2\}) \geq \frac{n}{3}-3+4=\frac{n}{3}+1$ and this is a contradiction to the assumption that $\gamma_I^p(C_{\mathrm{n}}\{1,2\})=\frac{n}{3}-1$, therefore no subgraph of $\{B_{\mathrm{i}}: 1 \leq i \leq \frac{n}{3}\}$ is of accumulated weight 0 if $n \equiv 0(mod\ 3)$ and we establish the lower bound:

$$\gamma_I^p(C_{\mathrm{n}}\{1,2\}) \geq \frac{n}{3} \qquad (1)$$

To prove that $\gamma_I^p(C_{\mathrm{n}}\{1,2\}) \leq \frac{n}{3}$, it is enough to conduct a PIDF of accumulated weight $\frac{n}{3}$ on $C_{\mathrm{n}}\{1,2\}$. Let $f': V \rightarrow \{0,1,2\}$ and for $1 \leq i \leq n$:

$$f'(v_{\mathrm{i}})=\begin{cases}1 \ if\ i \equiv 1(mod\ 3); \\ 0\ otherwise.\end{cases}.$$

We notice that f' is a PIDF on $(C_{\mathrm{n}}\{1,2\})$ and $w(f')=0|V_0|+1|V_1|=0(\frac{2n}{3})+1(\frac{n}{3})=\frac{n}{3}$. Which means if $n \equiv 0(mod\ 3)$;

$$\gamma_I^p(C_{\mathrm{n}}\{1,2\}) \leq \frac{n}{3} \qquad (2)$$

From (1) and (2) we conclude that $\gamma_I^p(C_{\mathrm{n}}\{1,2\})=\frac{n}{3}$ if $n \equiv 0(mod\ 3)$.

Case 2. $n \equiv 1(mod\ 3)$. We use the same segmentation established in Case 1 on $V-\{v_{\mathrm{n}}\}$, which means the $\left\lfloor\frac{n}{3}\right\rfloor=\frac{n-1}{3}$ blocks are defied as $B_{\mathrm{i}}=\{v_{3\mathrm{i}-2}, v_{3\mathrm{i}-1}, v_{3\mathrm{i}}\}: 1 \leq i \leq \frac{n-1}{3}$ while v_{n} forms a mini-block (denoted M) on its own. Let $f: V(G) \rightarrow\{0,1,2\}$ be an arbitrary PIDF on $C_{\mathrm{n}}\{1,2\}$ and let us assume that $\gamma_I^p(C_{\mathrm{n}}\{1,2\})=\frac{n-1}{3}$, by following the same argument of Case 1 we conclude that for $2 \leq i \leq \frac{n-10}{3}$; $f(B_{\mathrm{i}})+f(B_{\mathrm{i}+1})+f(B_{\mathrm{i}+2}) \geq 3$ and

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$$\sum_{i=2}^{i=\frac{n-4}{3}} f(B_i) \geq \frac{n-7}{3} \qquad (3)$$

Now we study the minimum possible weight of $f(B_1) + f(B_{\frac{n-1}{3}}) + f(M)$. It is obvious from our previous argument that $f(B_1) + f(B_{\frac{n-1}{3}}) + f(M) \geq 2$. Let us now assume that $f(B_1) + f(B_{\frac{n-1}{3}}) + f(M) = 2$. There are $\binom{7}{2} = 21$ different configurations to distribute two vertices of V_1 on $B_1 \cup B_{\frac{n-1}{3}} \cup M$. The following table shows twelve of these configurations taking into consideration that each one of the remaining nine configurations is symmetric to one of the first nine configurations listed in Table 1. Each double underlined zero of Table 1 represents an imperfect vertex.

Table 1: Different configurations to distribute two vertices of V_1 on $B_1 \cup B_{\frac{n-1}{3}} \cup M$

Configuration Number	$B_{\frac{n-1}{3}} \cup M \cup B_1$
1	$0\underline{\underline{00}}$-$\underline{\underline{0}}$-011
2	$0\underline{\underline{00}}$-$\underline{\underline{0}}$-101
3	$00\underline{\underline{0}}$-1-001
4	001-$\underline{\underline{0}}$-001
5	$01\underline{\underline{0}}$-$\underline{\underline{0}}$-$\underline{\underline{0}}01$
6	$00\underline{\underline{0}}$-0-110
7	$00\underline{\underline{0}}$-1-010
8	001-0-010
9	000-1-100
10	001-0-100
11	$01\underline{\underline{0}}$-0-$\underline{\underline{0}}10$
12	$10\underline{\underline{0}}$-$\underline{\underline{0}}$-$\underline{\underline{0}}01$

We notice from Table 1 that configurations 8, 9, 10 are the only configurations that do not directly produce imperfect vertices on $B_{\frac{n-1}{3}} \cup M \cup B_1$. We discuss these three configurations as follows:

Configuration 8: 001-0-010.

As shown in Figure 2, when configuration 8 is applied:

- $f(v_{n-4}) = 1$ or else $\sum_{v \in N(v_{n-2})} f(v) = 1$ which means $f(v_{n-5}) = 0$ or else $\sum_{v \in N(v_{n-3})} f(v) = 3$.
- $f(v_4) + f(v_5) = 1$ or else $\sum_{v \in N(v_3)} f(v) = 1, 3, 4, 5$ if $f(v_3) + f(v_4) = 0, 2, 3, 4$ respectively.
- If $f(v_{n-6}) = 1$ then $f(B_{\frac{n-7}{3}}) = 2$. However, $f(B_{\frac{n-10}{3}}) \geq 1$ or else $\sum_{v \in N(v_{n-7})} f(v) = 1$.
- If $f(v_{n-4}) = 1$, $f(v_{n-5}) = f(v_{n-6}) = 0$ then $f(v_{n-7}) = 1$ or else $\sum_{v \in N(v_{n-5})} f(v) = 1$, the same argument applies to $f(v_{n-10})$ and the minimal pattern continues as …-001-001-001. We call this pattern: Left Pattern.
- If $f(v_6) = 1$ then $f(B_2) = 2$.

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- If $f(v_4)=1$, $f(v_5)=f(v_6)=0$ then $f(v_7)=1$ or else $\sum_{v\in N(v_5)} f(v)=1$, the same applies to $f(v_{10})$ and the minimal pattern continues as 100-100-100-…, we call this pattern: Right Pattern 1 and it is illustrated in Figure 2.a.
- If $f(v_5)=1$, $f(v_4)=f(v_6)=0$ then $f(v_8)=1$ or else $\sum_{v\in N(v_6)} f(v)=1$, the same applies to $f(v_{11})$ and the minimal pattern continues as 010-010-010-…, we call this pattern: Right Pattern 2 and it is shown in Figure 2.b.

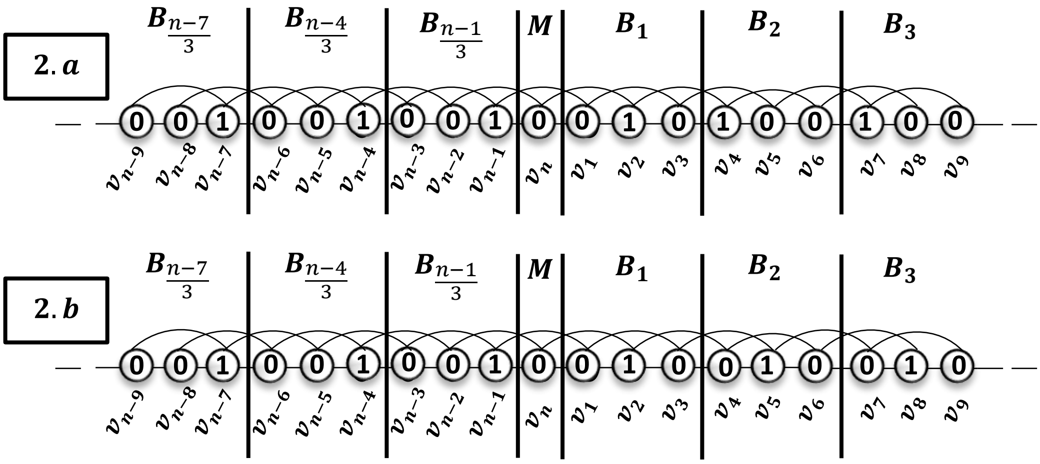

Figure 2. Applying configuration 8 on $B_{\frac{n-1}{3}} \cup M \cup B_1$ when $n \equiv 1 (mod\ 3)$.

In order to achieve minimality, $f(v_6)=f(v_{n-6})=0$, which means we have two patterns applied to $V-\{v_i: 2\le i\le \frac{n-4}{3}\}$ at the same time (a right pattern and the one left pattern). However, in both cases a block of the left pattern must be adjacent to a block of a right pattern. Figure 3.a shows that if $B_i \in$ Right Pattern 1 and $B_{i+1} \in$ Left Pattern then $\sum_{v\in N(v_{3i})} f(v)=\sum_{v\in N(v_{3i+1})} f(v)=1$. Figure 3.b shows that if $B_i \in$ Right Pattern 2 and $B_{i+1} \in$ Left Pattern then $\sum_{v\in N(v_{3i})} f(v)=1$.

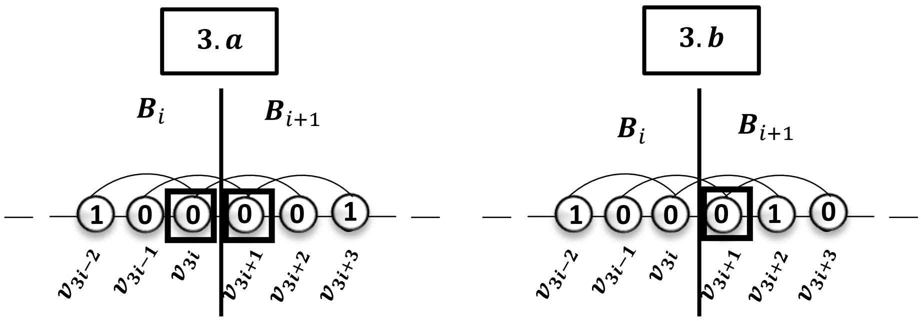

Figure 3. The pattern interactions of configuration 8.

Configuration 9: 000-1-100.

Figure 4 shows that when configuration 9 is applied, $f(v_{n-5})=f(v_{n-4})=1$ or else $\sum_{v\in N(v_{n-3})} f(v)<2$. This means either $f(v_{n-6})=1$ which makes $\sum_{v\in V} v \ge \frac{n-10}{3}+3+2$

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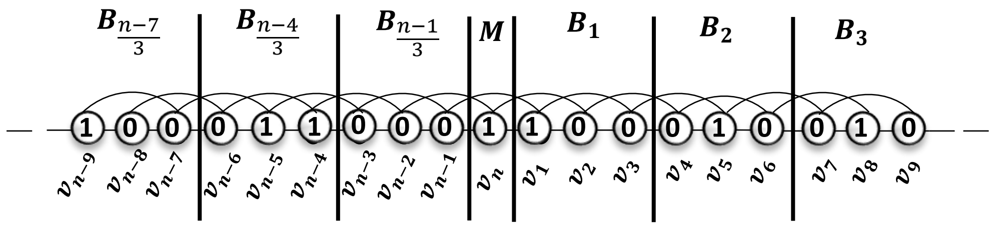

Figure 4. Applying configuration 9 on $B_{\frac{n-1}{3}} \cup M \cup B_1$ when $n \equiv 1(mod\ 3)$.

and that is a contradiction, or $f(v_{n-6}) = 0$ which means $f(v_{n-8}) = f(v_{n-7}) = 0$ or else $\sum_{v\in N(v_{n-6})} f(v) > 2$. Therefore, $f(v_{n-9}) = 1$ and thus the pattern continues as …-100-100 (Left Pattern). In a similar way, $f(v_4) + f(v_5) = 1$ or else $\sum_{v\in N(v_3)} f(v) \neq 2$. However, $f(v_4) = 0$ or else $\sum_{v\in N(v_2)} f(v) = 3$. Therefore, $f(v_5) = 1$ and the pattern continues as 010-010-… (Right Pattern). As Figure 3.b shows, an imperfect vertex (v_{3i}) is obtained when these two patterns meet i.e., when $B_i \in$ Right Pattern and $B_{i+1} \in$ Left Pattern.

Configuration 10: 001-0-100.

As shown in Figure 5 when configuration 10 is applied, $f(v_{n-4}) = 1$ otherwise, $\sum_{v\in N(v_{n-2})} f(v) \neq 2$. The same argument applies to v_{n-4} and the left pattern continues as:

Left Pattern: …-001-001.

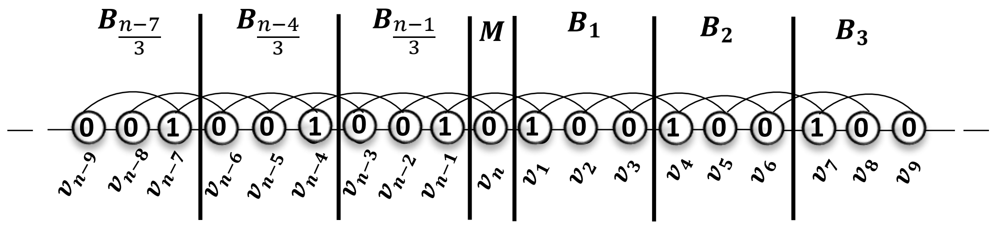

Figure 5. Applying configuration 10 on $B_{\frac{n-1}{3}} \cup M \cup B_1$ when $n \equiv 1(mod\ 3)$.

In a similar way, $f(v_4) = 1$ or else $\sum_{v\in N(v_2)} f(v) \neq 2$ and the right pattern continues as:

Right Pattern: 100-100-…

As shown in Figure 3.a, two imperfect vertices (v_{3i}, v_{3i+1}) are obtained when these two patterns meet i.e., when $B_i \in$ Right Pattern and $B_{i+1} \in$ Left Pattern.

It is obvious that replacing the two vertices of weight 1 by one vertex of weight 2 does not change the outcome of any configuration. From all the above we conclude that $f(B_1) + f(B_{\frac{n-1}{3}}) + f(M) > 2$, which means:

$$f(B_1) + f(B_{\frac{n-1}{3}}) + f(M) \geq 3 \qquad (4)$$

From (3) and (4) we obtain the lower bound $\gamma_I^p(C_n\{1,2\}) \geq \frac{n-7}{3} + 3 = \left\lceil \frac{n}{3} \right\rceil$ if $n \equiv 1(mod\ 3)$.

To prove that $\gamma_I^p(C_n\{1,2\}) \leq \left\lceil \frac{n}{3} \right\rceil$ we conduct a PIDF of weight $\left\lceil \frac{n}{3} \right\rceil$ on $C_n\{1,2\}$. WE choose this PIDF to be $f': V \to \{0,1,2\}$; for $1 \leq i \leq n$: $f'(v_i) = \begin{cases} 1\ if\ i \equiv 1(mod\ 3); \\ 0\ otherwise. \end{cases}$

$w(f') = 0|V_0| + 1|V_1| = 0(\frac{2n-2}{3}) + 1(\frac{n+2}{3}) = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil$. Therefore, $\gamma_I^p(C_n\{1,2\}) \leq \left\lceil \frac{n}{3} \right\rceil$. By comparing the lower and the upper bounds, we prove that $\gamma_I^p(C_n\{1,2\}) = \left\lceil \frac{n}{3} \right\rceil$ if $n \equiv 1(mod\ 3)$.

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Case 3. $n \equiv 2(mod\ 3)$. Let $f: V(G) \to \{0,1,2\}$ be an arbitrary PIDF on $C_n\{1,2\}$ and let us assume that $\gamma_I^p(C_n\{1,2\}) \leq \frac{n-2}{3}$. We use the same segmentation of Case 1 on $V - \{v_{n-1}, v_n\}$, therefore $B_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}: 1 \leq i \leq \frac{n-2}{3}$ and for $2 \leq i \leq \frac{n-11}{3}$; $f(B_i) + f(B_{i+1}) + f(B_{i+2}) \geq 3$ which means:

$$\sum_{i=2}^{i=\frac{n-5}{3}} f(B_i) \geq \frac{n-8}{3} \tag{5}$$

It is obvious that $f(B_1) + f(B_{\frac{n-2}{3}}) + f(M) \geq 2$. By constructing the $\binom{8}{2} = 28$ different possible configurations to distribute two vertices of V_1 on $B_1 \cup B_{\frac{n-1}{3}} \cup M$ which is of cardinality 8, we notice that 26 of them directly produce at least one imperfect vertex. We discuss the two remaining configurations:

Configuration 001-00-100**:** Figure 6.a shows that:

3. $f(v_4) = 1$ or else $\sum_{v \in N(v_2)} f(v) \neq 2$.
4. $f(v_{n-5}) = 1$ or else $\sum_{v \in N(v_{n-3})} f(v) \neq 2$.
5. $f(v_5) = 0$ or else $\sum_{v \in N(v_3)} f(v) \neq 2$.
6. $f(v_{n-6}) = 0$ or else $\sum_{v \in N(v_{n-4})} f(v) \neq 2$.
7. $f(v_6) = f(v_{n-7}) = 0$ or else $\sum_{i=2}^{i=\frac{n-5}{3}} f(B_i) \geq \frac{n-5}{3}$ which makes $\gamma_I^p(C_n\{1,2\}) \geq \frac{n+1}{3}$.
8. $f(v_{10}) = f(v_7) = 1$ and the pattern continues as: 100-100-… (Right Pattern).
9. $f(v_{n-11}) = f(v_{n-8}) = 1$ and the pattern continues as: …-001-001 (Left Pattern).

Configuration 000-11-000**:** Figure 6.b shows that:

10. $f(v_4) + f(v_5) = 2$ or else $\sum_{v \in N(v_2)} f(v) \neq 2$.
11. $f(v_{n-6}) + f(v_{n-5}) = 2$ or else $\sum_{v \in N(v_{n-3})} f(v) \neq 2$.
12. $f(v_9) = f(v_{n-10}) = 1$ or else $\sum_{v \in N(v_7)} f(v) \neq 2$; $\sum_{v \in N(v_{n-8})} f(v) \neq 2$ (respectively).
13. $f(v_8) = f(v_{n-9}) = 0$ or else $\sum_{v \in N(v_6)} f(v) \neq 2$; $\sum_{v \in N(v_{n-7})} f(v) \neq 2$ (respectively).
14. $f(v_{10}) = 1$ or else $\sum_{v \in N(v_8)} f(v) \neq 2$. Similarly, $f(v_{13}) = 1$ and the pattern continues as: 100-100… (Right Pattern).
15. $f(v_{n-11}) = 1$ or else $\sum_{v \in N(v_9)} f(v) \neq 2$. Similarly, $f(v_{n-14}) = 1$ and the pattern continues as: …-001-001 (Left Pattern).

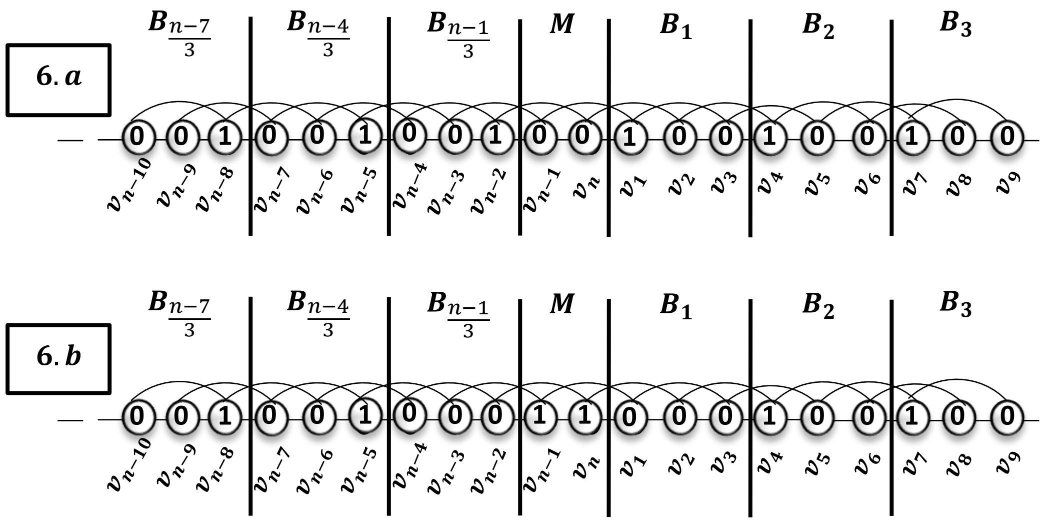

Figure 6. Applying configurations 001-00-100 and 000-11-000 on $B_{\frac{n-1}{3}} \cup M \cup B_1$ when $n \equiv 2(mod\ 3)$.

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For both patterns and as shown in Figure 3.a, if $B_{\rm i} \in$ Right Pattern and $B_{\rm i+1} \in$ Left Pattern then $\sum_{v\in N(v_{3\rm i})} f(v) = \sum_{v\in N(v_{3\rm i+1})} f(v) = 1$.

It is obvious that replacing the two vertices of weight 1 by one vertex of weight 2 does not change the outcome of any configuration. From all the above we conclude that:

$$f(B_1) + f(B_{\frac{n-2}{3}}) + f(M) \geq 3 \tag{6}$$

From (5) and (6) we prove that $\gamma_I^p(C_{\rm n}\{1,2\}) \geq \frac{n-8}{3} + 3 = \left\lceil\frac{n}{3}\right\rceil$ when $n \equiv 2(mod\ 3)$. Now we conduct the function $f': V \to \{0,1\}$; for $1 \leq i \leq n$: $f'(v_{\rm i}) = \begin{cases} 1\ if\ i \equiv 1(mod\ 3); \\ 0\ otherwise. \end{cases}$

Since f' is a PIDF of weight $\sum_{v\in V} f'(v) = w(f') = 0|V_0| + 1|V_1| = 0(\frac{2n-4}{3}) + 1(\frac{n+1}{3}) = \frac{n+1}{3} = \left\lceil\frac{n}{3}\right\rceil$ on $C_{\rm n}\{1,2\}$ we conclude that $\gamma_I^p(C_{\rm n}\{1,2\}) \leq \left\lceil\frac{n}{3}\right\rceil$ when $n \equiv 2(mod\ 3)$. Therefore $\gamma_I^p(C_{\rm n}\{1,2\}) = \left\lceil\frac{n}{3}\right\rceil$ when $n \equiv 2(mod\ 3)$. From all the previous cases we obtain that for $n \geq 5$; $\gamma_I^p(C_{\rm n}\{1,2\}) = \left\lceil\frac{n}{3}\right\rceil$.■

Theorem 2. *For* $n \geq 7$, let $C_{\rm n}\{1,3\}$ be a circulant graph,

i. $\gamma_I^p(C_{\rm n}\{1,3\}) = \frac{2n}{5}\ \ if\ n \equiv 0(mod\ 5)$.

ii. $\gamma_I^p(C_{\rm n}\{1,3\}) \leq \begin{cases} 2\left\lfloor\frac{n}{5}\right\rfloor + 1\ \ if\ n \equiv 1(mod\ 5); \\ 2\left\lfloor\frac{n}{5}\right\rfloor + 2\ \ if\ n \equiv 2,3,4(mod\ 5). \end{cases}$

Proof: We consider the following cases for n:

Case 1. $n \equiv 0(mod\ 5)$. We start by dividing $V(C_{\rm n}\{1,3\})$ into $\frac{n}{5}$ blocks each of which consists of five vertices and we denote them as $B_{\rm i} = \{v_{5\rm i-4}, v_{5\rm i-3}, v_{5\rm i-2}, v_{5\rm i-1}, v_{5\rm i}\}$: $1 \leq i \leq \frac{n}{5}$. Now Let $f: V(G) \to \{0,1,2\}$ be an arbitrary PIDF conducted on $C_{\rm n}\{1,3\}$. First, we will establish the lower bound $\gamma_I^p(C_{\rm n}\{1,3\}) \geq \frac{2n}{5}$ according to f. We assume that $\gamma_I^p(C_{\rm n}\{1,3\}) \leq \frac{2n}{5} - 1$. This means for at least one $B_{\rm x} \in \left\{B_{\rm i}: 1 \leq i \leq \frac{n}{5}\right\}$; $f(B_{\rm x}) = 1$. We discuss all the possible configurations that include exactly one vertex of V_1 on $B_{\rm x}$:

Configuration 00001**:** Figure 7.a demonstrates this configuration, we notice that:

- $f(v_{5\rm x+1}) + f(v_{5\rm x+2}) + f(v_{5\rm x-5}) = 3$. i.e., $f(v_{5\rm x+1}) = f(v_{5\rm x+2}) = f(v_{5\rm x-5}) = 1$. Otherwise $v_{5\rm x-2}$, $v_{5\rm x-1}$ are imperfect.
- $f(v_{5\rm x-6}) = 1$ or else $v_{5\rm x-3}$ is imperfect.
- $f(v_{5\rm x-7}) = 1$ or else $v_{5\rm x-4}$ is imperfect.
- If $f(v_{5\rm x+3}) = f(v_{5\rm x+4}) = f(v_{5\rm x+5}) = 0$ then $f(B_{\rm x+2}) \geq 2$. Otherwise, at least one vertex of $v_{5\rm x+4}$, $v_{5\rm x+5}$ is imperfect.
- $f(B_{\rm x-2}) \geq 2$. Otherwise, at least one vertex of $v_{5\rm x-11}, v_{5\rm x-10}, v_{5\rm x-9}$ is imperfect.

We conclude that $f(B_{\rm x-2}) + f(B_{\rm x-1}) + f(B_{\rm x}) + f(B_{\rm x+1}) + f(B_{\rm x+2}) \geq 2 + 3 + 1 + 3 + 2 = 11$ if $B_{\rm x}$ follows configuration 00001.

Configuration 00010**:** Figure 7.b demonstrates this configuration. We observe that:

- $f(v_{5\rm x-6}) = 2$. Otherwise, $v_{5\rm x-3}$ is imperfect.
- $f(v_{5\rm x-5}) + f(v_{5\rm x+1}) + f(v_{5\rm x+3}) = 2$. Otherwise, at least one of $v_{5\rm x-2}$,$v_{5\rm x}$ is imperfect.
- $f(v_{5\rm x-7}) + f(v_{5\rm x-5}) = 1$. Otherwise, $v_{5\rm x-4}$ is imperfect.
- If $f(v_{5\rm x-5}) = 0$ then $f(v_{5\rm x-9}) = f(v_{5\rm x-8}) = f(v_{5\rm x-7}) = 0$ or else $f(B_{\rm x-1}) \geq 4$. This means $f(v_{5\rm x-8}) = f(v_{5\rm x-10}) = 0$ or else $v_{5\rm x-7}$ is imperfect. It can also be concluded that $f(v_{5\rm x-12}) = 0$ or else $v_{5\rm x-9}$ is imperfect. This means $f(v_{5\rm x-13}) + f(v_{5\rm x-11}) \geq 2$ or else $v_{5\rm x-10}$ is imperfect. We conclude that $f(B_{\rm x-2}) \geq 2$. A similar argument can be made if $f(v_{5\rm x-7}) = 0$.

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- $f(v_{5x+1}) + f(v_{5x+3}) = 1$ or else v_{5x} is imperfect. Let us assume that $f(B_{x+1}) = 1$, then v_{5x+1} is imperfect if $f(v_{5x+1}) = 0$ and $f(v_{5x+3}) = 1$. Let us assume that $B_{x+1} = 10000$, then $f(v_{5x+6}) + f(v_{5x+7}) + f(v_{5x+8}) \geq 3$ with the assigned weights 1-1-1. This means $f(B_{x+2}) \geq 3$.

We conclude that $f(B_{x-2}) + f(B_{x-1}) + f(B_x) + f(B_{x+1}) + f(B_{x+2}) \geq 2 + 3 + 1 + 1 + 3 + 2 = 10$ if B_x follows configuration 00010.

Configuration 01000: is symmetric to configuration 00010.

Configuration 10000: is symmetric to configuration 00001.

Configuration 00100: demonstrated in Figure 7.c. We notice that:

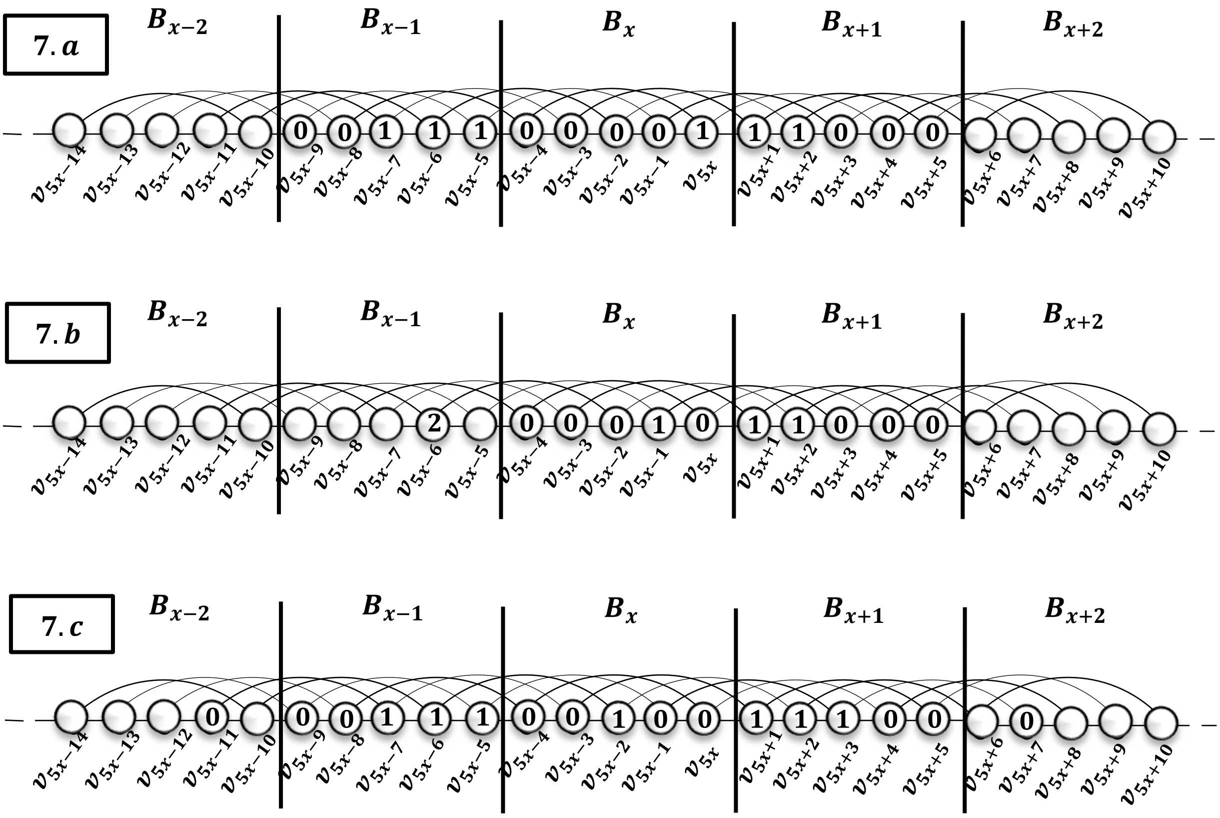

Figure 7. All different configurations that include exactly one vertex of V_1 on B_x.

- $f(v_{5x-6}) = f(v_{5x+2}) = 1$. Otherwise, v_{5x-3}, v_{5x-1} are imperfect (respectively).
- $f(v_{5x-7}) + f(v_{5x-5}) = 2$. Otherwise, v_{5x-4} is imperfect.
- $f(v_{5x+1}) + f(v_{5x+3}) = 2$. Otherwise, v_{5x} is imperfect.
- $f(v_{5x-9}) = f(v_{5x-8}) = 0$. Otherwise, $f(B_{x-1}) \geq 4$. This means: $f(v_{5x-11}) = 0$ or else v_{5x-8} is imperfect, $f(v_{5x-12}) + f(v_{5x-10}) = 1$ or else v_{5x-9} is imperfect.
  If $f(v_{5x-10}) = 1$ and $f(v_{5x-12}) = 0$ then $f(v_{5x-14}) = 1$ or else v_{5x-11} is imperfect.

  If $f(v_{5x-10}) = 0$ and $f(v_{5x-12}) = 1$ then $f(v_{5x-14}) = 1$ or else v_{5x-11} is imperfect.

  We conclude that $f(B_{x-2}) \geq 2$ when $f(v_{5x-9}) = f(v_{5x-8}) = 0$.

- $f(v_{5x+4}) = f(v_{5x+5}) = 0$. Otherwise, $f(B_{x+1}) \geq 4$. This means: $f(v_{5x+7}) = 0$ or else v_{5x+4} is imperfect, $f(v_{5x+6}) + f(v_{5x+8}) = 1$ or else v_{5x+5} is imperfect.
  If $f(v_{5x+6}) = 1$ and $f(v_{5x+8}) = 0$ then $f(v_{5x+10}) = 1$ or else v_{5x+7} is imperfect.

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If $f(v_{5x+6}) = 0$ and $f(v_{5x+8}) = 0$ then $f(v_{5x+10}) = 1$ or else v_{5x+7} is imperfect.
Therefore, $f(B_{x+2}) \geq 2$ when $f(v_{5x+4}) = f(v_{5x+5}) = 0$.

We conclude that $f(B_{x-2}) + f(B_{x-1}) + f(B_x) + f(B_{x+1}) + f(B_{x+2}) \geq 2 + 3 + 1 + 3 + 2 = 11$ if B_x follows configuration 00100. From all the above and without loss of generality, we observe that the weight of any five consecutive blocks is at least 10. Therefore, $\gamma_I^p(C_n\{1,3\}) = \sum_{i=1}^{i=\frac{n}{5}} f(B_i) \geq \frac{2n}{5}$ if $n \equiv 0(mod\ 5)$. Now we establish the lower bound by conducting a PIDF of weight $\frac{2n}{5}$ on $C_n\{1,3\}$. Let this PIDF be $f': V \to \{0,1,2\}$; for $1 \leq i \leq n$ then $f'(v_i) = \begin{cases} 1\ if\ i \equiv 0,1(mod\ 5); \\ 0\ otherwise. \end{cases}$

$$w(f') = \sum_{v \in V} f'(v) = 0|V_0| + 1|V_1| = 0\left(\frac{3n}{5}\right) + 1\left(\frac{2n}{5}\right) = \frac{2n}{5}.$$

Therefore, $\gamma_I^p(C_n\{1,3\}) \leq \frac{2n}{5}$ if $n \equiv 0(mod\ 5)$. By comparing the lower and the upper bounds we conclude that $\gamma_I^p(C_n\{1,3\}) = \frac{2n}{5}$ if $n \equiv 0(mod\ 5)$.

For the remaining cases, we also establish the upper bounds for $\gamma_I^p(C_n\{1,3\})$ by conducting a PIDF (denoted f') of weight $\gamma_I^p(C_n\{1,3\})$ for each case.

Case 2. $n \equiv 1(mod\ 5)$. The PIDF (f') is defined as:

$f': V \to \{0,1,2\}$; for $1 \leq i \leq n$ then $f'(v_i) = \begin{cases} 1\ if\ i \equiv 0,1(mod\ 5); \\ 0\ otherwise. \end{cases}$

$$w(f') = \sum_{v \in V} f'(v) = 0|V_0| + 1|V_1| = 0\left(\frac{3(n-1)}{5}\right) + 1\left(\left(\frac{2(n-1)}{5}\right) + 1\right) = 2\left\lfloor\frac{n}{5}\right\rfloor + 1.$$

We conclude that $\gamma_I^p(C_n\{1,3\}) \leq 2\left\lfloor\frac{n}{5}\right\rfloor + 1$ if $n \equiv 1(mod\ 5)$.

Case 3. $n \equiv 2(mod\ 5)$. The PIDF is:

$f': V \to \{0,1,2\}$; for $1 \leq i \leq n$ then $f'(v_i) = \begin{cases} 1\ if\ i \equiv 0,1(mod\ 5)\ \ or\ \ i = n; \\ 0\ \ otherwise. \end{cases}$

$$w(f') = 0|V_0| + 1|V_1| = 0\left(\frac{3(n-2)}{5}\right) + 1\left(\left(\frac{2(n-2)}{5}\right) + 2\right) = 2\left\lfloor\frac{n}{5}\right\rfloor + 2.$$

This means $\gamma_I^p(C_n\{1,3\}) \leq 2\left\lfloor\frac{n}{5}\right\rfloor + 2$ if $n \equiv 2(mod\ 5)$.

Case 4. $n \equiv 3(mod\ 5)$. We conduct the following PIDF on $C_n\{1,3\}$:

$f': V \to \{0,1,2\}$; for $1 \leq i \leq n$ then $f'(v_i) = \begin{cases} 1\ if\ i \equiv 0,1(mod\ 5)\ \ or\ \ i = n; \\ 0\ \ otherwise. \end{cases}$

$$w(f') = 0|V_0| + 1|V_1| = 0\left(\frac{3(n-3)}{5}\right) + 1\left(\left(\frac{2(n-3)}{5}\right) + 2\right) = 2\left\lfloor\frac{n}{5}\right\rfloor + 2.$$

Therefore $\gamma_I^p(C_n\{1,3\}) \leq 2\left\lfloor\frac{n}{5}\right\rfloor + 2$ if $n \equiv 3(mod\ 5)$.

Case 5. $n \equiv 4(mod\ 5)$. Let the PIDF be:

$f': V \to \{0,1,2\}$; for $1 \leq i \leq n$ then $f'(v_i) = \begin{cases} 1\ if\ i \equiv 0,1(mod\ 5)\ \ or\ \ i = n; \\ 0\ \ otherwise. \end{cases}$

$$w(f') = 0|V_0| + 1|V_1| = 0\left(\frac{3(n-4)}{5}\right) + 1\left(\left(\frac{2(n-4)}{5}\right) + 2\right) = 2\left\lfloor\frac{n}{5}\right\rfloor + 2.$$

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This means $\gamma_I^p(C_n\{1,3\}) \le 2\left\lfloor\frac{n}{5}\right\rfloor + 2$ if $n \equiv 4(mod\ 5)$.

From all five cases we conclude the requested.■

2.2. The perfect Italian domination number of generalized Petersen graph $P(n,2)$.

In this section, we determine $\gamma_I^p(P(n,2))$ for $n \ge 5$.

Theorem 3. For $n \ge 7$, let $P(n,2)$ be a generalized Petersen graph,

$$\gamma_I^p(P(n,2)) = \begin{cases} \left\lceil\frac{4n}{5}\right\rceil & if\ n \equiv 0,3,4(mod\ 5); \\ \left\lceil\frac{4n}{5}\right\rceil + 1 & if\ n \equiv 1,2(mod\ 5). \end{cases}$$

Proof: As an immediate consequence of Proposition 1 and Proposition 3, we obtain the lower bound:

$$\gamma_I^p(P(n,2)) \ge \gamma_I(P(n,2)) = \begin{cases} \left\lceil\frac{4n}{5}\right\rceil & if\ n \equiv 0,3,4(mod\ 5); \\ \left\lceil\frac{4n}{5}\right\rceil + 1 & if\ n \equiv 1,2(mod\ 5). \end{cases}$$

Now we establish the upper bound $\gamma_I^p(P(n,2)) \le \gamma_I(P(n,2))$ by conducting a PIDF of weight $\gamma_I(P(n,2))$ on $P(n,2)$.

Case 1. $n \equiv 0(mod\ 5)$. Let $f': V \to \{0,1,2\}$; for $1 \le i \le n$:

$$f'(u_i) = \begin{cases} 0\ if\ i \equiv 1,3,4(mod\ 5); \\ 1\ if\ i \equiv 0,2(mod\ 5). \end{cases}$$

$$f'(v_i) = \begin{cases} 0\ if\ i \equiv 0,1,2(mod\ 5); \\ 1\ if\ i \equiv 3,4(mod\ 5). \end{cases}$$

We notice that f' is a PIDF of weight:

$$w(f') = 0|V_0| + 1|V_1| = 0\left(\frac{3n}{5} + \frac{3n}{5}\right) + 1\left(\frac{2n}{5} + \frac{2n}{5}\right) = \frac{4n}{5} = \left\lceil\frac{4n}{5}\right\rceil.$$

This means $\gamma_I^p(P(n,2)) \le \left\lceil\frac{4n}{5}\right\rceil$ if $n \equiv 0(mod\ 5)$.

Case 2. $n \equiv 1(mod\ 5)$. We choose $f': V \to \{0,1,2\}$; for $1 \le i \le n$:

$$f'(u_i) = \begin{cases} 0\ if\ i \equiv 1,3,4(mod\ 5)\ and\ i \ne n; \\ 1\ if\ i \equiv 0,2(mod\ 5)\ or\ i = n. \end{cases}$$

$$f'(v_i) = \begin{cases} 0\ if\ i \equiv 0,1,2(mod\ 5) and\ i \ne n-1; \\ 1\ if\ i \equiv 3,4(mod\ 5)\ or\ i = n-1. \end{cases}$$

f' is a PIDF of weight:

$w(f') = 0|V_0| + 1|V_1|$

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$$= 0\left(\frac{3(n-6)}{5} + \frac{3(n-6)}{5} + 6\right) + 1\left(\frac{2(n-6)}{5} + \frac{2(n-6)}{5} + 6\right) = \frac{4n+6}{5} = \left\lceil\frac{4n}{5}\right\rceil + 1.$$

Therefore, $\gamma_I^p(P(n,2)) \leq \left\lceil\frac{4n}{5}\right\rceil + 1$ if $n \equiv 1(mod\ 5)$.

Case 3. $n \equiv 2(mod\ 5)$. The chosen PIDF is $f': V \to \{0,1,2\}$; for $1 \leq i \leq n$:

$$f'(u_i) = \begin{cases} 0\ if\ i \equiv 1,3,4(mod\ 5)\ and\ i \neq n-1; \\ 1\ if\ i \equiv 0,2(mod\ 5)\ or\ i = n-1. \end{cases}$$

$$f'(v_i) = \begin{cases} 0\ if\ i \equiv 0,1,2(mod\ 5) and\ i \neq n-2; \\ 1\ if\ i \equiv 3,4(mod\ 5)\ or\ i = n-2. \end{cases}$$

$w(f') = 0|V_0| + 1|V_1|$

$$= 0\left(\frac{3(n-7)}{5} + \frac{3(n-7)}{5} + 7\right) + 1\left(\frac{2(n-7)}{5} + \frac{2(n-7)}{5} + 7\right) = \frac{4n+7}{5} = \left\lceil\frac{4n}{5}\right\rceil + 1.$$

This means $\gamma_I^p(P(n,2)) \leq \left\lceil\frac{4n}{5}\right\rceil + 1$ if $n \equiv 2(mod\ 5)$.

Case 4. $n \equiv 3(mod\ 5)$. The PIDF is $f': V \to \{0,1,2\}$; for $1 \leq i \leq n-3$:

$$f'(u_i) = \begin{cases} 0\ if\ i \equiv 1,3,4(mod\ 5); \\ 1\ if\ i \equiv 0,2(mod\ 5)\ . \end{cases}$$

$$f'(v_i) = \begin{cases} 0\ if\ i \equiv 0,1,2(mod\ 5); \\ 1\ if\ i \equiv 3,4(mod\ 5). \end{cases}$$

$$f'(u_{n-2}) = f'(u_{n-1}) = f'(v_n) = 0;$$

$$f'(u_n) = f'(u_{n-2}) = f'(u_{n-1}) = 0;$$

$w(f') = 0|V_0| + 1|V_1|$

$$= 0\left(\frac{3(n-3)}{5} + \frac{3(n-3)}{5} + 3\right) + 1\left(\frac{2(n-3)}{5} + \frac{2(n-3)}{5} + 3\right) = \frac{4n+3}{5} = \left\lceil\frac{4n}{5}\right\rceil.$$

Therefore, $\gamma_I^p(P(n,2)) \leq \left\lceil\frac{4n}{5}\right\rceil$ if $n \equiv 3(mod\ 5)$.

Case 5. $n \equiv 4(mod\ 5)$. The PIDF is $f': V \to \{0,1,2\}$; for $1 \leq i \leq n-4$:

$$f'(u_i) = \begin{cases} 0\ if\ i \equiv 1,3,4(mod\ 5); \\ 1\ if\ i \equiv 0,2(mod\ 5)\ . \end{cases}$$

$$f'(v_i) = \begin{cases} 0\ if\ i \equiv 0,1,2(mod\ 5); \\ 1\ if\ i \equiv 3,4(mod\ 5). \end{cases}$$

$$f'(u_{n-2}) = f'(u_{n-1}) = f'(v_{n-3}) = f'(v_n) = 0;$$

$$f'(u_{n-3}) = f'(u_n) = f'(v_{n-2}) = f'(v_{n-1}) = 1;$$

$w(f') = 0|V_0| + 1|V_1|$

$$= 0\left(\frac{3(n-4)}{5} + \frac{3(n-4)}{5} + 4\right) + 1\left(\frac{2(n-4)}{5} + \frac{2(n-4)}{5} + 4\right) = \frac{4n+4}{5} = \left\lceil\frac{4n}{5}\right\rceil.$$

Which means $\gamma_I^p(P(n,2)) \leq \left\lceil\frac{4n}{5}\right\rceil$ if $n \equiv 4(mod\ 5)$.

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From all the previous cases we establish the lower bound $\gamma_I^p(P(n,2)) \le \gamma_I(P(n,2))$ and by comparing the upper and lower bounds we conclude that for $n \ge 7$:

$$\gamma_I^p(P(n,2)) = \gamma_I(P(n,2)) = \begin{cases} \left\lceil \frac{4n}{5} \right\rceil \; if \; n \equiv 0,3,4(mod\ 5); \\ \left\lceil \frac{4n}{5} \right\rceil + 1 \; if \; n \equiv 1,2(mod\ 5). \end{cases} \blacksquare$$

2.3. The perfect Italian domination numbers of Strong grids $P_m \boxtimes P_n$.

In this section we determine $\gamma_I^p(P_2 \boxtimes P_n)$ and $\gamma_I^p(P_3 \boxtimes P_n)$ for arbitrary $n \ge 2$, then we introduce an upper bound for $\gamma_I^p(P_m \boxtimes P_n)$ when $m, n \ge 2$ are arbitraries.

Note 6: We denote the rows of $P_m \boxtimes P_n$ by $R_i: 1 \le i \le m$. We also denote the columns by $CO_j: 1 \le j \le n$ and we denote the vertex of row i and column j by (i,j).

Theorem 4. For $n \ge 2$, let $P_2 \boxtimes P_n$ be a strong grid graph;

$$\gamma_I^p(P_2 \boxtimes P_n) = 2\left\lceil \frac{n}{3} \right\rceil.$$

Proof: We consider the following cases for n:

Case 1. $n \equiv 0(mod\ 3)$. We divide $P_2 \boxtimes P_n$ into $\frac{n}{3}$ blocks each of which consists of six vertices. We denote these blocks by $B_j: 1 \le j \le \frac{n}{3}$ so that:

$$B_j = \{(1,3j-2), (1,3j-1), (1,3j), (2,3j-2), (2,3j-1), (2,3j)\}$$

Let $f: V \to \{0,1,2\}$ be an arbitrary PIDF conducted on $P_2 \boxtimes P_n$. For any $B_j: 1 \le j \le \frac{n}{3}$. We notice that:

- If $f(1,3j-1) = 0$ then $f(B_j) = 2$. Otherwise, $(1,3j-1)$ is imperfect. The same argument can be applied if $f(2,3j-1) = 0$.
- If $f(1,3j-1) + f(2,3j-1) \ge 2$ it is obvious that $f(B_j) \ge 2$.

We conclude that $f(B_j) \ge 2$ for any $1 \le j \le \frac{n}{3}$. This means $\sum_{v \in V} f(v) = \sum_{j=1}^{j=\frac{n}{3}} f(B_j) \ge 2\left(\frac{n}{3}\right) = \frac{2n}{3}$. Therefore:

$$\gamma_I^p(P_2 \boxtimes P_n) \ge \frac{2n}{3} \quad (7)$$

Let $M_1 = \{(2,3j-1): 1 \le j \le \frac{n}{3}\}$, let $f': V \to \{0,1,2\}$ so that for $1 \le i \le 2; 1 \le j \le n$:

$$f'((i,j)) = \begin{cases} 2 \; if \; (i,j) \in M_1; \\ 0 \; otherwise. \end{cases}$$

We notice that f' is a PIDF on $(P_2 \boxtimes P_n)$ and $w(f') = 0|V_0| + 2|V_2| = 0\left(\frac{5n}{3}\right) + 2\left(\frac{n}{3}\right) = \frac{2n}{3}$. We conclude that $\gamma_I^p(P_2 \boxtimes P_n) \le \frac{2n}{3}$. Figure 8 illustrates that $\gamma_I^p(P_2 \boxtimes P_{12}) \le 8$. From (7) we obtain $\gamma_I^p(P_2 \boxtimes P_n) = \frac{2n}{3} = 2\left\lceil \frac{n}{3} \right\rceil$ if $n \equiv 0(mod\ 3)$.

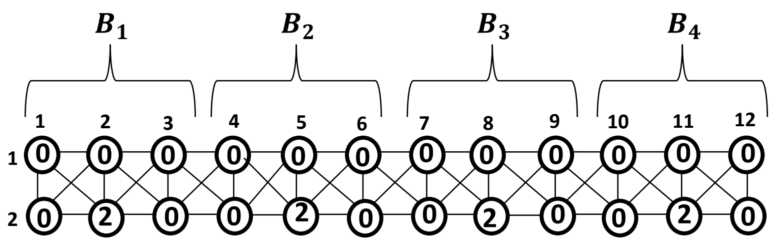

Figure 8. $\gamma_I^p(P_2 \boxtimes P_{12}) \le 8$.

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Case 2. $n \equiv 1(mod\ 3)$. We divide the graph induced by $\{(i,j): 1 \le i \le 2; 3 \le j \le n-2\}$ into $\frac{n-4}{3}$ blocks denoted by $B_j: 1 \le j \le \frac{n-4}{3}$ so that:

$$B_j = \{(1,3j), (1,3j+1), (1,3j+2), (2,3j), (2,3j+1), (2,3j+2)\}$$

The remining eight vertices of $P_2 \boxtimes P_n$ form the two mini-blocks:

$A_1 = \{(1,1), (1,2), (2,1), (2,2)\}$; $A_2 = \{(1,n-1), (1,n), (2,n-1), (2,n-1)\}$. Now let $f: V \to \{0,1,2\}$ be an arbitrary PIDF conducted on $P_2 \boxtimes P_n$. We can directly conclude from Case 1 that $\sum_{j=1}^{j=\frac{n-4}{3}} f(B_j) \ge 2\left(\frac{n-4}{3}\right)$, which means:

$$\gamma_I^p(P_2 \boxtimes P_n) \ge 2\left(\frac{n-4}{3}\right) + f(A_1) + f(A_2) \tag{8}$$

We notice that:

- If $f((1,1)) = f((2,1)) = 0$ then $f((1,2)) + f((2,2)) = 2$. Otherwise, (1,1) and (2,1) are both imperfect.
- If $f((1,1)) = 0$ and $f((2,1)) = 1$, then $f((1,2)) + f((2,2)) = 1$ or else (1,1) is imperfect. The same applies if $f((1,1)) = 1$ and $f((2,1)) = 0$.
- If $f((1,1)) + f((2,1)) \ge 2$ then obviously $f(A_1) \ge 2$.

We conclude that $f(A_1) \ge 2$ and the same argument applies to A_2, i.e., $f(A_2) \ge 2$. Therefore and from (8), we obtain the lower bound $\gamma_I^p(P_2 \boxtimes P_n) \ge 2\left(\frac{n-4}{3}\right) + 2 + 2 = 2\left(\frac{n+2}{3}\right) = 2\left\lceil\frac{n}{3}\right\rceil$.

Let $M_2 = \{(2,3j+1): 0 \le j \le \frac{n-1}{3}\}$, we define the following PIDF $f': V \to \{0,1,2\}$ so that for $1 \le i \le 2; 1 \le j \le n$:

$$f'((i,j)) = \begin{cases} 2 \ if\ (i,j) \in M_2; \\ 0\ otherwise. \end{cases}$$

$$w(f') = 0\left(5\left(\frac{n-4}{3}\right) + 3 + 3\right) + 2\left(\left(\frac{n-4}{3}\right) + 1 + 1\right) = 2\left(\frac{n+2}{3}\right) = 2\left\lceil\frac{n}{3}\right\rceil.$$

Therefore, $\gamma_I^p(P_2 \boxtimes P_n) \le 2\left\lceil\frac{n}{3}\right\rceil$. Figure 9 shows that $\gamma_I^p(P_2 \boxtimes P_{10}) \le 8$. By comparing the lower and the upper bounds we prove that $\gamma_I^p(P_2 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$ if $n \equiv 1(mod\ 3)$.

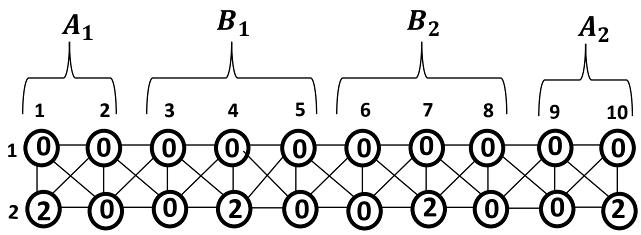

Figure 9. $\gamma_I^p(P_2 \boxtimes P_{10}) \le 8$.

Case 3. $n \equiv 2(mod\ 3)$. We divide the graph induced by $\{(i,j): 1 \le i \le 2; 1 \le j \le n-2\}$ into $\frac{n-2}{3}$ blocks denoted by $B_j: 1 \le j \le \frac{n-2}{3}$ so that:

$$B_j = \{(1,3j-2), (1,3j-1), (1,3j), (2,3j-2), (2,3j-1), (2,3j)\}$$

the remaining four vertices of $P_2 \boxtimes P_n$ form the mini-block $A_2 = \{(1,n-1), (1,n),$

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$(2, n-1), (2, n)\}$. From Case 1 we immediately conclude that $\sum_{j=1}^{j=\frac{n-2}{3}} f(B_j) \geq 2(\frac{n-2}{3})$, we also found in Case 2 that $f(A_2) \geq 2$. Therefore $\gamma_I^p(P_2 \boxtimes P_n) \geq 2\left(\frac{n-2}{3}\right) + 2 = 2\left(\frac{n+1}{3}\right) = 2\left\lceil\frac{n}{3}\right\rceil$. Let $M_3 = \{(2, 3j-1): 1 \leq j \leq \frac{n+1}{3}\}$. We define the following PIDF (denoted f') so that for $1 \leq i \leq 2; 1 \leq j \leq n$:

$$f'((i,j)) = \begin{cases} 2 \; if \; (i,j) \in M_3; \\ 0 \; otherwise. \end{cases}$$

$$w(f') = 0\left(5\left(\frac{n-2}{3}\right) + 3\right) + 2\left(\left(\frac{n-2}{3}\right) + 1\right) = 2\left(\frac{n+1}{3}\right) = 2\left\lceil\frac{n}{3}\right\rceil.$$

This means $\gamma_I^p(P_2 \boxtimes P_n) \leq 2\left\lceil\frac{n}{3}\right\rceil$ and thus $\gamma_I^p(P_2 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$ if $n \equiv 2(mod\ 3)$. From all cases we prove the requested.■

Theorem 5. For $n \geq 2$, let $P_3 \boxtimes P_n$ be a strong grid graph;

$$\gamma_I^p(P_3 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$$

Proof:

Case $n \equiv 0(mod\ 3)$. The blocks $B_j: 1 \leq j \leq \frac{n}{3}$ are defined as $B_j = \{(1,3j-2),$
$(1,3j-1), (1,3j), (2,3j-2), (2,3j-1), (2,3j), (3,3j-2), (3,3j-1), (3,3j)\}$. For every B_j we notice that:

- If $f\big((2,3j-1)\big) = 0$ then $f\big(B_j\big) = 2$ or else $(2,3j-1)$ is imperfect.
- If $f\big((2,3j-1)\big) = 1$ and $f\big((1,3j-1)\big) = 0$ then $f\big((1,3j-2)\big) + f\big((1,3j)\big) + f\big((2,3j-2)\big) + f\big((2,3j)\big) = 1$ or else $(1,3j-1)$ is imperfect.
- If $f\big((2,3j-1)\big) = 1$ and $f\big((3,3j-1)\big) = 0$ then $f\big((3,3j-2)\big) + f\big((3,3j)\big) + f\big((3,3j-2)\big) + f\big((3,3j)\big) = 1$ or else $(3,3j-1)$ is imperfect.
- If $f\big((2,3j-1)\big) = 2$ then $f\big(B_j\big) \geq 2$.

Therefore, $f\big(B_j\big) \geq 2$ for any $1 \leq j \leq \frac{n}{3}$. The rest of this proof is very similar to the proof of Theorem 4. We will only mention the main sets and functions for each case taking into consideration that the proof is exactly the same as Theorem 4:

Case $n \equiv 0(mod\ 3)$:

- $M_1 = \{(2,3j-1): 1 \leq j \leq \frac{n}{3}\}$.
- PIDF: $f': V \to \{0,1,2\}$ so that for $1 \leq i \leq 2; 1 \leq j \leq n$:

$$f'((i,j)) = \begin{cases} 2 \; if \; (i,j) \in M_1; \\ 0 \; otherwise. \end{cases}$$

- $w(f') = 2\left\lceil\frac{n}{3}\right\rceil$.
- $\gamma_I^p(P_3 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$.

Case $n \equiv 1(mod\ 3)$:

- For $1 \leq j \leq \frac{n-4}{3}$:

$$B_j = \{(1,3j), (1,3j+1), (1,3j+2), (2,3j), (2.3j+1), (2,3j+2), (3,3j), (3,3j+1), (3,3j+2)\}.$$

- $A_1 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$.
- $A_2 = \{(1, n-1), (1, n), (2, n-1), (2, n), (3, n-1), (3, n)\}$.
- If $f\big((2,1)\big) = 0$ then $f(A_1) = 2$.

  If $f\big((2,1)\big) = 1$ and $f\big((1,1)\big) = 0$ then $f\big((1,2)\big) + f\big((2,2)\big) = 1$.

  If $f\big((2,1)\big) = 1$ and $f\big((3,1)\big) = 0$ then $f\big((2,2)\big) + f\big((3,2)\big) = 1$.

  If $f\big((2,1)\big) = 2$ then $f(A_1) \geq 2$.

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We conclude that $f(A_1) \geq 2$ and similarly, $f(A_2) \geq 2$.

- $\gamma_I^p(P_3 \boxtimes P_n) = \sum_{j=1}^{j=\frac{n-4}{3}} f(B_j) + f(A_1) + f(A_2) \geq 2\left(\frac{n-4}{3}\right) + 2 + 2 = 2\left\lceil\frac{n}{3}\right\rceil$.
- $M_2 = \{(2,3j+1): 0 \leq j \leq \frac{n-1}{3}\}$.
- PIDF: $f': V \to \{0,1,2\}$ so that for $1 \leq i \leq 2; 1 \leq j \leq n$:

$$f'((i,j)) = \begin{cases} 2 \text{ } if \text{ } (i,j) \in M_2; \\ 0 \text{ } otherwise. \end{cases}$$

- $w(f') = 2\left\lceil\frac{n}{3}\right\rceil$.
- $\gamma_I^p(P_3 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$.

Case $n \equiv 2(mod\ 3)$:

- For $1 \leq j \leq \frac{n-2}{3}$:

$B_j = \{(1,3j-2),(1,3j-1),(1,3j),(2,3j-2),(2,3j-1),(2,3j),(3,3j-2),$
$(3,3j-1),(3,3j)\}$.

- For $1 \leq j \leq \frac{n-2}{3}$; $f(B_j) \geq 2$.
- $A_2 = \{(1,n-1),(1,n),(2,n-1),(2,n),(3,n-1),(3,n)\}$.
- $f(A_2) \geq 2$.
- $\gamma_I^p(P_3 \boxtimes P_n) = \sum_{j=1}^{j=\frac{n-2}{3}} f(B_j) + f(A_2) \geq 2\left(\frac{n-2}{3}\right) + 2 = 2\left\lceil\frac{n}{3}\right\rceil$.
- $M_3 = \{(2,3j-1): 1 \leq j \leq \frac{n+1}{3}\}$.
- PIDF: $f': V \to \{0,1,2\}$ so that for $1 \leq i \leq 2; 1 \leq j \leq n$:

$$f'((i,j)) = \begin{cases} 2 \text{ } if \text{ } (i,j) \in M_3; \\ 0 \text{ } otherwise. \end{cases}$$

- $w(f') = 2\left\lceil\frac{n}{3}\right\rceil$.
- $\gamma_I^p(P_3 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$.

From all the cases we conclude that $\gamma_I^p(P_3 \boxtimes P_n) = 2\left\lceil\frac{n}{3}\right\rceil$ for $n \geq 2$.■

Theorem 6. For $m,n \geq 2$, let $P_m \boxtimes P_n$ be a strong grid graph. We define $k_1 = 3-(n\ mod\ 3)$, $k_2 = 3-(m\ mod\ 3)$, then $\gamma_I^p(P_m \boxtimes P_n) \leq \frac{2mn+2k_1m+2k_2n+2k_1k_2}{9}$.

Proof: To establish this upper bound, it is enough to conduct $f': V \to \{0,1,2\}$, a PIDF of weight $\frac{2mn+2k_1m+2k_2n+2k_1k_2}{9}$ on $P_m \boxtimes P_n$ for all the cases of m,n. For $1 \leq i \leq m$, $1 \leq j \leq n$ let f' be:

$$f'((i,j)) = \begin{cases} 2 \text{ } if \text{ } (i,j) \in M; \\ 0 \text{ } otherwise. \end{cases}$$

Now we give $M, w(f')$ for all cases of m,n:

Case 1. $m,n \equiv 0(mod\ 3)$.

$M = \left\{(3i-1,3j-1): 1 \leq i \leq \frac{m}{3}; 1 \leq j \leq \frac{n}{3}\right\}$, $w(f') = \frac{2mn}{9}$.

Case 2. $m \equiv 1(mod\ 3)$ and $n \equiv 0(mod\ 3)$.

$M = \left\{(3i+1,3j-1): 0 \leq i \leq \frac{m-1}{3}; 1 \leq j \leq \frac{n}{3}\right\}$, $w(f') = \frac{2mn+4n}{9}$.

Case 3. $m \equiv 2(mod\ 3)$ and $n \equiv 0(mod\ 3)$.

$M = \left\{(3i-1,3j-1): 1 \leq i \leq \frac{m+1}{3}; 1 \leq j \leq \frac{n}{3}\right\}$, $w(f') = \frac{2mn+2n}{9}$.

Case 4. $m \equiv 0(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

$M = \left\{(3i-1,3j+1): 1 \leq i \leq \frac{m}{3}; 0 \leq j \leq \frac{n-1}{3}\right\}$, $w(f') = \frac{2mn+4m}{9}$.

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Case 5. $m, n \equiv 1(mod\ 3)$.

$M = \left\{(3i+1, 3j+1): 0 \le i \le \frac{m-1}{3}; 0 \le j \le \frac{n-1}{3}\right\}$, $w(f') = \frac{2mn+4m+4n+8}{9}$.

Case 6. $m \equiv 2(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

$M = \left\{(3i-1, 3j+1): 1 \le i \le \frac{m+1}{3}; 0 \le j \le \frac{n-1}{3}\right\}$, $w(f') = \frac{2mn+4m+2n+4}{9}$.

Case 7. $m \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

$M = \left\{(3i, 3j-1): 1 \le i \le \frac{m}{3}; 1 \le j \le \frac{n+1}{3}\right\}$, $w(f') = \frac{2mn+2m}{9}$.

Case 8. $m \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

$M = \left\{(3i+1, 3j+1): 0 \le i \le \frac{m-1}{3}; 1 \le j \le \frac{n+1}{3}\right\}$, $w(f') = \frac{2mn+2m+4n+4}{9}$.

Case 9. $m, n \equiv 2(mod\ 3)$.

$M = \left\{(3i-1, 3j-1): 1 \le i \le \frac{m+1}{3}; 1 \le j \le \frac{n+1}{3}\right\}$, $w(f') = \frac{2mn+2m+2n+2}{9}$.

From all these cases we conclude that $\gamma_I^p(P_m \boxtimes P_n) \le \frac{2mn+2k_1m+2k_2n+2k_1k_2}{9}$ for $m, n \ge 2$ and with $k_1 = 3 - (n\ mod\ 3)$, $k_2 = 3 - (m\ mod\ 3)$.■

2.4. The perfect Italian domination number of Jahangir graph $J_{s,m}$.

In this section, we determine the perfect Italian domination number of Jahangir graph $J_{s,m}$ for any $s \ge 2$ and $m \ge 3$.

Theorem 7. For $s \ge 2$ and $m \ge 3$, let $J_{s,m}$ be Jahangir graph:

$$\gamma_I^p(J_{s,m}) = \begin{cases} \frac{ms}{2} + 1 & if\ s\ is\ even; \\ \frac{m(s-1)}{2} + 1 & if\ s\ is\ odd. \end{cases}$$

Proof:

We implied earlier that the set of 3-degree vertices of C_{sm} i.e., $\{v_{1+is}: 1 \le i \le m-1\}$ is denoted by R. We notice that the vertices of R divide $J_{s,m}$ into m subgraphs (each of which consists of $s+2$ vertices). For $1 \le i \le m-1$ we denote these subgraphs by:

$$OA_i = \{v_{1+(i-1)s}, v_{2+(i-1)s}, \dots, v_{is}, v_{1+is}, v_{sm+1}\}$$

taking into consideration that $OA_m = \{v_{1+(m-1)s}, v_{2+(m-1)s}, \dots, v_{sm}, v_1, v_{sm+1}\}$. Figure 10 illustrates OA_1, OA_2, OA_3 on $J_{4,3}$.

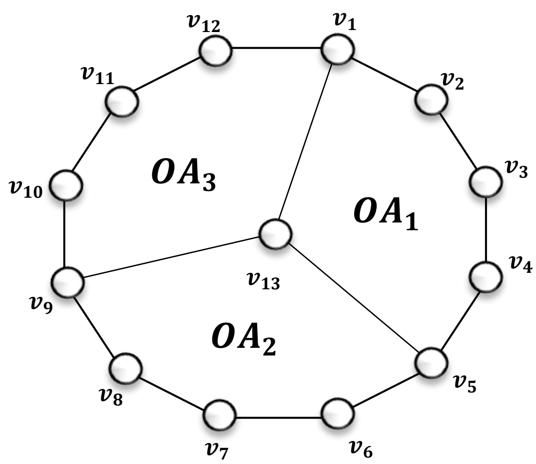

Figure 10. OA_1, OA_2, OA_3 on $J_{4,3}$.

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We also notice that the vertices of R divide C_{sm} into m paths of length $s-1$. We denote them by $P_{s-1}^{(i)}: 1 \le i \le m$ where $P_{s-1}^{(k)} = \{v_{2+(k-1)s}, \dots, v_{ks}\}$. We consider the following cases for s:

Case 1. s *is even*. Let us discuss the perfect Italian domination of two consecutive subgraphs OA_i, OA_{i+1}. Let $f: V \to \{0,1,2\}$ be an arbitrary PIDF conducted on $J_{s,m}$, we notice the following observations:

:

- If $f(v_{2+(i-1)s}) = f(v_{4+(i-1)s}) = \dots = f(v_{is-2}) = f(v_{is}) = 0$ and $f(v_{3+(i-1)s}) = f(v_{5+(i-1)s}) = \dots = f(v_{is-3}) = f(v_{is-1}) = 1$ then by Proposition 2; $f(P_{s-3}^{(i)}) = \left\lceil \frac{s-2}{2} \right\rceil = \frac{s}{2} - 1$. However, this means $f(v_{1+(i-1)s}) = f(v_{1+is}) = 1$ or else $v_{2+(i-1)s}, v_{is}$ is imperfect (respectively). The same argument applies to $P_{s-1}^{(i+1)}$. We will call this configuration 1.
- If $f(v_{2+(i-1)s}) = f(v_{4+(i-1)s}) = \dots = f(v_{is-2}) = f(v_{is}) = 0$ and $f(v_{3+(i-1)s}) = f(v_{5+(i-1)s}) = \dots = f(v_{is-3}) = f(v_{is-1}) = 1$ then by Proposition 2; $f(P_{s-1}^{(i)}) = \left\lceil \frac{s}{2} \right\rceil = \frac{s}{2}$. Therefore $f(v_{1+(i-1)s}) + f(v_{1+is}) \ge 0$. The same argument applies to $P_{s-1}^{(i+1)}$. We will call this configuration 2.
- If configuration 1 is applied, $f(v_{sm+1}) \ge 1$ or else $f(v_{sm+1}) = 0$ and $\sum_{u \in N(v_{sm+1})} f(u) > 2$ which is a contradiction.
- If configuration 2 is applied and $f(v_{1+(i-1)s}) + f(v_{1+is}) < 2$ then at least one of them is of weight zero which means $f(v_{sm+1}) = 0$, therefore two vertices of R must be of collective weight two and the rest must be of collective weight zero.

It is obvious that using vertices of weight two does not change these observations, without loss of generality, we conclude that:

If configuration 1 is applied on the entire graph, then the corresponding PIDF is:

$$f'(v_i) = \begin{cases} 1 \; if \; i = 2k+1 \; when \; 0 \le k \le \dfrac{sm}{2}; \\ 0 \; if \; i = 2k \; when \; 1 \le k \le \dfrac{sm}{2}. \end{cases}$$

which is of weight $w(f') = 0\left(\frac{sm}{2}\right) + 1\left(\frac{sm}{2} + 1\right) = \frac{sm}{2} + 1$. Therefore, $\gamma_I^p(J_{s,m}) \le \frac{sm}{2} + 1$.

If configuration 2 is applied on the entire graph, then the corresponding PIDF is:

$$f''(v_i) = \begin{cases} 0 \; if \; i = 2k+1 \; when \; 1 \le k \le \dfrac{sm}{2} \; and \; i \ne 1+s\,; \\ 1 \; if \; i = 2k \; when \; 1 \le k \le \dfrac{sm}{2} \; or \; i \in \{1, 1+s\}. \end{cases}$$

which is of weight $w(f'') = 0\left(\frac{sm}{2} - 1\right) + 1\left(\frac{sm}{2} + 2\right) = \frac{sm}{2} + 2$ and $\gamma_I^p(J_{s,m}) \le \frac{sm}{2} + 2$. We notice that configuration 1 is more optimal, thus $\gamma_I^p(J_{s,m}) = \frac{sm}{2} + 1$ if $s \ge 2$, $m \ge 3$ and s is even.

Case 2. s is odd. Let $f: V \to \{0,1,2\}$ be an arbitrary PIDF conducted on $J_{s,m}$. It is obvious that if $f(v_{sm+1}) = 0$, then the $f(R) = 2$. This means at least $m-2$ vertices of R are of weight zero. When studying f on a subgraph OA_i we notice the following:

If $f(v_{2+(i-1)s}) = f(v_{4+(i-1)s}) = \dots = f(v_{is-3}) = f(v_{is-1}) = 0$ and $f(v_{3+(i-1)s}) = f(v_{5+(i-1)s}) = \dots = f(v_{is-2}) = f(v_{is}) = 1$ then by Proposition 2; $f(P_{s-1}^{(i)}) \le \left\lceil \frac{s}{2} \right\rceil = \frac{s}{2}$. However, this means $f(v_{1+(i-1)s}) = 1$ or else $v_{2+(i-1)s}$ is imperfect. In a similar way, if $f(v_{2+(i-1)s}) = f(v_{4+(i-1)s}) = \dots = f(v_{is-3}) = f(v_{is-1}) = 1$ and $f(v_{3+(i-1)s}) = f(v_{5+(i-1)s}) = \dots = f(v_{is-2}) = f(v_{is}) = 0$ then $f(v_{1+is}) = 1$ or else v_{is} is imperfect. We conclude that, for $1 \le i \le m$ in order to assign the vertices of $P_{s-1}^{(i)}$ the minimal weighted possible assignment which is 010101..01, then:

- All vertices of R must be of weight 1. Otherwise, if an arbitrary vertex $v_{1+is} \in R$ is of weight zero, then either v_{is}, v_{is+2} is of weight zero and adjacent to two vertices of a collective weight (one), which is a contradiction.
- If $f(v_{sm+1}) = 0$ then v_{sm+1} is imperfect because $\sum_{u \in N(v_{sm+1})} f(u) = m$.

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We conclude that $\gamma_I^p(J_{s,m}) = mf(P_{s-1}^{(i)}) + f(R) + f(v_{sm+1}) \geq m\left(\frac{s-1}{2}\right) + m + 1 = m\left(\frac{s+1}{2}\right) + 1$. Now we establish the upper bound by conducting a PIDF of collective weight $m\left(\frac{s+1}{2}\right) + 1$ on $J_{s,m}$. First, for $1 \leq i \leq m$ we divide each path $P_{s-1}^{(i)}$ into two sets EP_i and OP_i defined as:

$$EP_i = \left\{v_{2l+(i-1)s} : 1 \leq l \leq \frac{s-1}{2}\right\};\ OP_i = \left\{v_{2l-1+(i-1)s} : 2 \leq l \leq \frac{s-1}{2}\right\};$$

We also define:

$$EP = \bigcup_{i=1}^{i=m} EP_i\,;\ OP = \bigcup_{i=1}^{i=m} OP_i.$$

We define the following PIDF (denoted f') on $J_{s,m}$ so that for $1 \leq i \leq sm+1$:

$$f'(v_i) = \begin{cases} 1\ if\ v_i \in \{OP \cup R \cup \{v_{sm+1}\}\}; \\ 0\ v_i \in EP. \end{cases}$$

$w(f') = |OP| + |R| + 1 = \frac{sm}{2} + m + 1 = m\left(\frac{s+1}{2}\right) + 1$, therefore $\gamma_I^p(J_{s,m}) \leq m\left(\frac{s+1}{2}\right) + 1$. Figure 11 shows that $\gamma_I^p(J_{5,4}) \leq 13$.

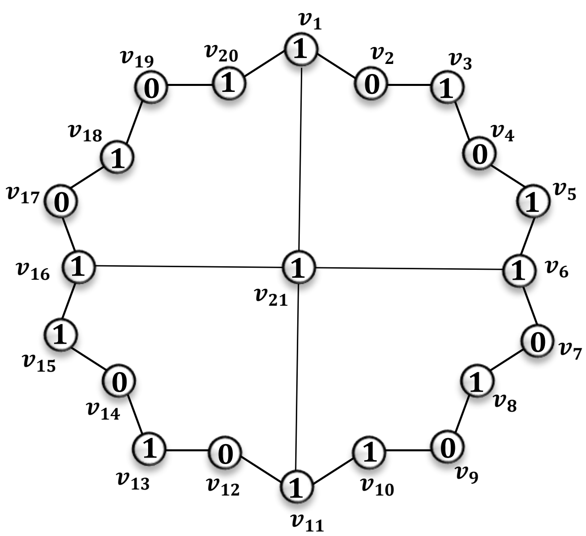

Figure 11. $\gamma_I^p(J_{5,4}) \leq 13$.

3. Conclusions

In this paper, we studied the perfect Italian domination problem on some graph classes. We determined the perfect Italian domination number of the circulant graph $C_n\{1,2\}$ for $n \geq 5$ and introduced an upper bound for the perfect Italian domination number $C_n\{1,3\}$ when $n \geq 7$. We also found this parameter for generalized Petersen graph $P(n,2)$ when $n \geq 5$. We determined $\gamma_I^p(P_2 \boxtimes P_n)$ and $\gamma_I^p(P_3 \boxtimes P_n)$ for arbitrary $n \geq 2$, then we introduced an upper bound for $\gamma_I^p(P_m \boxtimes P_n)$ when $m, n \geq 2$ are arbitraries. Finally, we determined the perfect Italian domination number of Jahangir graph $J_{s,m}$ for arbitrary $s \geq 2$ and $m \geq 3$.

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