Abelian subgroups based on neutrosophic sets

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Abstract

The notion of a neutrosophic Abelian subgroup of a group is introduced. The characterizations of a neutrosophic Abelian subgroup are investigated. We show that the homomorphic preimage of a neutrosophic Abelian subgroup of a group is a neutrosophic Abelian subgroup, and the onto homomorphic image of a neutrosophic Abelian subgroup of a group is a neutrosophic Abelian subgroup.

Keywords: neutrosophic group; neutrosophic Abelian subgroup; neutrosophic cyclic subgroup.

1 Introduction

Zadeh[5] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov[1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache came up with the word "neutrosophic", which comes from the words "neutrosophic" (French neuter, Latin neuter, neutral, and Greek Sophia, skill or wisdom), which means "knowledge of neutral thought". This third/neutral part is what makes "fuzzy/intuitionistic" logic/set different from "neutrosophic" logic/set; it is the part that is not clear or known (besides the truth). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). The notion of the neutrosophic set, which Smarandache developed,[3,4] extends the notions of the classic set and fuzzy set, intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts (refer to the site `http://fs.gallup.unm.edu/neutrosophy.htm`).

In this paper, we introduce the notion of a neutrosophic Abelian subgroup of a group. The characterizations of a neutrosophic Abelian subgroup ideal are investigated. We show that the homomorphic preimage of a neutrosophic Abelian subgroup of a group is a neutrosophic Abelian subgroup, and the onto homomorphic image of a neutrosophic Abelian subgroup of a group is a neutrosophic Abelian subgroup.

Definition 1.1. Let X be a nonempty set. The neutrosophic set[3] on X is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu : X \to [0, 1]$ is a truth membership function, $\gamma : X \to [0, 1]$ is an indeterminate membership function, and $\psi : X \to [0, 1]$ is a false membership function. The neutrosophic fuzzy set in (1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

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2 Abelian subgroups based on neutrosophic set

We start this section with the neutrosophic normalizer and neutrosophic centralizer and show that the neutrosophic normalizer (centralizer) is a subgroup of the group. We also prove that this newly defined centralizer is a normal subgroup of the neutrosophic normalizer and investigate some fundamental algebraic properties of these situations. We also introduce the notion of a neutrosophic Abelian (cyclic) group, prove that every neutrosophic subgroup is a neutrosophic Abelian group analogue to classical group theory, and also discuss their properties.

Definition 2.1. Let $\mathcal{G}$ be a group and $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set of $\mathcal{G}$. Then A is said to be a neutrosophic subgroup of $\mathcal{G}$ if the following conditions hold:

$$(\forall m, n \in \mathcal{G}) \begin{pmatrix} \mu_A(mn) \geq \mu_A(m) \wedge \mu_A(n) \\ \gamma_A(mn) \geq \gamma_A(m) \wedge \gamma_A(n) \\ \psi_A(mn) \leq \psi_A(m) \vee \psi_A(n) \end{pmatrix}, \tag{2}$$

$$(\forall m \in \mathcal{G}) \begin{pmatrix} \mu_A(m^{-1}) = \mu_A(m) \\ \gamma_A(m^{-1}) = \gamma_A(m) \\ \psi_A(m^{-1}) = \psi_A(m) \end{pmatrix}. \tag{3}$$

Equivalently, a neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ of a group $\mathcal{G}$ is said to be a neutrosophic subgroup of $\mathcal{G}$ if and only if

$$(\forall m, n \in \mathcal{G}) \begin{pmatrix} \mu_A(mn^{-1}) \geq \mu_A(m) \wedge \mu_A(n) \\ \gamma_A(mn^{-1}) \geq \gamma_A(m) \wedge \gamma_A(n) \\ \psi_A(mn^{-1}) \leq \psi_A(m) \wedge \gamma_A(n) \end{pmatrix}. \tag{4}$$

Definition 2.2. Let $\mathcal{G}$ be a group and $A = (\mu_A, \gamma_A, \psi_A)$ a neutrosophic subgroup of $\mathcal{G}$. Let $N(A) = \{a \in \mathcal{G} \mid \mu_A(a^{-1}xa) = \mu_A(x), \gamma_A(a^{-1}xa) = \gamma_A(x), \psi_A(a^{-1}xa) = \psi_A(x) \text{ for all } x \in \mathcal{G}\}$. Then $N(A)$ is called the neutrosophic fuzzy normalizer of A in $\mathcal{G}$.

Definition 2.3. A neutrosophic subgroup $A = (\mu_A, \gamma_A, \psi_A)$ of a group $\mathcal{G}$ is said to be a neutrosophic normal subgroup of $\mathcal{G}$ if

$$(\forall m, n \in \mathcal{G}) \begin{pmatrix} \mu_A(mn) = \mu_A(nm) \\ \gamma_A(mn) = \gamma_A(nm) \\ \psi_A(mn) = \psi_A(nm) \end{pmatrix}. \tag{5}$$

Equivalently, a neutrosophic subgroup $A = (\mu_A, \gamma_A, \psi_A)$ of a group $\mathcal{G}$ is said to be neutrosophic normal if and only if

$$(\forall m, n \in \mathcal{G}) \begin{pmatrix} \mu_A(n^{-1}mn) = \mu_A(m) \\ \gamma_A(n^{-1}mn) = \gamma_A(m) \\ \psi_A(n^{-1}mn) = \psi_A(m) \end{pmatrix}. \tag{6}$$

Theorem 2.4. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $\mathcal{G}$. Then*

(1) *$N(A)$ is a subgroup of $\mathcal{G}$.*

(2) *A is a neutrosophic normal subgroup of $\mathcal{G}$ if and only if $N(A) = \mathcal{G}$.*

(3) *A is a neutrosophic normal subgroup of the group $N(A)$.*

Proof. (1) Let $a, b \in N(A)$. Then we have

$$(\forall x \in \mathcal{G}) \begin{pmatrix} \mu_A(a^{-1}xa) = \mu_A(x) \\ \gamma_A(a^{-1}xa) = \gamma_A(x) \\ \psi_A(a^{-1}xa) = \psi_A(x) \end{pmatrix}, \tag{7}$$

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$$(\forall x \in \mathcal{G}) \begin{pmatrix} \mu_A(b^{-1}xb) = \mu_A(x) \\ \gamma_A(b^{-1}xb) = \gamma_A(x) \\ \psi_A(b^{-1}xb) = \psi_A(x) \end{pmatrix}. \tag{8}$$

Put $y = a^{-1}xa$ in (3) and using (7), we get

$$\begin{aligned} \mu_A(b^{-1}a^{-1}xab) = \mu_A(a^{-1}xa) = \mu_A(x), \\ \gamma_A(b^{-1}a^{-1}xab) = \gamma_A(a^{-1}xa) = \gamma_A(x), \\ \psi_A(b^{-1}a^{-1}xab) = \psi_A(a^{-1}xa) = \psi_A(x). \end{aligned}$$

That is,

$$\begin{aligned} \mu_A((ab)^{-1}x(ab)) = \mu_A(x), \\ \gamma_A((ab)^{-1}x(ab)) = \gamma_A(x), \\ \psi_A((ab)^{-1}x(ab)) = \psi_A(x). \end{aligned}$$

Thus, $ab \in N(A)$. Next, change x to x^{-1} in (7), we get

$$\begin{aligned} \mu_A(a^{-1}x^{-1}a) = \mu_A(x^{-1}) = \mu_A(x), \\ \gamma_A(a^{-1}x^{-1}a) = \gamma_A(x^{-1}) = \gamma_A(x), \\ \psi_A(a^{-1}x^{-1}a) = \psi_A(x^{-1}) = \psi_A(x). \end{aligned}$$

That is,

$$\begin{aligned} \mu_A((axa^{-1})^{-1}) = \mu_A(axa^{-1}) = \mu_A(x), \\ \gamma_A((axa^{-1})^{-1}) = \gamma_A(axa^{-1}) = \gamma_A(x), \\ \psi_A((axa^{-1})^{-1}) = \psi_A(axa^{-1}) = \psi_A(x). \end{aligned}$$

Thus,

$$\begin{aligned} \mu_A((a^{-1})^{-1}x(a^{-1})) = \mu_A(x), \\ \gamma_A((a^{-1})^{-1}x(a^{-1})) = \gamma_A(x), \\ \psi_A((a^{-1})^{-1}x(a^{-1})) = \psi_A(x). \end{aligned}$$

Then $a^{-1} \in N(A)$. Hence, $N(A)$ is a subgroup of $\mathcal{G}$.

(2) Obviously, when $N(A) = \mathcal{G}$, then $\mu_A(a^{-1}xa) = \mu_A(x)$, $\gamma_A(a^{-1}xa) = \gamma_A(x)$, and $\psi_A(a^{-1}xa) = \psi_A(x)$ for all $x, a \in \mathcal{G}$. Hence, A is a neutrosophic normal subgroup of $\mathcal{G}$.

Conversely, assume that A is a neutrosophic normal subgroup of $\mathcal{G}$. Then $\mu_A(a^{-1}xa) = \mu_A(x)$, $\gamma_A(a^{-1}xa) = \gamma_A(x)$, and $\psi_A(a^{-1}xa) = \psi_A(x)$ for all $x, a \in \mathcal{G}$, that is, the set $\{a \in \mathcal{G} \mid \mu_A(a^{-1}xa) = \mu_A(x)$, $\gamma_A(a^{-1}xa) = \gamma_A(x), \psi_A(a^{-1}xa) = \psi_A(x)$ for all $x \in \mathcal{G}\} = \mathcal{G}$. Hence, $N(A) = \mathcal{G}$.

(3) Let $a, b \in N(A)$. Then $\mu_A(a^{-1}xa) = \mu_A(x)$, $\gamma_A(a^{-1}xa) = \gamma_A(x)$, and $\psi_A(a^{-1}xa) = \psi_A(x)$ for all $x \in \mathcal{G}$. Putting $x = ab$, we get $\mu_A(ab) = \mu_A(a^{-1}aba) = \mu_A(ba)$, $\gamma_A(ab) = \gamma_A(a^{-1}aba) = \gamma_A(ba)$, and $\psi_A(ab) = \psi_A(a^{-1}aba) = \psi_A(ba)$. Hence, A is a neutrosophic normal subgroup of $N(A)$. □

Definition 2.5. Let $\mathcal{G}$ be a group and $A = (\mu_A, \gamma_A, \psi_A)$ a neutrosophic subgroup of $\mathcal{G}$. Let

$$C(A) = \{a \in \mathcal{G} \mid \mu_A([a, x]) = \mu_A(e), \gamma_A([a, x]) = \gamma_A(e), \psi_A([a, x]) = \psi_A(e) \text{ for all } x \in \mathcal{G}\}.$$

Then $C(A)$ is called the neutrosophic centralizer of A in $\mathcal{G}$, where $[x, y]$ is the commutator of the two elements x and y in $\mathcal{G}$, that is, $[x, y] = x^{-1}y^{-1}xy$.

Theorem 2.6. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $\mathcal{G}$. Then*

(1) *$C(A)$ is a subgroup of $\mathcal{G}$.*

(2) *$C(A)$ is a normal subgroup of $N(A)$.*

Proof. (1) Clearly, $C(A) \neq \emptyset$ as $e \in C(A)$. Let $a, b \in C(A)$. Then

$$\begin{aligned} \mu_A([a, x]) = \mu_A(e), \gamma_A([a, x]) = \gamma_A(e), \psi_A([a, x]) = \psi_A(e), \\ \mu_A([b, x]) = \mu_A(e), \gamma_A([b, x]) = \gamma_A(e), \psi_A([b, x]) = \psi_A(e) \end{aligned}$$

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hold for all $x, y \in \mathcal{G}$, that is,

$$\mu_A(a^{-1}x^{-1}ax) = \mu_A(e), \gamma_A(a^{-1}x^{-1}ax) = \gamma_A(e), \psi_A(a^{-1}x^{-1}ax) = \psi_A(e), \tag{9}$$

$$\mu_A(b^{-1}y^{-1}by) = \mu_A(e), \gamma_A(b^{-1}y^{-1}by) = \gamma_A(e), \psi_A(b^{-1}y^{-1}by) = \psi_A(e). \tag{10}$$

Putting $y = a^{-1}za$ in (10), we have

$$\begin{aligned}
\mu_A(b^{-1}a^{-1}z^{-1}aba^{-1}za) = \mu_A(e) &\Rightarrow \mu_A((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za) = \mu_A(e)\\
&\Rightarrow \mu_A((ab)^{-1}z^{-1}(ab)z) = \mu_A(e),\\
\gamma_A(b^{-1}a^{-1}z^{-1}aba^{-1}za) = \gamma_A(e) &\Rightarrow \gamma_A((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za)) = \gamma_A(e)\\
&\Rightarrow \gamma_A((ab)^{-1}z^{-1}(ab)z) = \gamma_A(e),\\
\psi_A(b^{-1}a^{-1}z^{-1}aba^{-1}za) = \psi_A(e) &\Rightarrow \psi_A((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za)) = \psi_A(e)\\
&\Rightarrow \psi_A((ab)^{-1}z^{-1}(ab)z) = \psi_A(e).
\end{aligned}$$

Hence, $ab \in C(A)$. Also, from (9), we have

$$\begin{aligned}
\mu_A(e) = \mu_A(a^{-1}x^{-1}ax) = \mu_A((a^{-1}x^{-1}ax)) = \mu_A(x^{-1}a^{-1}xa),\\
\gamma_A(e) = \gamma_A(a^{-1}x^{-1}ax) = \gamma_A((a^{-1}x^{-1}ax)) = \gamma_A(x^{-1}a^{-1}xa),\\
\psi_A(e) = \psi_A(a^{-1}x^{-1}ax) = \psi_A((a^{-1}x^{-1}ax)) = \psi_A(x^{-1}a^{-1}xa).
\end{aligned}$$

That is,

$$\begin{aligned}
\mu_A(x^{-1}a^{-1}xa) = \mu_A(e),\\
\gamma_A(x^{-1}a^{-1}xa) = \gamma_A(e),\\
\psi_A(x^{-1}a^{-1}xa) = \psi_A(e).
\end{aligned}$$

Putting $x = ta^{-1}$, we get

$$\begin{aligned}
\mu_A(at^{-1}a^{-1}ta^{-1}a) = \mu_A(at^{-1}a^{-1}t) = \mu_A(e),\\
\gamma_A(at^{-1}a^{-1}ta^{-1}a) = \gamma_A(at^{-1}a^{-1}t) = \gamma_A(e),\\
\psi_A(at^{-1}a^{-1}ta^{-1}a) = \psi_A(at^{-1}a^{-1}t) = \psi_A(e).
\end{aligned}$$

Thus, $a^{-1} \in C(A)$. Hence, $C(A)$ is a subgroup of $\mathcal{G}$.

(2) Let $a \in C(A)$ and $b \in N(A)$. We shall show that $b^{-1}ab \in C(A)$. Now,

$$(\forall x \in \mathcal{G}) \begin{pmatrix} \mu_A(a^{-1}x^{-1}ax) = \mu_A(e) \\ \gamma_A(a^{-1}x^{-1}ax) = \gamma_A(e) \\ \psi_A(a^{-1}x^{-1}ax) = \psi_A(e) \end{pmatrix}, \tag{11}$$

$$(\forall y \in \mathcal{G}) \begin{pmatrix} \mu_A(b^{-1}y^{-1}by) = \mu_A(e) \\ \gamma_A(b^{-1}y^{-1}by) = \gamma_A(e) \\ \psi_A(b^{-1}y^{-1}by) = \psi_A(e) \end{pmatrix}. \tag{12}$$

Put $y = a^{-1}x^{-1}ax$ in (12) and using (11), we have

$$\begin{aligned}
\mu_A(b^{-1}a^{-1}x^{-1}axb) = \mu_A(a^{-1}x^{-1}ax) = \mu_A(e),\\
\gamma_A(b^{-1}a^{-1}x^{-1}axb) = \gamma_A(a^{-1}x^{-1}ax) = \gamma_A(e),\\
\psi_A(b^{-1}a^{-1}x^{-1}axb) = \psi_A(a^{-1}x^{-1}ax) = \psi_A(e).
\end{aligned}$$

Again, putting $x = bzb^{-1}$ above, we have

$$\begin{aligned}
\mu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abzb^{-1}b) = \mu_A(e),\\
\gamma_A(b^{-1}a^{-1}bz^{-1}b^{-1}abzb^{-1}b) = \gamma_A(e),\\
\psi_A(b^{-1}a^{-1}bz^{-1}b^{-1}abzb^{-1}b) = \psi_A(e).
\end{aligned}$$

That is,

$$\begin{aligned}
\mu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = \mu_A(e),\\
\gamma_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = \gamma_A(e),\\
\psi_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = \psi_A(e).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu_A((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = \mu_A(e),\\
\gamma_A((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = \gamma_A(e),\\
\psi_A((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = \psi_A(e).
\end{aligned}$$

So, $b^{-1}ab \in C(A)$. Hence, $C(A)$ is a normal subgroup of $N(A)$. □

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Proposition 2.7. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of a group $\mathcal{G}$. Let $N = \{a \in \mathcal{G} \mid \mu_A(a) = \mu_A(e), \gamma_A(a) = \gamma_A(e), \psi_A(a) = \psi_A(e)\}$. Then $N \subseteq C(A)$.*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic normal subgroup of group $\mathcal{G}$. Therefore, $\mu_A(y^{-1}xy) = \mu_A(x)$, $\gamma_A(y^{-1}xy) = \gamma_A(x)$, and $\psi_A(y^{-1}xy) = \psi_A(x)$ for all $x, y \in \mathcal{G}$. Let $a \in N$. Then $\mu_A(a) = \mu_A(e)$, $\gamma_A(a) = \gamma_A(e)$, and $\psi_A(a) = \psi_A(e)$. Now,

$$\begin{aligned} \mu_A([a,x]) &= \mu_A(a^{-1}x^{-1}ax) \\ &\geq \mu_A(a^{-1}) \wedge \mu_A(x^{-1}ax) \\ &= \mu_A(a) \wedge \mu_A(a) \\ &= \mu_A(e) \wedge \mu_A(e) \\ &= \mu_A(e). \end{aligned}$$

Thus, $\mu_A([a,x]) = \mu_A(e)$; similarly, we can show that $\gamma_A([a,x]) = \gamma_A(e)$ and $\psi_A([a,x]) = \psi_A(e)$. Thus, $a \in C(A)$. Hence, $N \subseteq C(A)$. □

Definition 2.8. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $\mathcal{G}$. Then $A = (\mu_A, \gamma_A, \psi_A)$ is called a neutrosophic Abelian subgroup of $\mathcal{G}$ if $C_{\alpha,\beta,\delta}(A) = \{x \in X \mid \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \delta\}$ is an Abelian subgroup of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0,1]$ with $0 < \alpha + \beta + \delta \leq 1$.

Theorem 2.9. *Let A be a neutrosophic subgroup of a group $\mathcal{G}$. Then A is a neutrosophic subgroup of $\mathcal{G}$ if and only if $C_{\alpha,\beta,\delta}(A)$ is a subgroup of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0,1]$ with $\alpha + \beta + \delta \leq 1$.*

Proof. Clearly, $C_{\alpha,\beta,\delta}$ is nonempty as $e \in C_{\alpha,\beta,\delta}$. For $C_{\alpha,\beta,\delta}$ to be a subgroup of $\mathcal{G}$, we shall show that for $x, y \in C_{\alpha,\beta,\delta}$, $xy^{-1} \in C_{\alpha,\beta,\delta}$. Let $x, y \in C_{\alpha,\beta,\delta}$. Then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \geq \beta$, $\psi_A(x) \leq \delta$ and $\mu_A(y) \geq \alpha$, $\gamma_A(y) \geq \beta$, $\psi_A(y) \leq \delta$. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic subgroup of $\mathcal{G}$, we have

$$\begin{aligned} &\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y^{-1})\} = \min\{\mu_A(x), \mu_A(y)\} \geq \min\{\alpha, \alpha\} = \alpha, \\ &\gamma_A(xy^{-1}) \geq \min\{\gamma_A(x), \gamma_A(y^{-1})\} = \min\{\gamma_A(x), \gamma_A(y)\} \geq \min\{\beta, \beta\} = \beta, \\ &\psi_A(xy^{-1}) \leq \max\{\psi_A(x), \psi_A(y^{-1})\} = \max\{\psi_A(x), \psi_A(y)\} \leq \max\{\delta, \delta\} = \delta. \end{aligned}$$

Therefore, $xy^{-1} \in C_{\alpha,\beta,\delta}$. Hence, $C_{\alpha,\beta,\delta}$ is a subgroup of $\mathcal{G}$. □

Remark 2.10. [2] Every subgroup of an Abelian group is Abelian.

Theorem 2.11. *If $\mathcal{G}$ is an Abelian group, then every neutrosophic subgroup of $\mathcal{G}$ is a neutrosophic Abelian subgroup of $\mathcal{G}$.*

Proof. Given that $\mathcal{G}$ is an Abelian group. Then $xy = yx$ holds for all $x, y \in \mathcal{G}$. Since A is a neutrosophic Abelian subgroup of $\mathcal{G}$ and by Theorem 2.9, we have $C_{\alpha,\beta,\delta}(A)$ is a subgroup of $\mathcal{G}$. In view of Remark 2.10, we know that $C_{\alpha,\beta,\delta}(A)$ is an Abelian subgroup of $\mathcal{G}$. By using the definition of neutrosophic Abelian subgroup, we conclude that A is a neutrosophic Abelian subgroup group $\mathcal{G}$. □

The following example leads us to note that the converse of Theorem 2.11 may not be true.

Example 2.12. Consider $\mathcal{G} = S_3 = \{i, (12), (13), (23), (123), (132)\}$ be the symmetric group. Consider the neutrosophic set A of $\mathcal{G}$ defined by

$$\mu_A(x) = \begin{cases} 0.9 & \text{if } x = i \\ 0 & \text{if } x^2 = i \\ 0.05 & \text{if } x^3 = i, \end{cases}$$

$$\gamma_A(x) = \begin{cases} 0.1 & \text{if } x = i \\ 0 & \text{if } x^2 = i \\ 0.05 & \text{if } x^3 = i, \end{cases}$$

$$\psi_A(x) = \begin{cases} 0 & \text{if } x = i \\ 0.03 & \text{if } x^2 = i \\ 0.04 & \text{if } x^3 = i, \end{cases}$$

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where $x \in \mathcal{G}$ and i is the identity element of $\mathcal{G}$. Clearly, A is a neutrosophic subgroup of $\mathcal{G}$. Moreover, all $C_{\alpha,\beta,\delta}(A)$ are Abelian subgroups of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. Hence, A is a neutrosophic Abelian subgroup of $\mathcal{G}$, but $\mathcal{G}$ is a non-Abelian group.

Theorem 2.13. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic Abelian subgroup of $\mathcal{G}$. Then the set $H = \{a \in \mathcal{G} \mid \mu_A(ab) = \mu_A(ba), \gamma_A(ab) = \gamma_A(ba), \psi_A(ab) = \psi_A(ba)\ \forall\, b \in \mathcal{G}\}$ is an Abelian subgroup of $\mathcal{G}$.*

Proof. Since $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic Abelian subgroup of group $\mathcal{G}$, $C_{\alpha,\beta,\delta}(A)$ is an Abelian subgroup of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha+\beta+\delta \leq 1$. We shall show that H is an Abelian subgroup of $\mathcal{G}$. Clearly, $H \neq \emptyset$ as $e \in H$. Let $a, b \in H$. Then $\mu_A(ax) = \mu_A(xa)$, $\gamma_A(ax) = \gamma_A(xa)$, $\psi_A(ax) = \psi_A(xa)$ and $\mu_A(ax) = \mu_A(xa)$, $\gamma_A(ax) = \gamma_A(xa)$, $\psi_A(ax) = \psi_A(xa)$ for all $x \in \mathcal{G}$. Now, for $x \in \mathcal{G}$, we have $\mu_A((ab)x) = \mu_A(a(bx)) = \mu_A((bx)a) = \mu_A(b(xa)) = \mu_A((xa)b) = \mu_A(x(ab))$, $\gamma_A((ab)x) = \gamma_A(a(bx)) = \gamma_A((bx)a) = \gamma_A(b(xa)) = \gamma_A((xa)b) = \gamma_A(x(ab))$ and $\psi_A((ab)x) = \psi_A(a(bx)) = \psi_A((bx)a) = \psi_A(b(xa)) = \psi_A((xa)b) = \psi_A(x(ab))$. Hence, $ab \in H$. Also, let $a \in H$. We shall show that $a^{-1} \in H$. Since $a \in H$, we have $\mu_A(ax) = \mu_A(xa), \gamma_A(ax) = \gamma_A(xa)$, and $\psi_A(ax) = \psi_A(xa)$ hold for all $x \in \mathcal{G}$ $(\star)$. We shall show that $\mu_A(a^{-1}y) = \mu_A(ya^{-1}), \gamma_A(a^{-1}y) = \gamma_A(ya^{-1})$, and $\psi_A(a^{-1}y) = \psi_A(ya^{-1})$ hold for all $y \in \mathcal{G}$. Putting $x = y^{-1}$ in $(\star)$, we get $\mu_A(ay^{-1}) = \mu_A(y^{-1}a)$, $\gamma_A(ay^{-1}) = \gamma_A(y^{-1}a)$, and $\psi_A(ay^{-1}) = \psi_A(y^{-1}a)$. Now, $\mu_A(a^{-1}y) = \mu_A((a^{-1}y)^{-1}) = \mu_A(y^{-1}a) = \mu_A(ay^{-1}) = \mu_A((ay^{-1})^{-1}) = \mu_A(ya^{-1})$. Similarly, we can show that $\gamma_A(a^{-1}y) = \gamma_A(ya^{-1})$ and $\psi_A(a^{-1}y) = \psi_A(ya^{-1})$ hold for all $y \in \mathcal{G}$. Thus, $a^{-1} \in H$. So H is a subgroup of $\mathcal{G}$. Next, we show that H is an Abelian subgroup of $\mathcal{G}$. Let $a, b \in H$. Without loss of generality, let $\mu_A(a) = \alpha, \gamma_A(a) = \beta$, $\psi_A(a) \leq 1 - (\alpha + \beta)$ and $\mu_A(b) = \alpha_1, \gamma_A(b) = \beta_1$, $\psi_A(a) \leq 1 - (\alpha_1 + \beta_1)$. Then $a \in C_{\alpha,\beta,1-(\alpha+\beta)}(A)$, $b \in C_{\alpha_1,\beta_1,1-(\alpha_1+\beta_1)}(A)$. Let $\alpha < \alpha_1$ and $\beta < \beta_1$. Then $\mu_A(b) = \alpha_1 > \alpha$, $\gamma_A(b) = \beta_1 > \beta$ and $\psi_A(b) \leq 1-(\alpha_1+\beta_1) < 1-(\alpha+\beta)$, so $b \in C_{\alpha,\beta,1-(\alpha+\beta)}(A)$. Thus, $a, b \in C_{\alpha,\beta,1-(\alpha+\beta)}(A)$ and so $ab = ba$. Hence, H is an Abelian subgroup of $\mathcal{G}$. □

Proposition 2.14. (1) *If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic Abelian subgroup of a group $\mathcal{G}$, then A is also a neutrosophic normal subgroup of $\mathcal{G}$.*

(2) *The sets H and $C(A)$ are same, that is, $C(A) = H$.*

Proof.

$$\begin{aligned} C(A) &= \{a \in \mathcal{G} : \mu_A([a,x]) = \mu_A(e), \gamma_A([a,x]) = \gamma_A(e), \\ &\quad \psi_A([a,x]) = \psi_A(e) \text{ for all } x \in \mathcal{G}\} \\ &= \{a \in \mathcal{G} : \mu_A(a^{-1}x^{-1}ax) = \mu_A(e), \gamma_A(a^{-1}x^{-1}ax) = \gamma_A(e), \\ &\quad \psi_A(a^{-1}x^{-1}ax) = \psi_A(e) \text{ for all } x \in \mathcal{G}\} \\ &= \{a \in \mathcal{G} : \mu_A((xa)^{-1}ax) = \mu_A(e), \gamma_A((xa)^{-1}ax) = \gamma_A(e), \\ &\quad \psi_A((xa)^{-1}ax) = \psi_A(e) \text{ for all } x \in \mathcal{G}\} \\ &= \{a \in \mathcal{G} : \mu_A(xa) = \mu_A(ax), \gamma_A(xa) = \gamma_A(ax), \\ &\quad \psi_A(xa) = \psi_A(ax) \text{ for all } x \in \mathcal{G}\} \\ &= H. \end{aligned}$$

□

Theorem 2.15. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic Abelian subgroup of a group $\mathcal{G}$. Then $C(A)$ is an Abelian subgroup of $\mathcal{G}$.*

Theorem 2.16. *Let $A = (\mathcal{G}_1, \mu_A, \gamma_A, \psi_A)$ and $B = (\mathcal{G}_2, \mu_B, \gamma_B, \psi_B)$ be two neutrosophic subgroups of a group $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $A \times B$ is a neutrosophic Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$ if and only if both A and B are neutrosophic Abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.*

Proof. First, let A and B be neutrosophic Abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $C_{\alpha,\beta,\delta}(A)$ and $C_{\alpha,\beta,\delta}(B)$ are Abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$, so $C_{\alpha,\beta,\delta}(A) \times C_{\alpha,\beta,\delta}(B)$ is an Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$. But $C_{\alpha,\beta,\delta}(A \times B) = C_{\alpha,\beta,\delta}(A) \times C_{\alpha,\beta,\delta}(B)$. Therefore, $C_{\alpha,\beta,\delta}(A \times B)$ is an Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. Thus, $A \times B$ is a neutrosophic Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$.

Conversely, let $A \times B$ be a neutrosophic Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$. Then $C_{\alpha,\beta,\delta}(A \times B)$ is an Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$, that is, $C_{\alpha,\beta,\delta}(A) \times C_{\alpha,\beta,\delta}(B)$ is an Abelian subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$. Thus, $C_{\alpha,\beta,\delta}(A)$ and $C_{\alpha,\beta,\delta}(B)$ are Abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Hence, A and B are neutrosophic Abelian subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. □

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Definition 2.17. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $\mathcal{G}$. Then A is called a neutrosophic cyclic subgroup of $\mathcal{G}$ if $C_{\alpha,\beta,\delta}(A)$ is a cyclic subgroup of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$.

Remark 2.18. [2] Every subgroup of a cyclic group is cyclic.

Theorem 2.19. *If $\mathcal{G}$ is a cyclic group, then every neutrosophic subgroup of $\mathcal{G}$ is a neutrosophic cyclic subgroup of $\mathcal{G}$.*

Proof. Given that $\mathcal{G}$ is a cyclic group. Then $\mathcal{G} = \langle x \rangle$ for some $x \in \mathcal{G}$. Let A be a neutrosophic subgroup of $\mathcal{G}$. Since A is a neutrosophic Abelian subgroup of $\mathcal{G}$ and by Theorem 2.9, we have $C_{\alpha,\beta,\delta}(A)$ is a subgroup of $\mathcal{G}$. In view of Remark 2.10, we know that $C_{\alpha,\beta,\delta}(A)$ is a cyclic subgroup of $\mathcal{G}$. By using the definition of neutrosophic cyclic subgroup, we conclude that A is a neutrosophic cyclic subgroup group $\mathcal{G}$. □

The following example leads us to note that the converse of Theorem 2.19 may not be true.

Example 2.20. Consider $\mathcal{G} = \langle a, b \mid a^3 = b^2 = e, bab^{-1} = a^{-1} \rangle$ be dihedral group of order six. Consider the neutrosophic set A of $\mathcal{G}$ defined by

$$\mu_A(x) = \begin{cases} 0.9 & \text{if } x = e \\ 0 & \text{if } x^2 = e \\ 0.05 & \text{if } x^3 = e, \end{cases}$$

$$\gamma_A(x) = \begin{cases} 0.1 & \text{if } x = e \\ 0 & \text{if } x^2 = e \\ 0.05 & \text{if } x^3 = e, \end{cases}$$

$$\psi_A(x) = \begin{cases} 0 & \text{if } x = e \\ 0.01 & \text{if } x^2 = e \\ 0.05 & \text{if } x^3 = e, \end{cases}$$

where $x \in \mathcal{G}$ and e is the identity element of $\mathcal{G}$. Clearly, A is a neutrosophic subgroup of $\mathcal{G}$. Moreover, all $C_{\alpha,\beta,\delta}(A)$ are cyclic subgroups of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha+\beta+\delta \leq 1$. Hence, A is a neutrosophic cyclic subgroup of $\mathcal{G}$, but $\mathcal{G}$ is not a cyclic group.

Proposition 2.21. *If $\mathcal{G}$ be a cyclic group, then every neutrosophic subgroup of $\mathcal{G}$ is a neutrosophic cyclic subgroup of $\mathcal{G}$.*

Proof. Let $\mathcal{G} = \langle x \rangle$ be a cyclic group, and let A be any neutrosophic subgroup of $\mathcal{G}$. Then

$$\begin{gathered} \mu_A(x^n) \geq \mu_A(x^{n-1}) \geq \mu_A(x^{n-2}) \geq \ldots \geq \mu_A(x), \\ \gamma_A(x^n) \geq \gamma_A(x^{n-1}) \geq \gamma_A(x^{n-2}) \geq \ldots \geq \gamma_A(x), \\ \psi_A(x^n) \leq \psi_A(x^{n-1}) \leq \psi_A(x^{n-2}) \leq \ldots \leq \psi_A(x) \end{gathered}$$

hold for all $n \in \mathbb{N}$. Therefore, if $x^m \in C_{\alpha,\beta,\delta}(A)$ for some $m \in \mathbb{N}$, then $x^m, x^{m+1}, x^{m+2}, \ldots \in C_{\alpha,\beta,\delta}(A)$, that is, $C_{\alpha,\beta,\delta}(A) = \langle x^{-1} \rangle$, which is a cyclic subgroup of $\mathcal{G}$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. Hence, A is a neutrosophic cyclic subgroup of $\mathcal{G}$. □

Theorem 2.22. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be homomorphism of a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$. Let B be a neutrosophic Abelian subgroup of $\mathcal{G}_2$. Then $h^{-1}(B)$ is a neutrosophic Abelian subgroup of $\mathcal{G}_1$.*

Proof. Let B be a neutrosophic Abelian subgroup of $\mathcal{G}_2$. Therefore, $C_{\alpha,\beta,\delta}(B)$ is an Abelian subgroup of $\mathcal{G}_2$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. Then $C_{\alpha,\beta,\delta}(h^{-1}(B)) = h^{-1}(C_{\alpha,\beta,\delta}(B)) = \{x \in \mathcal{G}_1 \mid h(x) \in C_{\alpha,\beta,\delta}(B)\}$. Let $x_1, x_2 \in C_{\alpha,\beta,\delta}(h^{-1}(B))$. Then $h(x_1), h(x_2) \in C_{\alpha,\beta,\delta}(B)$. Then

$$\begin{gathered} \mu_{h^{-1}(B)}(x_1) \geq \alpha, \gamma_{h^{-1}(B)}(x_1) \geq \beta, \psi_{h^{-1}(B)}(x_1) \leq \delta, \\ \mu_{h^{-1}(B)}(x_2) \geq \alpha, \gamma_{h^{-1}(B)}(x_2) \geq \beta, \psi_{h^{-1}(B)}(x_2) \leq \delta. \end{gathered}$$

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That is,

$$\mu_B(h(x_1)) \geq \alpha, \gamma_B(h(x_1)) \geq \beta, \psi_B(h(x_1)) \leq \delta,$$
$$\mu_B(h(x_2)) \geq \alpha, \gamma_B(h(x_2)) \geq \beta, \psi_B(h(x_2)) \leq \delta.$$

This implies that

$$\min\{\mu_B(h(x_1)), \mu_B(h(x_2))\} \geq \alpha,$$
$$\min\{\gamma_B(h(x_1)), \gamma_B(h(x_2))\} \geq \beta,$$
$$\max\{\psi_B(h(x_1)), \psi_B(h(x_2))\} \leq \delta.$$

Hence,

$$\mu_B(h(x_1)h(x_2^{-1})) \geq \min\{\mu_B(h(x_1)), \mu_B(h(x_2))\} \geq \alpha,$$
$$\gamma_B(h(x_1)h(x_2^{-1})) \geq \min\{\gamma_B(h(x_1)), \gamma_B(h(x_2))\} \geq \beta,$$
$$\psi_B(h(x_1)h(x_2^{-1})) \leq \max\{\psi_B(h(x_1)), \psi_B(h(x_2))\} \leq \delta.$$

Therefore,

$$\mu_B(h(x_1)h(x_2^{-1})) \geq \alpha, \gamma_B(h(x_1)h(x_2^{-1})) \geq \beta, \psi_B(h(x_1)h(x_2^{-1})) \leq \delta.$$

It follows that

$$\begin{aligned} h(x_1)h(x_2^{-1}) \in C_{\alpha,\beta,\delta}(B) &\Rightarrow h(x_1x_2^{-1}) \in C_{\alpha,\beta,\delta}(B) \\ &\Rightarrow x_1x_2^{-1} \in h^{-1}(C_{\alpha,\beta,\delta}(B)) \\ &\Rightarrow x_1x_2^{-1} \in C_{\alpha,\beta,\delta}(h^{-1}(B)). \end{aligned}$$

Hence, $C_{\alpha,\beta,\delta}(h^{-1}(B))$ is a subgroup of $\mathcal{G}_1$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha+\beta+\delta \leq 1$. As $C_{\alpha,\beta,\delta}(h^{-1}(B))$ is an Abelian subgroup of $\mathcal{G}_2$, we have $h(x_1)h(x_2) = h(x_2)h(x_1)$. This implies that $h(x_1x_2) = h(x_2x_1)$ and so $\mu_B(h(x_1x_2)) = \mu_B(hx_1x_2))$, $\gamma_B(h(x_1x_2)) = \gamma_B(hx_1x_2))$, and $\psi_B(h(x_1x_2)) = \psi_B(h(x_2x_1))$. It follows that $\mu_{h^{-1}(B)}(x_1x_2) = \mu_{h^{-1}(B)}(x_2x_1)$, $\gamma_{h^{-1}(B)}(x_1x_2) = \gamma_{h^{-1}(B)}(x_2x_1)$, and $\psi_{h^{-1}(B)}(x_1x_2) = \psi_{h^{-1}(B)}(x_2x_1)$. Hence, $x_1x_2 = x_2x_1$. Thus, $C_{\alpha,\beta,\delta}(h^{-1}(B))$ is an Abelian subgroup of $\mathcal{G}_1$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. Hence, $h^{-1}(B)$ is a neutrosophic Abelian subgroup of $\mathcal{G}_1$. □

Theorem 2.23. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a surjective homomorphism of a group $\mathcal{G}_1$ onto a group $\mathcal{G}_2$. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic Abelian subgroup of $\mathcal{G}_1$. Then $h(A)$ is a neutrosophic Abelian subgroup of $\mathcal{G}_2$.*

Proof. Since A is a neutrosophic Abelian subgroup of $\mathcal{G}_1$, we have $C_{\alpha,\beta,\delta}(A)$ is an Abelian subgroup of $\mathcal{G}_1$ for all $\alpha, \beta, \delta \in (0, 1]$ with $0 < \alpha + \beta + \delta \leq 1$. We shall show that $h(A)$ is a neutrosophic Abelian subgroup of $\mathcal{G}_2$. For this, we show will that $C_{\alpha,\beta,\delta}(h(A))$ is an Abelian subgroup of $\mathcal{G}_2$. Let $y_1, y_2 \in C_{\alpha,\beta,\delta}(h(A))$. Then there exist $x_1, x_2 \in \mathcal{G}_1$ such that $h(x_1) = y_1$, $h(x_2) = y_2$. Then

$$\mu_{h(A)}(y_1) \geq \alpha, \gamma_{h(A)}(y_1) \geq \beta, \psi_{h(A)}(y_1) \leq \delta,$$
$$\mu_{h(A)}(y_2) \geq \alpha, \gamma_{h(A)}(y_2) \geq \beta, \psi_{h(A)}(y_2) \leq \delta.$$

Since $h(C_{\alpha,\beta,\delta}(A)) \subseteq C_{\alpha,\beta,\delta}(h(A))$, there exist $x_1, x_2 \in \mathcal{G}_1$ such that

$$\mu_A(x_1) \geq \mu_{h(A)}(y_1) \geq \alpha, \gamma_A(x_1) \geq \gamma_{h(A)}(y_1) \geq \beta, \psi_A(x_1) \leq \psi_{h(A)}(y_1) \leq \delta,$$
$$\mu_A(x_2) \geq \mu_{h(A)}(y_2) \geq \alpha, \gamma_A(x_2) \geq \gamma_{h(A)}(y_2) \geq \beta, \psi_A(x_2) \leq \psi_{h(A)}(y_2) \leq \delta.$$

This implies that

$$\min\{\mu_A(x_1), \mu_A(x_2)\} \geq \alpha,$$
$$\min\{\gamma_A(x_1), \gamma_A(x_2)\} \geq \beta,$$
$$\max\{\psi_A(x_1), \psi_A(x_2)\} \leq \delta.$$

Hence,

$$\mu_A(y_1y_2^{-1}) \geq \min\{\mu_A(y_1), \mu_A(y_2)\} \geq \alpha,$$
$$\gamma_A(y_1y_2^{-1}) \geq \min\{\gamma_A(y_1), \gamma_A(y_2)\} \geq \beta,$$
$$\psi_A(y_1y_2^{-1}) \leq \max\{\psi_A(y_1), \psi_A(y_2)\} \leq \delta.$$

Therefore,

$$\mu_A(y_1y_2^{-1}) \geq \alpha, \gamma_A(y_1y_2^{-1}) \geq \beta, \psi_A(y_1y_2^{-1}) \leq \delta.$$

It follows that

$$\begin{aligned} y_1y_2^{-1} \in C_{\alpha,\beta,\delta}(A) &\Rightarrow h(y_1y_2^{-1}) \in h(C_{\alpha,\beta,\delta}(A)) \subseteq C_{\alpha,\beta,\delta}(h(A)) \\ &\Rightarrow h(y_1)h(y_2^{-1}) \in C_{\alpha,\beta,\delta}(h(A)) \\ &\Rightarrow y_1y_2^{-1} \in C_{\alpha,\beta,\delta}(h(A)). \end{aligned}$$

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Hence, $C_{\alpha,\beta,\delta}(h^{-1}(B))$ is a subgroup of $\mathcal{G}_1$ for all $\alpha,\beta,\delta \in (0,1]$ with $0<\alpha+\beta+\delta \leq 1$. Let $h(x_1),h(x_2) \in C_{\alpha,\beta,\delta}(h(A))$. Then there exists $C_{\delta,\theta,\omega}(A)$ such that $x_1,x_2 \in C_{\delta,\theta,\omega}(A)$, where $\delta,\theta,\omega \in (0,1]$ and $0 < \delta+\theta+\omega \leq 1$. Since $C_{\alpha,\beta,\delta}(A)$ is an Abelian group, we get $x_1x_2 = x_2x_1$ and so $h(x_1)h(x_2) = h(x_1x_2) = h(x_2x_1) = h(x_2)h(x_1)$, that is, $y_1y_2 = y_2y_1$. Thus, $C_{\alpha,\beta,\delta}(h(A))$ is an Abelian subgroup of $\mathcal{G}_2$. Hence, $h(A)$ is a neutrosophic Abelian subgroup of $\mathcal{G}_2$. □

Theorem 2.24. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a homomorphism of a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$. Let B be a neutrosophic cyclic subgroup of $\mathcal{G}_2$. Then $h^{-1}(B)$ is neutrosophic cyclic subgroup of $\mathcal{G}_1$.*

Proof. Since B is neutrosophic cyclic subgroup of $\mathcal{G}_2$, we have $C_{\alpha,\beta,\delta}(B)$ is a cyclic subgroup of $\mathcal{G}_2$ for all $\alpha,\beta,\delta \in (0,1]$ with $0<\alpha+\beta+\delta \leq 1$. Let $C_{\alpha,\beta,\delta}(B) = \langle g_2 \rangle$ for some $g_2 \in \mathcal{G}_2$. Now, for $g_2 \in \mathcal{G}_2$, there exists $g_1 \in \mathcal{G}_1$ such that $h(g_1) = g_2$. Thus, $C_{\alpha,\beta,\delta}(B) = \langle f(g_1) \rangle$. So, $h^{-1}(C_{\alpha,\beta,\delta}) = C_{\alpha,\beta,\delta}(h^{-1}(B)) = \langle g_1 \rangle$. Hence, $h^{-1}(B)$ is a neutrosophic cyclic subgroup of $\mathcal{G}_1$. □

Theorem 2.25. *Let $h : \mathcal{G}_1 \to \mathcal{G}_2$ be a surjective homomorphism of a group $\mathcal{G}_1$ onto a group $\mathcal{G}_2$. Let A be a neutrosophic cyclic subgroup of $\mathcal{G}_1$. Then $h(A)$ is a neutrosophic cyclic subgroup of $\mathcal{G}_2$.*

Proof. Let A be a neutrosophic cyclic subgroup of $\mathcal{G}_1$. Therefore, $C_{\alpha,\beta,\delta}(A)$ is a cyclic subgroup of $\mathcal{G}_1$ for all $\alpha,\beta,\delta \in (0,1]$ with $0<\alpha+\beta+\delta \leq 1$. We shall show that $h(A)$ is a neutrosophic cyclic subgroup of $\mathcal{G}_2$. Let $g \in C_{\alpha,\beta,\delta}(f(A))$. As h is surjective, therefore, let $g = h(g_1)$ for some $g_1 \in \mathcal{G}_1$. As $g_1 \in \mathcal{G}_1$, we can find one $C_{\alpha,\beta,\delta}(A)$ which exists for all $g_1 \in \mathcal{G}_1$ and hence, for all $g \in C_{\alpha,\beta,\delta}(h(A))$ such that $g_1 \in C_{\alpha,\beta,\delta}(A)$. Since $C_{\alpha,\beta,\delta}(A)$ is a cyclic subgroup of $\mathcal{G}_1$, let $C_{\alpha,\beta,\delta}(A) = \langle g_1 \rangle$. So, $g_1 = g^n$. Thus, $g = h(g_1)h((g_1)^n) = (h(g_1))^n$, that is, $C_{\alpha,\beta,\delta}(h(A))$ is a cyclic subgroup of $\mathcal{G}_2$. Hence, $h(A)$ is a neutrosophic cyclic subgroup of $\mathcal{G}_2$. □

Definition 2.26. The support of a neutrosophic set A of X is defined to be

$$\text{supp}_X(A) = \{x \in X \mid \mu_A(x) > 0, \gamma(x) > 0, \psi_A(x) < 1\}.$$

Clearly, $\text{supp}_X(A)$ is $\bigcup\{C_{\alpha,\beta,\delta}(A) \mid$ for all $\alpha,\beta,\delta \in (0,1]$ such that $\alpha+\beta+\delta \leq 1\}$.

Proposition 2.27. *For a function $f : X \to Y$ and neutrosophic sets A and B of X and Y, respectively, we have*

(1) $f(\text{supp}_X(A)) \subseteq \text{supp}_Y(f(A))$, *equivalently holds if f is bijective,*

(2) $f^{-1}(\text{supp}_Y(B)) = \text{supp}_X(f^{-1}(B))$.

Proposition 2.28. *If A is a non-zero neutrosophic subgroup of a group $\mathcal{G}$, then $\text{supp}_G(A)$ is a subgroup of $\mathcal{G}$.*

The following example shows that the converse of Proposition 2.28 is untrue.

Example 2.29. Let $\mathcal{G} = (\mathbb{R},+)$ be a group of real numbers under addition. Define the neutrosophic set A on $\mathcal{G}$ by

$$\mu_A(x) = \begin{cases} 0.31 & \text{if } x = 0 \\ 0.72 & \text{if } x \in \mathbb{Q}-\{0\} \\ 0 & \text{if } x \in \mathbb{R}-\mathbb{Q}, \end{cases}$$

$$\gamma_A(x) = \begin{cases} 0.21 & \text{if } x = 0 \\ 0.62 & \text{if } x \in \mathbb{Q}-\{0\} \\ 0 & \text{if } x \in \mathbb{R}-\mathbb{Q}, \end{cases}$$

$$\psi_A(x) = \begin{cases} 0.51 & \text{if } x = 0 \\ 0.22 & \text{if } x \in \mathbb{Q}-\{0\} \\ 1 & \text{if } x \in \mathbb{R}-\mathbb{Q}. \end{cases}$$

Clearly, A is not a neutrosophic subgroup of $\mathcal{G}$, but $\text{supp}_G(A) = \mathbb{Q}$ is a subgroup of $\mathcal{G}$.

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Proposition 2.30. *If A is a neutrosophic normal subgroup of a group $\mathcal{G}$, then $\text{supp}_G(A)$ is a normal subgroup of $\mathcal{G}$.*

The following example shows that the converse of Proposition 2.30 is untrue.

Example 2.31. Let $G = S_3 = \{e, a, a, b, ab, ab\}$, where $b = c = a$ be the symmetric group on 3 symbols. Define the neutrosophic set A on $\mathcal{G}$ by

$$\mu_A(x) = \begin{cases} \frac{1}{2} & \text{if } x = e \\ \frac{1}{2} & \text{if } x = b \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

$$\gamma_A(x) = \begin{cases} \frac{1}{2} & \text{if } x = e \\ \frac{3}{4} & \text{if } x = b \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

$$\psi_A(x) = \begin{cases} 0 & \text{if } x = e \\ \frac{1}{3} & \text{if } x = b \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Clearly, A is a neutrosophic subgroup of $\mathcal{G}$ and $\text{supp}_G(A) = S_3$ is normal in $\mathcal{G}$. But A is not a neutrosophic normal subgroup of $\mathcal{G}$, for $C_{\frac{1}{2},\frac{1}{2},1} = \{x \in \mathcal{G} \mid \mu_A(x) \geq \frac{1}{2}, \gamma_A(x) \geq \frac{1}{2}, \psi_A(x) \leq 1\} = \{e, b\}$ is not normal in $\mathcal{G}$.

Theorem 2.32. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic subgroup of a group $\mathcal{G}$. Then A is a neutrosophic Abelian subgroup of $\mathcal{G}$ if and only if $\text{supp}_G(A)$ is an Abelian (cyclic) subgroup of $\mathcal{G}$.*

Proof. If $\text{supp}_G(A)$ is an Abelian subgroup of $\mathcal{G}$, then the result follows as $C_{\alpha,\beta,\delta} \subseteq \text{supp}_G(A)$ for $\alpha, \beta, \delta \in (0, 1]$ such that $\alpha + \beta + \delta < 1$.

Conversely, let A be a neutrosophic Abelian subgroup of $\mathcal{G}$. Let $a, b \in \text{supp}_G(A)$. Then $a \in C_{\alpha_1,\beta_1,\delta_1}(A)$ and $b \in C_{\alpha_2,\beta_2,\delta_2}(A)$ for some $\alpha_i, \beta_i, \delta_i \in (0, 1]$ such that $\alpha_i + \beta_i + \delta_i < 1$, where $i = 1, 2$.

Case i: When $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ and $\delta_1 > \delta_2$, $a, b \in C_{\alpha_1,\beta_1,\delta_1}(A)$ and $ab = ba$.

Case ii: When $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$ and $\delta_1 < \delta_2$, $a, b \in C_{\alpha_1,\beta_2,\delta_2}(A)$ and $ab = ba$.

Other cases can similarly be dealt with. That is, when A is a neutrosophic cyclic subgroup of $\mathcal{G}$, $\text{supp}_G(A)$ is cyclic and can be proved on the same lines. □

Definition 2.33. If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic set of a group $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then the restriction of A on H is denoted by $A|H$ is a neutrosophic set on H defined as

$$(A|H)(x) = (\mu_{A|H}(x), \gamma_{A|H}(x), \psi_{A|H}(x)),$$

where $\mu_{A|H}(x) = \mu_A(x)$, $\gamma_{A|H}(x) = \gamma_A(x)$ and $\psi_{A|H}(x) = \psi_A(x)$.

The proof of the following propositions is easy and hence omitted.

Proposition 2.34. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set of a group $\mathcal{G}$. Then we have the following:*

(1) *If A is a neutrosophic subgroup of $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then $A|H$ is a neutrosophic subgroup of H.*

(2) *If $A|H$ is the restriction of the neutrosophic set A of $\mathcal{G}$ on the subgroup H of $\mathcal{G}$, then $\text{supp}_H(A|H) = \text{supp}_G(A) \cap H$.*

(3) *IF A is a cyclic neutrosophic subgroup of $\mathcal{G}$ and H is a subgroup of $\mathcal{G}$, then $A|H$ is a cyclic neutrosophic subgroup of H if and only if H is a cyclic subgroup of $\mathcal{G}$.*

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