A New Approach for Solving Singularly Perturbed Boundary Value Problems by Using Exponential Spline Method

Ahmed R. Khlefha [*1]

[1] Department of Mathematics, College of Education, University of Sumer, Thi-Qar, Iraq.
Emails: arkdsh85@gmail.com

Abstract

This article presents the development of families of approaches for numerically solving singularly perturbed two-point boundary-value problems using exponential spline functions. The proposed approaches exhibit second-order and fourth-order accuracy and are suitable for both singular and non-singular problem scenarios. Numerical data are presented to demonstrate the efficacy of our methodologies and are compared with those proposed by various writers.

Keywords: Exponential spline functions; boundary value problems, absolute errors, Convergence Analysis.

1. Introduction

We study a boundary value problem for the second order that is singularly perturbed, of the following form [1]:

$$\varepsilon\tilde{u}'' = g(\ddot{\eta})\tilde{u} + r(\ddot{\eta}) \ , \quad \ddot{\eta} \in [a, b] \tag{1}$$

with boundary conditions

$$u(a) = \hat{\lambda}_1 \text{ and } u(b) = \hat{\lambda}_2 \tag{2}$$

Given constants λ_1 , and λ_2, a tiny positive parameter ε $0 \le \varepsilon \le 1$, and bounded continuous functions $g(b), r(x)$. The boundary value problem at both ends of the interval is influenced by the properties of the function $r(\ddot{\eta})$. These issues occur in various domains of applied mathematics and engineering. For instance, consider the arena-vier stokes flow characterized by high Reynolds numbers and heat transfer difficulties involving Peclet numbers. Due to the existence of boundary value concerns, we are encountering challenges in tackling these particular problems utilizing numerical approaches involving homogeneous arrays. To achieve a precise estimation, it is necessary to have a dense grid in the boundary layer region. This article presents a novel approach using exponential spline functions to approximate the solution of problem (1) with fourth-order accuracy while satisfying the boundary requirements (2). A multitude of numerical approaches have been devised that rely on solving singular perturbed value problems, particularly those with boundary values at one or both ends of the interval.El-Zahar and El-Kabeir used An innovative approach for resolving boundary value problems with singular perturbations. Phaneendra and Emineni [2] employed the Variable mesh nonpolynomial spline approach to address singular perturbation situations that display twin layers. Ali and Hadhoud [3] conducted a numerical investigation on self-adjoint singly perturbed two-point boundary value problems using the collocation method. They also provided an error estimation. Rashidinia and Mohammad [4] employed quintic spline algorithms to solve singularly perturbed boundary-value problems. Furthermore, numerous studies have detailed

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

the utilization of splines for the numerical resolution of singularly perturbed boundary value problems [5–11]. This article presents a novel approach to solving problem (1) by developing a variable mesh finite difference scheme that utilizes exponential splines. The method is proven to converge consistently. The concept involves employing a first-order continuity clause, variable mesh derivatives, exponential spline, and discretization equation at the interior nodes to address the problem. The key characteristic of our technology is its ability to achieve high resolution while maintaining computational efficiency and ease of implementation on a computer. This article introduces the exponential spline approach and presents the formulation of our spline function approximation and truncation error in section 2. The application of these methods to two instances is provided in section three, presents Convergence Analysis in section four, while the numerical solutions are explained in section five, Finally, the conclusion was mentioned in section six.

2. Derivation of the Method

To simplify, we choose the break point of the interval. $[a,b]$ at $c=\frac{3a+b}{4}$ and $c=\frac{a+3b}{4}$ to build a numerical approach for approximating the solution of issue (1). interval $[a,b]$ was partitioned into n equal subintervals using the point$x_i = a + ih, i = 0,1,2,\dots,n-1,n$, and n is an arbitrary positive integer here, $a=\ddot{\eta}_0$, $b=\ddot{\eta}_n$ and $h=\frac{b-a}{n}$. Let $u(\ddot{\eta})$ represent the exact solution and u_i represent an approximation to $u(\ddot{\eta}_i)$ obtained using the exponential spline $E_i(\ddot{\eta})$ that passes through the points $(\ddot{\eta}_i, u_i)$ and $(\ddot{\eta}_{i+1}, u_{i+1})$. It is not only necessary for $E_i(\ddot{\eta})$ to satisfy the interpolator's conditions at x_i and x_{i+1}, but also for the continuity of the first derivative to be fulfilled at the shared nodes $(\ddot{\eta}_i, u_i)$.The expression $E_i(\ddot{\eta})$ is written in the following format:

$$E_i(\ddot{\eta}) = a_i e^{\breve{\omega}(\ddot{\eta}-\ddot{\eta}_i)} + b_i e^{-\breve{\omega}(\ddot{\eta}-\ddot{\eta}_i)} + c_i(\ddot{\eta}-\ddot{\eta}_i) + d_i \tag{3}$$

where $a_i, b_i, c_i$and $d_i$are constants and $\breve{\omega}$ is free parameter to be determined later The exponential spline function$E(\ddot{\eta})$ of class $C^2[a,b]$ interpolated$u(\ddot{\eta})$ at the network points $x_i, i = 0,1,2,\dots,n$ depended on a parameter $\breve{\omega}$.

To derive an expression for the coefficient of equation (2) in term $\tilde{u}_{i+\frac{1}{2}}, D_i, N_{i+\frac{1}{2}}, \Psi_i$and Ψ_{i+1}, we first define:

$$E_i\left(\ddot{\eta}_{i+\frac{1}{2}}\right) = \tilde{u}_{i+\frac{1}{2}} \ , \ E_i^{(1)}(\ddot{\eta}_i) = \mathcal{J}_i$$

$$E_i^{(2)}(\ddot{\eta}_{i+1}) = \tilde{\mathcal{F}}_{i+\frac{1}{2}} \ , E_i^{(3)}(\ddot{\eta}_i) = \frac{1}{2}[\Psi_{i+1} + \Psi_i]$$

Through algebraic manipulation, we derive the following expression:

$$a_i = \frac{h^3(\Psi_{i+1}+\Psi_i)e^{\frac{-\theta}{2}} + 2h^2\theta\tilde{\mathcal{F}}_{i+\frac{1}{2}}}{2\theta^3\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}$$

$$b_i = \frac{2h^2\theta\tilde{\mathcal{F}}_{i+\frac{1}{2}} - h^3(\Psi_{i+1}+\Psi_i)e^{\frac{\theta}{2}}}{2\theta^3\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}$$

$$c_i = \frac{2\mathcal{J}_i\theta^2 - h^2(\Psi_{i+1}+\Psi_i)}{2\theta^2}$$

$$d_i = \frac{\theta^2\tilde{u}_{i+\frac{1}{2}} - h^2\tilde{\mathcal{F}}_{i+\frac{1}{2}}}{\theta^2} - \frac{h^2\left(2\mathcal{J}_i\theta^2 - h(\Psi_{i+1}+\Psi_i)\right)}{2\theta^2}$$

Where $\theta = \breve{\omega}h$ and $i = 0,1,2,\dots,n$.

By evaluating the derivative at point $(\ddot{\eta}_i, \tilde{u}_i)$,, namely when $E_{i-1}^{(m)}(\ddot{\eta}_i) = E_i^{(m)}(\ddot{\eta})$ and $m = 0,1.2$, we obtain the subsequent consistency relations for $i = 1,\dots,n$.

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

$$\mathcal{J}_i + \mathcal{J}_{i-1} = \frac{2h\left(2 - e^{\frac{\theta}{2}} - e^{\frac{-\theta}{2}}\right)\tilde{\mathcal{F}}_{i+\frac{1}{2}}}{\theta^2(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})} + \frac{2h\left(e^{\frac{\theta}{2}} - e^{\frac{-\theta}{2}} - e^{-\theta} - e^{\theta}\right)\tilde{\mathcal{F}}_{i-\frac{1}{2}}}{\theta^3(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})} - 2h\tilde{u}_{i+\frac{1}{2}}$$

$$-2h\tilde{u}_{i-\frac{1}{2}} + \frac{h^2\left(2\left(e^{\frac{\theta}{2}} - e^{\frac{-\theta}{2}}\right) + \theta\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)\right)(\Psi_{i-1} + 2\Psi_i + \Psi_{i+1})}{2\theta^3\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)} \tag{4}$$

$$\mathcal{J}_i + \mathcal{J}_{i-1} = \frac{h\left(e^{\frac{\theta}{2}} - e^{\frac{-\theta}{2}}\right)\tilde{\mathcal{F}}_{i-\frac{1}{2}}}{\theta} \tag{5}$$

and

$$\frac{h\left(\left(e^{\frac{\theta}{2}} - e^{\frac{-\theta}{2}}\right)\right)(\Psi_{i-1} + 2\Psi_i + \Psi_{i+1})}{2\theta^3(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})} = \frac{2N_{i+1}}{(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})} - \frac{(e^{\theta} + e^{-\theta})\tilde{\mathcal{F}}_{i-\frac{1}{2}}}{\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)} \tag{6}$$

From Equations. (4) to (6) we get the following:

$$\left(\frac{2(h^2+\theta^2) - h^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}{\theta^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}\right)\tilde{\mathcal{F}}_{i-\frac{3}{2}} + 2\left(\frac{\left((\theta^2+2) - 2\theta^2\right)(e^{\theta} + e^{-\theta}) - 2h^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}{2\theta^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}\right)\tilde{\mathcal{F}}_{i-\frac{1}{2}} +$$

$$\left(\frac{2(h^2+\theta^2) - h^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}{\theta^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}\right)\tilde{\mathcal{F}}_{i+\frac{1}{2}} \tag{7}$$

which can further be written as,

$$\tilde{u}_{i-\frac{3}{2}} - \tilde{u}_{i-\frac{1}{2}} + \tilde{u}_{i+\frac{1}{2}} = \hat{\mu}\tilde{\mathcal{F}}_{i-\frac{3}{2}} + 2\hat{\delta}\tilde{\mathcal{F}}_{i-\frac{1}{2}} + \hat{\mu}\tilde{\mathcal{F}}_{i+\frac{1}{2}} \tag{8}$$

Where

$$\hat{\mu} = \frac{2(h^2 + \theta^2) - h^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}{\theta^2(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})}$$

$$\hat{\delta} = \frac{\left((\theta^2 + 2) - 2\theta^2\right)\left(e^{\theta} + e^{-\theta}\right) - 2h^2\left(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}}\right)}{2\theta^2(e^{\frac{\theta}{2}} + e^{\frac{-\theta}{2}})}$$

Equation (8) provides $n-1$ algebraic linear equations for the unknowns $\tilde{u}_{i+\frac{1}{2}}, 0,1,2,\ldots,n-1$. To directly compute $\tilde{u}_{i+\frac{1}{2}}, 0,1,2,\ldots,n-1$, two additional equations are required, one at either end of the integration range. The two equations are derived using the Taylor series and the method of indeterminate coefficients. These equations exist.

$$2\tilde{u}_0 - 3\tilde{u}_{\frac{1}{2}} + \tilde{u}_{\frac{3}{2}} = \left(\Phi_0\tilde{\mathcal{F}}_0 + \Phi_1\tilde{\mathcal{F}}_{\frac{1}{2}} + \Phi_2\tilde{\mathcal{F}}_{\frac{3}{2}} + \Phi_3\tilde{\mathcal{F}}_{\frac{5}{2}}\right) \quad at\ i = 1 \tag{9}$$

$$2\tilde{u}_n - 3\tilde{u}_{n-\frac{1}{2}} + \tilde{u}_{n-\frac{1}{3}} = \left(\Phi_0\tilde{\mathcal{F}}_n + \Phi_1\tilde{\mathcal{F}}_{n-\frac{1}{2}} + \Phi_2\tilde{\mathcal{F}}_{n-\frac{3}{2}} + \Phi_3\tilde{\mathcal{F}}_{n-\frac{5}{2}}\right) at\ i = n \tag{10}$$

The local truncation errors associated with equations (8), (9), and (10) are denoted as t_i for $i = 1,2,\ldots.,\mathrm{n}-1$.

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

$$t_i = \begin{cases} -\frac{1}{64}h^4\tilde{u}_i^{(5)} + O(h^5), \quad i=1. \\ h^2(1-2\hat{\mu}-\hat{\delta})\tilde{u}_i^{(2)} + h^3\left(\frac{1}{2}-\hat{\mu}+\frac{\hat{\delta}}{2}\right)\tilde{u}_i^{(3)} + h^4\left(\frac{-80}{834}-\frac{7}{4}\hat{\mu}+\frac{\hat{\delta}}{8}\right)\tilde{u}_i^{(4)} \\ +h^3\left(\frac{1}{16}-\frac{13\hat{\mu}}{24}-\frac{\hat{\delta}}{48}\right)\tilde{u}_i^{(5)} + h^6\left(\frac{85}{6534}-\frac{82\hat{\mu}}{415}-\frac{\hat{\delta}}{415}\right)\tilde{u}_i^{(6)} + O(h^7), \quad i=2,3,\dots,n-1. \\ \frac{-h^4}{64}\tilde{u}_i^{(5)} + O(h^5), \quad i=n. \end{cases}$$

Equations (8) to (10) introduce a set of approaches based on the selection of $\hat{\mu}, \hat{\delta}$ and Φ. Class of methods for assigning values to $(\hat{\mu}, \hat{\delta}) = (0.5, 0.833)$.

1- Method with third order convergence for $(\Phi_0, \Phi_1, \Phi_2)(-0.0416, 0.687, 0.1041)$, the local truncation mistake is:

$$t_i = \begin{cases} -\frac{1}{96}h^5\tilde{u}_i^{(5)} + O(h^6), \quad i = 1, n \\ \frac{-1}{240}h^6\tilde{u}_i^{(6)} + O(h^7), \quad i = 2,3, \dots, n-1 \end{cases} \tag{11}$$

Method with fourth order convergence for $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) = \left(\frac{-1}{120}, \frac{5}{8}, \frac{7}{48}, \frac{-1}{80}\right)$, the local truncation mistake is:

$$t_i = \begin{cases} \frac{19}{5120}h^6\tilde{u}_i^{(6)} + O(h^7), \quad i = 1, n \\ \frac{-1}{240}h^6\tilde{u}_i^{(6)} + O(h^7), \quad i = 2,3, \dots, n-1 \end{cases} \tag{12}$$

3. Exponential Spline solution

The linear equations that are supplied by equations (8-10) are the foundation for the spline solution to the problem (1) with the condition defining the boundary (2). Let us now consider the following:

$\tilde{U} = \left(\tilde{u}_{i-\frac{1}{2}}\right)$, $\bar{U} = \left(\bar{\tilde{u}}_{i-\frac{1}{2}}\right)$, $T = \left(t_{i-\frac{1}{2}}\right)$ and $E = \left(e_{i-\frac{1}{2}}\right) = \left(\tilde{U}_{i-\frac{1}{2}} - \bar{\tilde{U}}_{i-\frac{1}{2}}\right)$ for $i = 1,2$, Be n-

If we consider column vectors in several dimensions, we may express the conventional matrix equations for the exponential spline approach as follows.

$\mathcal{D} = \bar{Z} + T$

$$\mathcal{D}\bar{\tilde{U}} = \bar{Z} \tag{13}$$

$\mathcal{D}(\tilde{U} - \bar{\tilde{U}}) = T$

$$\mathcal{D}E \doteq T \tag{14}$$

Also,

$\mathcal{D} = \Psi_0 + h^4\mathcal{H}\tilde{U}$

The three-band symmetric matrix N0 has the form:

$$\Psi_0 = \begin{bmatrix} -1 & 1 & 0 & \dots & & & 0 \\ 1 & -2 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & -2 & 1 & & & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & -2 & 1 \\ 0 & 0 & \cdots & & -1 & 1 & 1 \\ 0 & 0 & \cdots & & \dots & & \end{bmatrix}$$

The matrix $\mathcal{H}$ has the form:

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

$$\mathcal{H} = \begin{bmatrix} \Phi_0 & \Phi_1 & \Phi_2 & \Phi_3 & & 0 \\ \hat{\mu} & \delta & \mu & 0 & \cdots & 0 \\ 0 & \hat{\mu} & \delta & \mu & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 \quad 0 & \ddots & \ddots & \ddots & & \\ 0 \quad 0 & \cdots & \hat{\mu} & & \delta & \hat{\mu} \\ 0 \quad 0 & \cdots & \cdots & & \delta & \hat{\mu} \end{bmatrix}$$

Regarding vector Z,

$$\bar{Z}_i = \begin{cases} h\breve{\mathcal{G}} + h^2\left(\Phi_0 q_{\frac{1}{2}} + \Phi_1 q_{\frac{3}{2}} + \Phi_2 q_{\frac{5}{2}} + \Phi_3 q_{\frac{7}{2}}\right) \\ h^2\left(\mu q_{i-\frac{3}{2}} + \delta q_{i-\frac{1}{2}} + \mu q_{i+\frac{1}{2}}\right) \\ -h\breve{\mathcal{G}} + h^2\left(\Phi_0 q_{n-\frac{1}{2}} + \Phi_1 q_{n-\frac{3}{2}} + \Phi_2 q_{n-\frac{5}{2}} + \Phi_3 q_{n-\frac{7}{2}}\right) \end{cases}$$

Set $\Psi_0 = \mathcal{N}_0 + \mathcal{V}_0$

Where,

$$\mathcal{N}_0 = \begin{bmatrix} -3 & 1 & 0 & \ldots & & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 \quad 0 & \ddots & \ddots & \ddots & -2 & 1 \\ 0 \quad 0 & \cdots & -1 & & & \\ 0 \quad 0 & \cdots & \ldots & & 1 & -3 \end{bmatrix}$$

And

$$\mathcal{V}_0 = \begin{bmatrix} 2 & 0 & 0 & \ldots & & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 \quad 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 \quad 0 & \cdots & 0 & & & \\ 0 \quad 0 & \cdots & \ldots & & 0 & 2 \end{bmatrix}$$

4. Convergence Analysis

Our primary objective at the moment is to establish a limit for $\|E\|_\infty$. Let's revisit error equation (14) in (13) and express it in a different form.

$$E = \mathcal{D}^{-1}T = \left[\mathcal{N}_0 + \mathcal{V}_0 + h^2\mathcal{H}\tilde{U}\right]^{-1}T = \left(I + \mathcal{N}_0{}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\tilde{U}\right)\right)\mathcal{N}_0 T$$

This implies that

$$\|E\|_\infty = \left\|\left(I + \mathcal{N}_0{}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\tilde{U}\right)\right)^{-1}\right\|_\infty \left\|\mathcal{N}_0{}^{-1}\right\|_\infty \|T\|_\infty \tag{15}$$

To determine the bound on $\|E\|_\infty$,, we require the following two lemmas.

Lemma 1: If Σ is a square matrix of a certain size n and $\|\Sigma\| < 1$, then the $(1+\Sigma)^{-1}$ exists and. $\|(I+\Sigma)^{-1}\| \leq (1 - \|\Sigma\|)^{-1}$

Lemma 2: The matrix $(\mathcal{N}_0 + \mathcal{V}_0 + h^2\mathcal{H}\tilde{u})$ is nonsingular if $\|\tilde{u}\|_\infty < \frac{h^2 - 2\ell}{h^2\ell(2\mu+\delta)}$,where

$$\ell = \frac{1}{8}\left((a-b)^2 + h^2\right)\left(2 + h^2(2\mu+\delta)\left\|\tilde{U}\right\|_\infty\right)$$

Proof:

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

Since, $\mathcal{D} = \left(\mathcal{N}_0 + \mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right) = \left(I + {\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right)\mathcal{N}_0$ and the matrix $\mathcal{N}_0$ is nonsingular, to prove that $\mathcal{D}$ is nonsingular, it is sufficient to demonstrate

$\left(I + {\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right)\mathcal{N}_0$ nonsingular. Moreover,

$\left\|\widetilde{\mathrm{U}}\right\|_\infty \leq \|\tilde{u}\|_\infty = max_{a\leq x\leq b}|\tilde{u}(x)|$

$\left\|{\mathcal{N}_0}^{-1}\right\|_\infty < \frac{(a-b)^2+h^2}{8h^2}$

$\|\mathcal{V}_0\|_\infty = 2$

$\|\mathcal{H}\|_\infty = 2\hat{\mu} + \hat{\delta}$

Also

$\left\|{\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty \leq \left\|{\mathcal{N}_0}^{-1}\right\|_\infty\left(\left\|\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty\right) \leq \left\|{\mathcal{N}_0}^{-1}\right\|_\infty\left(\|\mathcal{V}_0\|_\infty + h^2\|\mathcal{H}\|_\infty\left\|\widetilde{\mathrm{U}}\right\|_\infty\right)$
(16)

Thus, replacing $\|\tilde{u}\|_\infty$, $\left\|{\mathcal{N}_0}^{-1}\right\|_\infty$, $\|\mathcal{V}_0\|_\infty$ and $\|\mathcal{H}\|_\infty$ in equation (16), we get

$$\left\|{\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty \leq \left(\frac{(a-b)^2 + h^2}{8h^2}\right)\left(2 + h^2\left(2\hat{\mu} + \hat{\delta}\right)\left\|\widetilde{\mathrm{U}}\right\|_\infty\right)$$

$$\|\tilde{u}\|_\infty < \frac{h^2-2\ell}{h^2\ell\left(2\hat{\mu}+\hat{\delta}\right)} \quad (17)$$

Therefor equation (17) leads

$\left\|{\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty \leq 1$

The no singularity of matrix $\mathcal{D}$ is demonstrated by lemma 1. Given that $\left\|{\mathcal{N}_0}^{-1}\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty \leq 1$, it may be inferred by applying lemma (1) and equation (14), we get that,

$\|E\|_\infty \leq \frac{\left\|{\mathcal{N}_0}^{-1}\right\|_\infty\|T\|_\infty}{1-\left\|{\mathcal{N}_0}^{-1}\right\|_\infty\left\|\left(\mathcal{V}_0+h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty}$.

From Equation. (11) we have

$\|T\|_\infty = \frac{427}{7560}h^2M_5,\ M_5 = max_{a\leq x\leq b}\left|y^{(5)}(\ddot{\eta}_i)\right|$

Then

$$\|E\|_\infty \leq \frac{\left\|{\mathcal{N}_0}^{-1}\right\|_\infty\|T\|_\infty}{1 - \left\|{\mathcal{N}_0}^{-1}\right\|_\infty\left\|\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty} \cong O(h^3)$$

Also, from Eq. (12) we have

$\|T\|_\infty = \frac{32}{675}h^6M_6,\ M_6 = max_{a\leq x\leq b}\left|y^{(6)}(\ddot{\eta}_i)\right|$

$$\|E\|_\infty \leq \frac{\left\|{\mathcal{N}_0}^{-1}\right\|_\infty\|T\|_\infty}{1 - \left\|{\mathcal{N}_0}^{-1}\right\|_\infty\left\|\left(\mathcal{V}_0 + h^2\mathcal{H}\widetilde{\mathrm{U}}\right)\right\|_\infty} \cong O(h^4)$$

5. Numerical Results

We solve two problems that are sensitive to small changes by using different values of h and ε. numerical solutions is computed and then evaluated against precise solutions at a particular grade point. Maple 22 is in charge of performing the calculations.

Example 1: Consider the singularly perturbed boundary value problem

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

$\varepsilon\tilde{u}^{(2)}(\ddot{\eta}) - \tilde{u}(\ddot{\eta}) = cos^2(\pi\ddot{\eta}) + 2\varepsilon\pi^2 \cos(2\,\pi\ddot{\eta})$

With boundary condition,

$\tilde{u}(0) = 0\ , \tilde{u}(1) = 0.$

The precise solution is as follows:

$$\tilde{u}(\ddot{\eta}) = \cos(\pi\ddot{\eta}) + x + \frac{e^{\left(\frac{x-1}{\sqrt{\varepsilon}}\right)} + e^{-\left(\frac{x}{\sqrt{\varepsilon}}\right)}}{1 + e^{-\left(\frac{1}{\sqrt{\varepsilon}}\right)}} - cos^2(\pi\ddot{\eta})$$

The problem was addressed using approach (8) by examining various numbers for n =16, 32, 64, 128, and 256, along with ε values of 0.0625, 0.03125, 0.015625, and 0.0078125. Tables 1-2, displays the highest absolute errors in solutions and juxtaposes them with the findings in [4], showcasing the accuracy of our method.

Example 2: Consider the singularly perturbed boundary value problem

$$\varepsilon\tilde{u}^{(2)}(\ddot{\eta}) - \big(1 + \ddot{\eta}(1-\ddot{\eta})\big)\tilde{u}(\ddot{\eta}) = -1 - \ddot{\eta}(1-\ddot{\eta}) - 2\sqrt{\varepsilon} + \ddot{\eta}(1-\ddot{\eta})^2 e^{\left(\frac{-\ddot{\eta}}{\sqrt{\varepsilon}}\right)}\big(2\sqrt{\varepsilon} - \ddot{\eta}^2(1-\ddot{\eta})\big)e^{\left(-\left(\frac{1-\ddot{\eta}}{\sqrt{\varepsilon}}\right)\right)}$$

With boundary condition,

$\tilde{u}(0) = 0\ , \tilde{u}(1) = 0.$

The precise solution is as follows

$$\tilde{u}(\ddot{\eta}) = 1 + \ddot{\eta}(x-1)e^{\left(\frac{-\ddot{\eta}}{\sqrt{\varepsilon}}\right)} - \ddot{\eta}e^{\left(-\left(\frac{1-\ddot{\eta}}{\sqrt{\varepsilon}}\right)\right)}.$$

This problem was addressed by applying procedure (8) using different values of n =16,32,64,128,256 and ε =0.0625,0.03125,0.015625,0.0078125, resulting in the precise solution. Table 3 lists the largest absolute errors in solutions and compares them with the results from [4] to showcase the accuracy of our approach.

Table 1: The maximum absolute errors in solutions of example1.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$
0.0625	8.02×10^{-8}	0.81×10^{-9}	2.93×10^{-10}	9.86×10^{-12}	2.95×10^{-17}
0.03125	3.24×10^{-7}	2.42×10^{-9}	7.94×10^{-10}	8.64×10^{-12}	5.86×10^{-17}
0.015625	5.71×10^{-7}	4.75×10^{-8}	8.85×10^{-10}	4.77×10^{-12}	8.99×10^{-17}
0.0078125	4.85×10^{-7}	2.03×10^{-8}	7.75×10^{-10}	7.51×10^{-12}	6.24×10^{-17}

Table 2: The maximum absolute errors in solutions of example1 ($n = 256$).

ε	Rashidinia [4]	Our method
0.0625	6.17×10^{-10}	2.95×10^{-17}
0.03125	3.02×10^{-10}	5.86×10^{-17}
0.015625	8.39×10^{-10}	8.99×10^{-17}
0.0078125	3.01×10^{-9}	6.24×10^{-17}

Table 3: The maximum absolute errors in solutions of example 2.

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$
0.0625	1.51×10^{-16}	3.62×10^{-17}	7.91×10^{-18}	5.94×10^{-20}	8.17×10^{-22}
0.03125	1.48×10^{-16}	2.81×10^{-17}	3.85×10^{-18}	4.77×10^{-20}	6.22×10^{-22}
0.015625	0.92×10^{-16}	3.92×10^{-17}	5.37×10^{-18}	3.72×10^{-19}	5.83×10^{-22}
0.0078125	0.97×10^{-16}	1.63×10^{-17}	6.63×10^{-18}	7.35×10^{-19}	8.24×10^{-22}

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024

Table 4: The maximum absolute errors in solutions of example2 ($n = 32$).

ε	Rashidinia [4]	our method
1×10^{-1}	1.846×10^{-4}	3.71×10^{-19}
1×10^{-2}	2.682×10^{-4}	3.90×10^{-19}
1×10^{-3}	2.642×10^{-4}	2.89×10^{-19}
1×10^{-4}	1.043×10^{-4}	2.78×10^{-19}

6. Conclusion

A new exponential spline technique is created to address singularly perturbed boundary value problems, taking into account the boundary condition. Our technique (8) was found to display third and fourth-order convergence throughout the convergence analysis. This article employs a first-degree exponential spline. The largest absolute errors in Tables 2 to 4 highlight the advantages of our method compared to earlier methods.

References

[1] El-Zahar ER, El-Kabeir SMM. A new method for solving singularly perturbed boundary value problems. Applied Mathematics & Information Sciences 2013;7:927.

[2] Phaneendra K, Emineni SP. Variable mesh non polynomial spline method for singular perturbation problems exhibiting twin layers. Journal of Physics: Conference Series, vol. 1344, IOP Publishing; 2019, p. 12011.

[3] Ali KK, Hadhoud AR, Shaalan MA. Numerical study of self-adjoint singularly perturbed two-point boundary value problems using collocation method with error estimation. Journal of Ocean Engineering and Science 2018;3:237–43.

[4] Rashidinia J, Mohammadi R, Moatamedoshariati SH. Quintic spline methods for the solution of singularly perturbed boundary-value problems. International Journal for Computational Methods in Engineering Science and Mechanics 2010;11:247–57.

[5] Ranjan R, Prasad HS. A novel exponentially fitted finite difference method for a class of 2nd order singularly perturbed boundary value problems with a simple turning point exhibiting twin boundary layers. Journal of Ambient Intelligence and Humanized Computing 2022;13:4207–21.

[6] Ranjan R, Prasad HS. A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shifts. Journal of Applied Mathematics and Computing 2021;65:403–27.

[7] Ahmed S, Jahan S, Ansari KJ, Shah K, Abdeljawad T. Wavelets collocation method for singularly perturbed differential–difference equations arising in control system. Results in Applied Mathematics 2024;21:100415.

[8] Rao SCS, Kumar M. Optimal B-spline collocation method for self-adjoint singularly perturbed boundary value problems. Applied Mathematics and Computation 2007;188:749–61.

[9] Ayalew M, Kiltu GG, Duressa GF. Fitted numerical scheme for second-order singularly perturbed differential-difference equations with mixed shifts. Abstract and Applied Analysis, vol. 2021, Hindawi Limited; 2021, p. 1–11.

[10] Gebeyaw W, Andargie A, Adamu G. Solving second order singularly perturbed delay differential equations with layer behavior via initial value method. Journal of Applied Mathematics & Informatics 2018;36:331–48.

[11] Rashidinia J, Mahmoodi Z. Non-polynomial spline solution of a singularly perturbed boundary-value problems. Int J Contemp Math Sciences 2007;2:1581–6.

Doi: https://doi.org/10.54216/GJMSA.0100202
Received: October 19, 2023 Revised: January 16, 2024 Accepted: March 18, 2024