Interval Valued Neutrosophic Subalgebra in INK-Algebra

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Abstract

This work presents the concept of interval-valued neutrosophic INK-subalgebras, also known as IVN INK-subalg's, which are the level and strong level neutrosophic INK-subalgebras. Next, we establish and validate a few theorems that establish the connection between these concepts and neutrosophic INK-subalgebras. We define the images and inverse images of IVN INK-subalgebras and study the transformations of the homomorphic images and inverse images of interval valud neutrosophic (briefly, IVN) INK-subalgebra into IVN INK-subalgebras.

Keywords: interval valued neutrosophic set; neutrosophic INK-subalgebra and interval valued neutrosophic INK-subalgebra.
AMS (2000) subject classification: 06F35, 03G10, 03B52.

1 Introduction

BCK-algebra and BCI-algebra are two classes of abstract algebras introduced by Imai and Iséki in 1966.[1] The BCK-algebra class is recognized as a legitimate subclass of the BCI-algebra class.[2,7] A concept known as d-algebra was presented by Neggers et al.,.[8] It is a generalization of $BCH/BCI/BCK$-algebras and generalizes several of the theorems covered in BCI-algebra. The concept of INK-algebras was first developed by Kaviyarasu et al.,.[3] The notions of fuzzy INK-algebras were introduced and applied to INK-algebras by Kaviyarasu et al.,.[5] The notion of fuzzy sets was first presented by Zadeh.[10] An $IVFS$, or fuzzy set with an interval-valued membership function, is Zadeh's expansion of the fuzzy set idea in.[10] He created a technique for approximative inference using his $IVFS$s, which he refers to as a $IVFS$. Interval-valued fuzzy INK-subalgebras and neutrosophic set in INK-algebra were defined by Kaviyarasu et al.[4,6] In the current work, we presented the idea of interval-valued neutrosophic INK-subalgebras (abbreviated IVN INK-subalgebras) of a INK-algebra and studied some of its features, utilizing the notion of interval-valued fuzzy set by Zadeh. We demonstrate that each INK-subalgebra of a INK-algebra U may be realized as a IV level INK-subalgebra of a IVN INK-subalgebra of U. From this, we derive several related findings that are specified in the abstract.

2 Preliminaries

Definition 2.1. [4] An algebra $(U, *, 0)$ is a non-null set U with a constant '0' and a single binary operation '$*$' is called INK-algebras if it satisfying the following conditions.

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

(i) $\alpha * \alpha = 0$

(ii) $\alpha * 0 = \alpha$ for $\alpha \in U$

(iii) $0 * \alpha = \alpha$

(iv) $(\beta * \alpha) * (\beta * \gamma) = (\alpha * \gamma)$ for all $\alpha, \beta, \gamma \in U$.

Definition 2.2. [4] Let U be a INK-algebra and $T \subseteq U$. Then T is said to a INK-subalgebra (briefly, INK-subalg) of U, if $\alpha * \beta \in T$, for all $\alpha,\ \beta \in U$.

Definition 2.3. [3] A mapping $f : U \longrightarrow V$ of INK-algebras is called a INK-homomorphism if $f(\alpha * \beta) = f(\alpha) * f(\beta)$ for all $\alpha, \beta \in U$.

Now, we go over a few fuzzy logic concepts (see[10]).

Let U be a set. A fuzzy set τ_1 in U is characterized by a membership function $\mu_{\tau_1} : U \longrightarrow [0, 1]$. Let f be a mapping from the set U to the set V and let τ_2 be a fuzzy set in V with membership function μ_{τ_2}. The inverse image of τ_2, denoted $f^{-1}(\tau_2)$, is the fuzzy set in U with membership function $\mu_{f^{-1}(\tau_2)}$ defined by $\mu_{f^{-1}(\tau_2)}(\alpha) = \mu_{\tau_2}(f(\alpha))$ for all $\alpha \in U$. Conversely, let τ_1 be a fuzzy set in U with membership function μ_{τ_1}. Then the image of τ_1, denoted by $f(\tau_1)$, is the fuzzy set in V such that:

$$\mu_{f(\tau_1)}(\beta) = \begin{cases} \sup\limits_{\gamma \in f^{-1}(\beta)} \mu_{\tau_1}(\gamma) & \text{if} \quad f^{-1}(\beta) = \{\alpha : f(\alpha) = \beta\} \neq 0 \\ 0, \quad \text{otherwise} \end{cases}$$

A fuzzy set τ_1 in the INK-algebra U with the membership function μ_{τ_1} is said to be have the sup property if for any subset $T \subseteq U$ there exists $\alpha_0 \in T$ such that $\mu_{\tau_1}(\alpha_0) = \underset{t \in T}{sup} \quad \mu_{\tau_1}(t)$.

Definition 2.4. [4] A fuzzy subset μ in a INK-algebra U is called a fuzzy INK-subalgebra (briefly, $FINK$-subalg) of U if $\mu(\alpha * \beta) \geq \min\{\mu(\alpha), \mu(\beta)\}$, for all $\alpha, \beta \in U$.

Definition 2.5. [9] A fuzzy subset μ in a INK-algebra U is called a anti fuzzy subalgebra[9] (briefly, $AFINK$-subalg) of U if $\mu(\alpha * \beta) \leq \max\{\mu(\alpha), \mu(\beta)\}$, for all $\alpha, \beta \in U$.

Definition 2.6. [9] Let μ be a fuzzy set of a set U. For a fixed $t \in [0, 1]$, the set $\mu_t = \{\alpha \in U/\mu(\alpha)\}$ is called an upper level of μ.

Definition 2.7. [9] Let U be a fuzzy subset of S. Then for $t \in [0, 1]$, the t-level cut of U is the set $U_t := \{x \in S | U(x) \geq t\}$.

Definition 2.8. [9] Let U be a fuzzy subset of a INK-algebra S. Then for $t \in [0, 1]$, the lower t-level cut of U is the set $U^t := \{x \in S | A(x) \leq t\}$. Clearly $U^1 = X$ and $U_t \cup U^t = X$ for $t \in [0, 1]$. If $t_1 < t_2$, then $U^{t_1} \subseteq U^{t_2}$.

An interval valued neutrosophic set (briefly, $IVNS$)[4] Λ in a non-empty set X is defined to be a structure $\Lambda = \{(\alpha, T_\Lambda(\alpha), I_\Lambda(\alpha), F_\Lambda(\alpha)) | \alpha \in X\}$, where $T_\Lambda : U \to \theta[0, 1]$, $I_\Lambda : U \to \theta[0, 1]$ and $F_\Lambda : U \to \theta[0, 1]$, which are called a truth membership function, an indeterminacy membership function and a falsity membership function respectively.

The intervals $T_\Lambda(\alpha)$, $I_\Lambda(\alpha)$ and $F_\Lambda(\alpha)$ denote the intervals of the degree of membership, indeterminacy and non-membership of the element α to the set $\theta[0, 1]$, respectively, where $T^-_\Lambda(\alpha) = [T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)]$, $I^-_\Lambda(\alpha) = [I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)]$ and $F^-_\Lambda(\alpha) = [F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)]$ for all $\alpha \in X$. Also, note that $\overline{T_\Lambda(\alpha)} = 1 - T_\Lambda(\alpha) = [1 - T^U_\Lambda(\alpha), 1 - T^L_\Lambda(\alpha)]$,

$\overline{I_\Lambda(\alpha)} = 1 - I_\Lambda(\alpha) = [1 - I^U_\Lambda(\alpha), 1 - I^L_\Lambda(\alpha)]$

and $\overline{F_\Lambda(\alpha)} = 1 - F_\Lambda(\alpha) = [1 - F^U_\Lambda(\alpha), 1 - F^L_\Lambda(\alpha)]$ for all $\alpha \in X$, where $(\alpha, \overline{T_\Lambda(\alpha)}, \overline{I_\Lambda(\alpha)}, \overline{F_\Lambda(\alpha)})$ represents the complement of α in Λ.

We define $\overline{\Lambda} = (\overline{T_\Lambda}, \overline{I_\Lambda}, \overline{F_\Lambda})$ as the complement of $\Lambda = (T_\Lambda, I_\Lambda, F_\Lambda)$. For the sake of simplicity, we shall use the symbol $\Lambda = (T_\Lambda, I_\Lambda, F_\Lambda)$ for the $IVNS$ is given by $\Lambda = \{(\alpha, T_\Lambda(\alpha), I_\Lambda(\alpha), F_\Lambda(\alpha)) | \alpha \in U\}$.

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

Definition 2.9. [6] A NS Λ in U is called a neutrosophic INK-subalgebra, (briefly, N INK-subalg) of U if it satisfies the following condition, for all $\alpha,\ \beta \in U$.

(i) $T_\Lambda(\alpha * \beta) \geq \min\{T_\Lambda(\alpha), T_\Lambda(\beta)\}$,

(ii) $I_\Lambda(\alpha * \beta) \leq \max\{I_\Lambda(\alpha), T_\Lambda(\beta)\}$,

(iii) $F_\Lambda(\alpha * \beta) \geq \min\{F_\Lambda(\alpha), F_\Lambda(\beta)\}$.

3 Interval-valued neutrosophic subalgebra in INK-algebra

In this section, we introduce the concept of IVN INK-subalgebra and investigate some related properties.

Definition 3.1. An $IVNS$ Λ in U is called an interval valued neutrosophic INK-subalgebra, (briefly, IVN INK-subalg) of U if satisfying the following condition, for all $\alpha, \beta \in U$

(i) $T^+_\Lambda(\alpha * \beta) \geq r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}$

(ii) $I^+_\Lambda(\alpha * \beta) \leq r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}$

(iii) $F^+_\Lambda(\alpha * \beta) \geq r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\}$.

Example 3.2. Assume that the set $U = \{0, \alpha, \beta\}$ has the following table:

$*$	0	α	β
0	0	α	β
α	α	0	β
β	β	α	0

Then U is a INK-subalgebra. We define an $IVNS$ $\Lambda = (T_\Lambda, I_\Lambda, F_\Lambda)$ as following:

$$T(0) = [0.6, 0.8], T(\alpha) = [0.4, 0.5], T(\beta) = [0.3, 0.4],$$
$$I(0) = [0.2, 0.3], I(\alpha) = [0.5, 0.6], I(\beta) = [0.4, 0.7],$$
$$F(0) = [0.6, 0.8], F(\alpha) = [0.4, 0.5], F(\beta) = [0.3, 0.4].$$

Hence Λ is an IVN-INK-subalgebra of U. The existence of a IVN INK-subalg of U for Λ is easily verified.

Proposition 3.3. Every IVN INK-subalgebra of U, satisfies the conditions

(i) $T^+_\Lambda(0) \geq T^+_\Lambda(\alpha)$

(ii) $I^+_\Lambda(0) \leq I^+_\Lambda(\alpha)$ and

(iii) $F^+_\Lambda(0) \geq F^+_\Lambda(\alpha)$ for all $\alpha \in U$.

Proof. Assume that Λ in IVN INK-subalgebra of U. Then for all $\alpha \in U$, we have

$$\begin{aligned} T^+_\Lambda(0) &= T^+_\Lambda(\alpha * \alpha) \\ &\geq r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\alpha)\} \\ &= r\min\{[T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)], [T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)]\} \\ &= [T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)] \\ &= T^+_\Lambda(\alpha). \end{aligned}$$

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

$$\begin{aligned}I^+_\Lambda(0) &= I^+_\Lambda(\alpha * \alpha)\\ &\leq r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\alpha)\}\\ &= r\max\{[I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)], [I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)]\}\\ &= [I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)]\\ &= I^+_\Lambda(\alpha).\end{aligned}$$

$$\begin{aligned}F^+_\Lambda(0) &= F^+_\Lambda(\alpha * \alpha)\\ &\geq r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\alpha)\}\\ &= r\min\{[F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)], [F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)]\}\\ &= [F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)]\\ &= F^+_\Lambda(\alpha).\end{aligned}$$

Hence Λ is a $IVN\ INK$-subalg of U. □

Theorem 3.4. Let Λ be a $IVN\ INK$-subalgebra of U. If there exists a sequence $\{\alpha_n\}$ in U such that

(i) $\lim\limits_{n\to\infty} T^+_\Lambda(\alpha_n) = [1, 1]$, then $T^+_\Lambda(0) = [1, 1]$

(ii) $\lim\limits_{n\to\infty} I^+_\Lambda(\alpha_n) = [1, 1]$, then $I^+_\Lambda(0) = [1, 1]$

(iii) $\lim\limits_{n\to\infty} F^+_\Lambda(\alpha_n) = [1, 1]$, then $F^+_\Lambda(0) = [1, 1]$

Proof. By Proposition 3.3, we have

$$T^+_\Lambda(0) \geq T^+_\Lambda(\alpha), \text{ for all } \alpha \in U.$$
$$\text{Then } T^+_\Lambda(0) \geq T^+_\Lambda(\alpha_n), \text{for every positive integer } n.$$
$$\text{Consider } [1, 1] \geq T^+_\Lambda(0) \geq \lim_{n\to\infty} T^+_\Lambda(\alpha_n) \geq [1, 1].$$

Hence, $T^+_\Lambda(0) = [1, 1]$.

$$I^+_\Lambda(0) \leq I^+_\Lambda(\alpha), \text{ for all } \alpha \in U.$$
$$\text{Then } I^+_\Lambda(0) \leq I^+_\Lambda(\alpha_n), \text{for every positive integer } n.$$
$$\text{Consider } [1, 1] \leq I^+_\Lambda(0) \leq \lim_{n\to\infty} I^+_\Lambda(\alpha_n) \leq [1, 1].$$

Hence $I^+_\Lambda(0) = [1, 1]$.

$$F^+_\Lambda(0) \geq F^+_\Lambda(\alpha), \ \forall\ \alpha \in U.$$
$$\text{Then } F^+_\Lambda(0) \geq F^+_\Lambda(\alpha_n), \text{for every positive integer } n.$$
$$\text{Consider } [1, 1] \geq F^+_\Lambda(0) \geq \lim_{n\to\infty} F^+_\Lambda(\alpha_n) \geq [1, 1].$$

Hence $F^+_\Lambda(0) = [1, 1]$. □

Theorem 3.5. An $IVNS\ \Lambda = \{T_\Lambda, I_\Lambda, F_\Lambda\}$ in U is a $IVN\ INK$-subalg of U iff $T^L_\Lambda, T^U_\Lambda, I^L_\Lambda, I^U_\Lambda$, F^L_Λ and F^U_Λ are INK-subalg of U.

Proof. Let T^L_Λ and T^U_Λ are $NINK$-subalg's of U and $\alpha, \beta \in U$. Then

$$\begin{aligned}T^+_\Lambda(\alpha * \beta) &\geq [T^L_\Lambda(\alpha * \beta), T^U_\Lambda(\alpha * \beta)]\\ &\geq [\min\{T^L_\Lambda(\alpha), T^L_\Lambda(\beta)\}, \min\{T^U_\Lambda(\alpha), T^U_\Lambda(\beta)\}]\\ &= r\min\{[T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)], [T^L_\Lambda(\beta), T^U_\Lambda(\beta)]\}\\ &= r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}.\end{aligned}$$

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

Again, let I^L_Λ and I^U_Λ are $NINK$-subalg's of U and $\alpha, \beta \in U$. Then

$$\begin{aligned} I^+_\Lambda(\alpha * \beta) &\leq [I^L_\Lambda(\alpha * \beta), I^U_\Lambda(\alpha * \beta)] \\ &\leq [\max\{I^L_\Lambda(\alpha), I^L_\Lambda(\beta)\}, \max\{I^U_\Lambda(\alpha), I^U_\Lambda(\beta)\}] \\ &= r\max\{[I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)], [I^L_\Lambda(\beta), I^U_\Lambda(\beta)]\} \\ &= r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}. \end{aligned}$$

Also, let F^L_Λ and F^U_Λ are $NINK$-subalg's of U and $\alpha, \beta \in U$. Then

$$\begin{aligned} F^+_\Lambda(\alpha * \beta) &\geq [F^L_\Lambda(\alpha * \beta), F^U_\Lambda(\alpha * \beta)] \\ &\geq [\min\{F^L_\Lambda(\alpha), F^L_\Lambda(\beta)\}, \min\{F^U_\Lambda(\alpha), F^U_\Lambda(\beta)\}] \\ &= r\min\{[F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)], [F^L_\Lambda(\beta), F^U_\Lambda(\beta)]\} \\ &= r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\}. \end{aligned}$$

Hence, Λ is an IVN-INK-subalg of U. Conversely, assume that Λ is a $IVN\ INK$-subalg of U. For any $\alpha,\ \beta \in U$, we have

$$\begin{aligned} [T^L_\Lambda(\alpha * \beta),\ T^U_\Lambda(\alpha * \beta)] &= T^+_\Lambda(\alpha * \beta) \\ &\geq r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\} \\ &= r\min\{[T^L_\Lambda(\alpha), T^U_\Lambda(\alpha)], [T^L_\Lambda(\beta), T^U_\Lambda(\beta)]\} \\ &= [\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}] \\ T^L_\Lambda(\alpha * \beta) &\geq [\min\{T^L_\Lambda(\alpha), T^L_\Lambda(\beta)\}] \\ T^U_\Lambda(\alpha * \beta) &\geq [\min\{T^U_\Lambda(\alpha), T^U_\Lambda(\beta)\}]. \end{aligned}$$

$$\begin{aligned} [I^L_\Lambda(\alpha * \beta), I^U_\Lambda(\alpha * \beta)] &= I^+_\Lambda(\alpha * \beta) \\ &\leq r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\} \\ &= r\max\{[I^L_\Lambda(\alpha), I^U_\Lambda(\alpha)], [I^L_\Lambda(\beta), I^U_\Lambda(\beta)]\} \\ &= [\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}] \\ I^L_\Lambda(\alpha * \beta) &\leq [\max\{I^L_\Lambda(\alpha), I^L_\Lambda(\beta)\}] \\ I^U_\Lambda(\alpha * \beta) &\leq [\max\{I^U_\Lambda(\alpha), I^U_\Lambda(\beta)\}]. \end{aligned}$$

$$\begin{aligned} [F^L_\Lambda(\alpha * \beta), F^U_\Lambda(\alpha * \beta)] &= F^+_\Lambda(\alpha * \beta) \\ &\geq r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\} \\ &= r\min\{[F^L_\Lambda(\alpha), F^U_\Lambda(\alpha)], [F^L_\Lambda(\beta), F^U_\Lambda(\beta)]\} \\ &= [\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\}] \\ F^L_\Lambda(\alpha * \beta) &\geq [\min\{F^L_\Lambda(\alpha), F^L_\Lambda(\beta)\}] \\ F^U_\Lambda(\alpha * \beta) &\geq [\min\{F^U_\Lambda(\alpha), F^U_\Lambda(\beta)\}]. \end{aligned}$$

Hence T^L_Λ, T^U_Λ, I^L_Λ, I^U_Λ, F^L_Λ and F^U_Λ are $NINK$-subalg's of U. □

Definition 3.6. Let $\Lambda^+_1 = \{T^+_{\Lambda_1}, I^+_{\Lambda_1}, F^+_{\Lambda_1}\}$ and $\Lambda^+_2 = \{T^+_{\Lambda_2}, I^+_{\Lambda_2}, F^+_{\Lambda_2}\}$ are two $IVNS$'s on a INK-algebra U, define the $IVNS$ $(\Lambda_1 \cap \Lambda_2)^+ = \{T^+_{\Lambda_1\cap\Lambda_2}, I^+_{\Lambda_1\cap\Lambda_2}, F^+_{\Lambda_1\cap\Lambda_2}\}$ on U by,

$$\begin{aligned} T^+_{\Lambda_1\cap\Lambda_2}(\alpha) &= r\min\{T^+_{\Lambda_1}(\alpha), T^+_{\Lambda_2}(\alpha)\}, \\ I^+_{\Lambda_1\cap\Lambda_2(\alpha)} &= r\max\{I^+_{\Lambda_1}(\alpha), I^+_{\Lambda_2}(\alpha)\} \\ F^+_{\Lambda_1\cap\Lambda_2(\alpha)} &= r\min\{F^+_{\Lambda_1}(\alpha), F^+_{\Lambda_2}(\alpha)\},\ \forall\, \alpha \in U. \end{aligned}$$

Then $(\Lambda_1 \cap \Lambda_2)^+ = \{T^+_{\Lambda_1\cap\Lambda_2}, I^+_{\Lambda_1\cap\Lambda_2}, F^+_{\Lambda_1\cap\Lambda_2}\}$ is called the intersection of $\Lambda^+_1 = \{T^+_{\Lambda_1}, I^+_{\Lambda_1}, F^+_{\Lambda_1}\}$ and $\Lambda^+_2 = \{T^+_{\Lambda_2}, I^+_{\Lambda_2}, F^+_{\Lambda_2}\}$.

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

Theorem 3.7. Let Λ_1 and Λ_2 are two $IVNINK$-subalg's of U. Then $\Lambda_1 \cap \Lambda_2$ is an $IVNINK$-subalg of U.

Proof. Let $\alpha, \beta \in \Lambda_1 \cap \Lambda_2$, then $\alpha, \beta \in \Lambda_1$ and Λ_2. Since, Λ_1 and Λ_2 are $IVNINK$-subalg's of U by Theorem 3.5, we have,

$$\begin{aligned}
T^+_{\Lambda_1\cap\Lambda_2}(\alpha*\beta) &= r\min\{T^+_{\Lambda_1}(\alpha*\beta), T^+_{\Lambda_2}(\alpha*\beta)\}\\
&\geq r\min\{r\min\{T^L_{\Lambda_1}(\alpha), T^U_{\Lambda_1}(\beta)\}, r\min\{T^L_{\Lambda_2}(\alpha), T^U_{\Lambda_2}(\beta)\}\\
&= r\min\{r\min\{T^L_{\Lambda_1}(\alpha), T^U_{\Lambda_2}(\alpha)\}, r\min\{T^L_{\Lambda_1}(\beta), T^U_{\Lambda_2}(\beta)\}\}\\
&= r\min\{T^+_{\Lambda_1\cap\Lambda_2}(\alpha), T^+_{\Lambda_1\cap\Lambda_2}(\beta)\}
\end{aligned}$$

$$\begin{aligned}
I^+_{\Lambda_1\cap\Lambda_2}(\alpha*\beta) &= r\max\{I^+_{\Lambda_1}(\alpha*\beta), I^+_{\Lambda_2}(\alpha*\beta)\}\\
&\leq r\max\{r\max\{I^L_{\Lambda_1}(\alpha), I^U_{\Lambda_1}(\beta)\}, r\max\{I^L_{\Lambda_2}(\alpha), I^U_{\Lambda_2}(\beta)\}\}\\
&= r\max\{r\max\{I^L_{\Lambda_1}(\alpha), I^U_{\Lambda_2}(\alpha)\}, r\max\{I^L_{\Lambda_1}(\beta), I^U_{\Lambda_2}(\beta)\}\}\\
&= r\max\{I^+_{\Lambda_1\cap\Lambda_2}(\alpha), I^+_{\Lambda_1\cap\Lambda_2}(\beta)\}
\end{aligned}$$

$$\begin{aligned}
F^+_{\Lambda_1\cap\Lambda_2}(\alpha*\beta) &= r\min\{F^+_{\Lambda_1}(\alpha*\beta), F^+_{\Lambda_2}(\alpha*\beta)\}\\
&\geq r\min\{r\min\{F^L_{\Lambda_1}(\alpha), F^U_{\Lambda_1}(\beta)\}, r\min\{F^L_{\Lambda_2}(\alpha), F^U_{\Lambda_2}(\beta)\}\\
&= r\min\{r\min\{F^L_{\Lambda_1}(\alpha), F^U_{\Lambda_2}(\alpha)\}, r\min\{F^L_{\Lambda_1}(\beta), F^U_{\Lambda_2}(\beta)\}\}\\
&= r\min\{F^+_{\Lambda_1\cap\Lambda_2}(\alpha), F^+_{\Lambda_1\cap\Lambda_2}(\beta)\}.
\end{aligned}$$

Hence $\Lambda_1 \cap \Lambda_2$ is $IVNINK$-subalg of U. □

Definition 3.8. Let $\Lambda_1^+ = \{T^+_{\Lambda_1}, I^+_{\Lambda_1}, F^+_{\Lambda_1}\}$ and $\Lambda_2^+ = \{T^+_{\Lambda_2}, I^+_{\Lambda_2}, F^+_{\Lambda_2}\}$ are two $IVNS$'s on a INK-algebra U, define the $IVNS \quad (\Lambda_1 \cup \Lambda_2)^+ = \{T^+_{\Lambda_1\cup\Lambda_2}, I^+_{\Lambda_1\cup\Lambda_2}, F^+_{\Lambda_1\cup\Lambda_2}\}$ on U by,

$$\begin{aligned}
T^+_{\Lambda_1\cup\Lambda_2}(\alpha) &= r\max\{T^+_{\Lambda_1}(\alpha), T^+_{\Lambda_2}(\alpha)\},\\
I^+_{\Lambda_1\cup\Lambda_2(\alpha)} &= r\min\{I^+_{\Lambda_1}(\alpha), I^+_{\Lambda_2}(\alpha)\}\\
F^+_{\Lambda_1\cup\Lambda_2(\alpha)} &= r\max\{F^+_{\Lambda_1}(\alpha), F^+_{\Lambda_2}(\alpha)\}
\end{aligned}$$

$\forall\ \alpha \in U$. Then $(\Lambda_1 \cup \Lambda_2)^+ = \{T^+_{\Lambda_1\cup\Lambda_2}, I^+_{\Lambda_1\cup\Lambda_2}, F^+_{\Lambda_1\cup\Lambda_2}\}$ is called the union of $\Lambda_1^+ = \{T^+_{\Lambda_1}, I^+_{\Lambda_1}, F^+_{\Lambda_1}\}$ and $\Lambda_2^+ = \{T^+_{\Lambda_2}, I^+_{\Lambda_2}, F^+_{\Lambda_2}\}$.

Theorem 3.9. Let Λ_1 and Λ_2 are two $IVN\ INK$-subalg's of U. Then $\Lambda_1 \cup \Lambda_2$ is an $IVN\ INK$-subalg of U.

Proof. Let $\alpha,\ \beta \in \Lambda_1 \cup \Lambda_2$, then $\alpha,\ \beta \in \Lambda_1$ and Λ_2. Since Λ_1 and Λ_2 are $IVN\ INK$-subalg's of U by Theorem 3.5, we have,

$$\begin{aligned}
T^+_{\Lambda_1\cup\Lambda_2}(\alpha*\beta) &= r\max\{T^+_{\Lambda_1}(\alpha*\beta), T^+_{\Lambda_2}(\alpha*\beta)\}\\
&\geq r\max\{r\min\{T^L_{\Lambda_1}(\alpha), T^U_{\Lambda_1}(\beta)\}, r\min\{T^L_{\Lambda_2}(\alpha), T^U_{\Lambda_2}(\beta)\}\}\\
&= r\min\{r\max\{T^L_{\Lambda_1}(\alpha), T^U_{\Lambda_2}(\alpha)\}, r\min\{T^L_{\Lambda_1}(\beta), T^U_{\Lambda_2}(\beta)\}\}\\
&= r\min\{T^+_{\Lambda_1\cup\Lambda_2}(\alpha), T^+_{\Lambda_1\cup\Lambda_2}(\beta)\}.
\end{aligned}$$

$$\begin{aligned}
I^+_{\Lambda_1\cup\Lambda_2}(\alpha*\beta) &= r\min\{I^+_{\Lambda_1}(\alpha*\beta), I^+_{\Lambda_2}(\alpha*\beta)\}\\
&\leq r\min\{r\max\{I^L_{\Lambda_1}(\alpha), I^U_{\Lambda_1}(\beta)\}, r\max\{I^L_{\Lambda_2}(\alpha), I^U_{\Lambda_2}(\beta)\}\}\\
&= r\max\{r\min\{I^L_{\Lambda_1}(\alpha), I^U_{\Lambda_2}(\alpha)\}, r\min\{I^L_{\Lambda_1}(\beta), I^U_{\Lambda_2}(\beta)\}\}\\
&= r\max\{I^+_{\Lambda_1\cup\Lambda_2}(\alpha), I^+_{\Lambda_1\cup\Lambda_2}(\beta)\}.
\end{aligned}$$

https://doi.org/10.54216/IJNS.240105
Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

$$\begin{aligned} F^+_{\Lambda_1\cup\Lambda_2}(\alpha*\beta) &= r\max\{F^+_{\Lambda_1}(\alpha*\beta), F^+_{\Lambda_2}(\alpha*\beta)\} \\ &\geq r\max\{r\min\{F^L_{\Lambda_1}(\alpha), F^U_{\Lambda_1}(\beta)\}, r\min\{F^L_{\Lambda_2}(\alpha), F^U_{\Lambda_2}(\beta)\}\} \\ &= r\min\{r\max\{F^L_{\Lambda_1}(\alpha), F^U_{\Lambda_2}(\alpha)\}, r\min\{F^L_{\Lambda_1}(\beta), F^U_{\Lambda_2}(\beta)\}\} \\ &= r\min\{F^+_{\Lambda_1\cup\Lambda_2}(\alpha), F^+_{\Lambda_1\cup\Lambda_2}(\beta)\}. \end{aligned}$$

Hence $\Lambda_1 \cup \Lambda_2$ is $IVN\ INK$-subalg of U. □

Corollary 3.10. Let $\{\Lambda_i/i \in \Lambda\}$ be a family of $IVN\ INK$-subalg's of U. Then $\bigcap_{i\in\Lambda}\Lambda_i$ is also an $IVN\ INK$-subalg of U.

Definition 3.11. Let Λ be an IVN in U and $[\alpha_1,\ \alpha_2] \in \theta[0,1]$. Then the IV level neutrosophic INK-subalgebra (briefly, $IVN\ INK$-subalg) $U(\Lambda;[\alpha_1,\alpha_2])$ of Λ and strong $IVN\ INK$-subalgebra (briefly, $sIVN\ INK$-subalg) $U(\Lambda;<[\alpha_1,\alpha_2])$ of U are defined as following:

(i) $U(\Lambda;[\alpha_1,\alpha_2]) := \{\alpha \in U | T^+_\Lambda(\alpha) \geq [\alpha_1,\alpha_2]\}$, $U(\Lambda;>,[\alpha_1,\alpha_2]) := \{\alpha \in U | T^+_\Lambda(\alpha) > [\alpha_1,\alpha_2]\}$.

(ii) $U(\Lambda;[\alpha_1,\alpha_2]) := \{\alpha \in U | I^+_\Lambda(\alpha) \leq [\alpha_1,\alpha_2]\}$, $U(\Lambda;<,[\alpha_1,\alpha_2]) := \{\alpha \in U | I^+_\Lambda(\alpha) < [\alpha_1,\alpha_2]\}$.

(iii) $U(\Lambda;[\alpha_1,\alpha_2]) := \{\alpha \in U | F^+_\Lambda(\alpha) \geq [\alpha_1,\alpha_2]\}$, $U(\Lambda;\geq,[\alpha_1,\alpha_2]) := \{\alpha \in U | F^+_\Lambda(\alpha), > [\alpha_1,\alpha_2]\}$.

Theorem 3.12. Let Λ_1 be an $IVFS$ of U and Λ_2 be the closure of image of $\Lambda = \{T_\Lambda, I_\Lambda, F_\Lambda\}$. Then the subsequent conditions are equivalent:

(i) Λ_1 is an $IVN\ INK$-subalg of U.

(ii) For all $[\alpha_1,\alpha_2] \in Im(\Lambda = \{T_\Lambda, I_\Lambda, F_\Lambda\})$, the non empty level subset $U(\Lambda;[\alpha_1,\alpha_2])$ of Λ_1 is a INK-subalg of U.

(iii) For all $[\alpha_1,\alpha_2] \in Im(\Lambda = \{T_\Lambda, I_\Lambda, F_\Lambda\})$, the non empty strong level subset $U(\Lambda;<,[\alpha_1,\alpha_2])$ of Λ_1 is a INK-subalg of U.

(iv) For all $[\alpha_1,\alpha_2] \in \theta[0,1]$, the non empty strong level subset $U(\Lambda_1;<,[\alpha_1,\alpha_2])$ of Λ_1 is a INK-subalg of U.

(v) For all $[\alpha_1,\alpha_2] \in \theta[0,1]$, the non empty level subset $U(\Lambda_1;[\alpha_1,\alpha_2])$ of Λ_1 is a INK-subalg of U.

Proof. (i) → (iv): Let Λ_1 be an $IVN\ INK$-subalg of U, $[\alpha_1,\alpha_2] \in \theta[0,1]$ and $\alpha,\beta \in U(\Lambda_1;<,[\alpha_1,\alpha_2])$, then we have

$$\begin{aligned} T^+_{\Lambda_1}(\alpha*\beta) &\geq r\min\{T^+_{\Lambda_1}(\alpha), T^+_{\Lambda_1}(\beta)\} \geq r\min\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2], \\ I^+_{\Lambda_1}(\alpha*\beta) &\leq r\max\{I^+_{\Lambda_1}(\alpha), I^+_{\Lambda_1}(\beta)\} \leq r\max\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2], \\ F^+_{\Lambda_1}(\alpha*\beta) &\geq r\min\{F^+_{\Lambda_1}(\alpha), F^+_{\Lambda_1}(\beta)\} \geq r\min\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2]. \end{aligned}$$

Thus $\alpha*\beta \in U(\Lambda_1;<,[\alpha_1,\alpha_2])$. Hence $U(\Lambda_1;<,[\alpha_1,\alpha_2])$ is a neutrosophic INK- subalg of U.

(iv) → (iii): It's evident.

(iii) → (ii): Let $[\alpha_1,\alpha_2] \in Im(\Lambda_1 = \{T_{\Lambda_1}, I_{\Lambda_1}, F_{\Lambda_1}\})$. Then $U(\Lambda_1;[\alpha_1,\alpha_2])$ is a non empty. Since $U(\Lambda_1;[\alpha_1,\alpha_2]) = \bigcap_{[\alpha_1,\alpha_2]<[\beta_1,\beta_2]} U(\Lambda_1;[\alpha_1,\alpha_2])$, where$[\beta_1,\beta_2] \in Im(\Lambda_1 = \{T_{\Lambda_1}, I_{\Lambda_1}, F_{\Lambda_1}\})|\Lambda_2$. Then by (iii) and Corollary 3.10, $U(\Lambda_1;[\alpha_1,\alpha_2])$ is a $IVN\ INK$-subalg of U.

(ii) → (v): Let $[\alpha_1,\ \alpha_2] \in \theta[0,1]$ and $U(\Lambda_1;[\alpha_1,\alpha_2])$ be nonempty. Suppose $\alpha,\beta \in U(\Lambda_1;[\alpha_1,\alpha_2])$. Let (i) $[\gamma_1,\gamma_2] = \min\{T_{\Lambda_1}(\alpha), T_{\Lambda_1}(\beta)\}$, it is clear that $[\gamma_1,\gamma_2] = \min\{T_{\Lambda_1}(\alpha), T_{\Lambda_1}(\beta)\} \geq \{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}$. Thus $\alpha,\ \beta \in U(\Lambda_1;[\gamma_1,\gamma_2])$ and $[\gamma_1,\gamma_2] \in Im(T_{\Lambda_1})$.

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

(ii) $[\gamma_1,\gamma_2]=\max\{I_{\Lambda_1}(\alpha),T_{\Lambda_1}(\beta)\}$, it is clear that $[\gamma_1,\gamma_2]=\max\{I_{\Lambda_1}(\alpha),I_{\Lambda_1}(\beta)\}\leq\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}$. Thus $\alpha,\beta\in U(\Lambda_1;[\gamma_1,\gamma_2])$ and $[\gamma_1,\gamma_2]\in Im(T_{\Lambda_1})$.

(iii) $[\gamma_1,\gamma_2]=\min\{F_{\Lambda_1}(\alpha),F_{\Lambda_1}(\beta)\}$, it is clear that $[\gamma_1,\gamma_2]=\min\{F_{\Lambda_1}(\alpha),F_{\Lambda_1}(\beta)\}\geq\{[\alpha_1,\alpha_2],[\alpha_1,\alpha_2]\}$. Thus $\alpha,\beta\in U(\Lambda_1;[\gamma_1,\gamma_2])$ and $[\gamma_1,\gamma_2]\in Im(F_{\Lambda_1})$, by (ii) $U(\Lambda_1;[\gamma_1,\gamma_2])$ is a $N\ INK$-subalg's of U, hence $\alpha*\beta\in U(\Lambda_1;[\gamma_1,\gamma_2])$. Then we have,

(i) $T^+_{\Lambda_1}(\alpha*\beta)\geq r\min\{T^+_{\Lambda_1}(\alpha),\ T^+_{\Lambda_1}(\beta)\}\geq\{[\gamma_1,\gamma_2],[\gamma_1,\gamma_2]\}=[\gamma_1,\gamma_2]\geq[\alpha_1,\alpha_2]$.

(ii) $I^+_{\Lambda_1}(\alpha*\beta)\leq r\max\{I^+_{\Lambda_1}(\alpha),I^+_{\Lambda_1}(\beta)\}\leq\{[\gamma_1,\gamma_2],[\gamma_1,\gamma_2]\}=[\gamma_1,\gamma_2]\leq[\alpha_1,\alpha_2]$.

(iii) $F^+_{\Lambda_1}(\alpha*\beta)\geq r\min\{F^+_{\Lambda_1}(\alpha),F^+_{\Lambda_1}(\beta)\}\leq\{[\gamma_1,\gamma_2],[\gamma_1,\gamma_2]\}=[\gamma_1,\gamma_2]\leq[\alpha_1,\alpha_2]$. Therefore $\alpha*\beta\in U(\Lambda_1;[\alpha_1,\alpha_2])$. Then $\alpha*\beta\in U(\Lambda_1;[\alpha_1,\alpha_2])$ is a $N\ INK$-subalg of U.

(v) $\rightarrow$ (i): Assume that the non empty set $U(\Lambda_1;[\alpha_1,\alpha_2])$ is a $N\ INK$-subalg of U, for every $[\alpha_1,\alpha_2]\in\theta[0,1]$. In contrary, let $\alpha_0,\ \beta_0\in U$ be such that

(i) $T^+_{\Lambda_1}(\alpha_0*\beta_0)<r\min\{T^+_{\Lambda_1}(\alpha_0),\ T^+_{\Lambda_1}(\beta_0)\}$. Let $T^+_{\Lambda_1}(\alpha_0)=[\delta_1,\delta_2],\ T^+_{\Lambda_1}(\beta_0)=[\delta_3,\delta_4]$ and $T^+_{\Lambda_1}(\alpha_0*\beta_0)=[\alpha_1,\alpha_2]$. Then $[\alpha_1,\ \alpha_2]<r\min\{[\delta_1,\delta_2],[\delta_3,\delta_4]\}=[\min\{\delta_1,\delta_3\},\min\{\delta_2,\delta_4\}]$. So $\alpha_1<\min\{\delta_1,\delta_3\}$ and $\alpha_2<\min\{\delta_2,\delta_4\}$.

$$\begin{aligned}\text{Consider, }[g_1,g_2]&=\frac{1}{2}T^+_{\Lambda_1}(\alpha_0*\beta_0)+r\min\{T^+_{\Lambda_1}(\alpha_0),T^+_{\Lambda_1}(\beta_0)\}.\\ \text{We get that }[g_1,g_2]&=\frac{1}{2}([\alpha_1,\alpha_2]+[\min\{\delta_1,\delta_3\},\min\{\delta_2,\delta_4\}])\\ &=[\frac{1}{2}(\alpha_1+\min\{\delta_1,\delta_3\}),\frac{1}{2}(\alpha_2+\min\{\delta_2,\delta_4\})].\\ \text{Therefore, }\min\{\delta_1,\delta_3\}>g_1&=\frac{1}{2}(\alpha_1+\min\{\delta_1,\delta_3\})>\alpha_1.\\ \min\{\delta_2,\delta_4\}>g_2&=\frac{1}{2}(\alpha_2+\min\{\delta_2,\delta_4\})>\alpha_2.\end{aligned}$$

Hence, $[\min\{\delta_1,\delta_3\},\min\{\delta_2,\delta_4\}]>[g_1,g_2]>[\alpha_1,\alpha_2]=T^+_{\Lambda_1}(\alpha_0\times\beta_0)$. So that $\alpha_0\times\beta_0\notin U(\Lambda_1;[\alpha_1,\alpha_2])$.

(ii) $I^+_{\Lambda_1}(\alpha_0*\beta_0)>r\max\{I^+_{\Lambda_1}(\alpha_0),\ I^+_{\Lambda_1}(\beta_0)\}$. Let $I^+_{\Lambda_1}(\alpha_0)=[\delta_1,\delta_2],\ I^+_{\Lambda_1}(\beta_0)=[\delta_3,\delta_4]$ and $I^+_{\Lambda_1}(\alpha_0*\beta_0)=[\alpha_1,\alpha_2]$. Then, $[\alpha_1,\alpha_2]>r\max\{[\delta_1,\delta_2],[\delta_3,\delta_4]\}=[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]$. So $\alpha_1<\max\{\delta_1,\delta_3\}$ and $\alpha_2<\max\{\delta_2,\delta_4\}$.

$$\begin{aligned}\text{Consider }[g_1,g_2]&=\frac{1}{2}I^+_{\Lambda_1}(\alpha_0*\beta_0)+r\max\{I^+_{\Lambda_1}(\alpha_0),I^+_{\Lambda_1}(\beta_0)\}.\\ \text{We get that }[g_1,g_2]&=\frac{1}{2}([\alpha_1,\alpha_2]+[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}])\\ &=[\frac{1}{2}(\alpha_1+\max\{\delta_1,\delta_3\}),\frac{1}{2}(\alpha_2+\max\{\delta_2,\delta_4\})].\\ \text{Therefore, }\max\{\delta_1,\delta_3\}<g_1&=\frac{1}{2}(\alpha_1+\max\{\delta_1,\delta_3\})<\alpha_1.\\ \max\{\delta_2,\delta_4\}<g_2&=\frac{1}{2}(\alpha_2+\max\{\delta_2,\delta_4\})<\alpha_2.\end{aligned}$$

Hence, $[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]<[g_1,g_2]<[\alpha_1,\alpha_2]=I^+_{\Lambda_1}(\alpha_0\times\beta_0)$. So that $\alpha_0\times\beta_0\notin U(\Lambda_1;[\alpha_1,\alpha_2])$.

(iii) $F^+_{\Lambda_1}(\alpha_0*\beta_0)<r\min\{F^+_{\Lambda_1}(\alpha_0),\ F^+_{\Lambda_1}(\beta_0)\}$. Let $F^+_{\Lambda_1}(\alpha_0)=[\delta_1,\delta_2],\ F^+_{\Lambda_1}(\beta_0)=[\delta_3,\delta_4]$ and $F^+_{\Lambda_1}(\alpha_0*\beta_0)=[\alpha_1,\alpha_2]$. Then $[\alpha_1,\alpha_2]>r\min\{[\delta_1,\delta_2],[\delta_3,\delta_4]\}=[\min\{\delta_1,\delta_3\},\min\{\delta_2,\delta_4\}]$. So, $\alpha_1<\min\{\delta_1,\delta_3\}$

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

and $\alpha_2 < \min\{\delta_2, \delta_4\}$.

$$\text{Consider } [g_1, g_2] = \frac{1}{2}F^+_{\Lambda_1}(\alpha_0 * \beta_0) + r\min\{F^+_{\Lambda_1}(\alpha_0), F^+_{\Lambda_1}(\beta_0)\}.$$

$$\text{We get that } [g_1, g_2] = \frac{1}{2}([\alpha_1, \alpha_2] + [\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}])$$

$$= [\frac{1}{2}(\alpha_1 + \min\{\delta_1, \delta_3\}),\ \frac{1}{2}(\alpha_2 + \min\{\delta_2, \delta_4\})].$$

$$\text{Therefore, } \min\{\delta_1, \delta_3\} > g_1 = \frac{1}{2}(\alpha_1 + \min\{\delta_1, \delta_3\}) > \alpha_1.$$

$$\min\{\delta_2, \delta_4\} > g_2 = \frac{1}{2}(\alpha_2 + \min\{\delta_2, \delta_4\}) > \alpha_2.$$

Hence, $[\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}] > [g_1, g_2] > [\alpha_1, \alpha_2] = F^+_{\Lambda_1}(\alpha_0 \times \beta_0)$. So that $\alpha_0 \times \beta_0 \notin U(\Lambda_1; [\alpha_1, \alpha_2])$, which is a contraction. Since

(i) $T^+_{\Lambda_1}(\alpha_0) = [\delta_1, \delta_2] \geq [\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}] > [g_1, g_2]$
$T^+_{\Lambda_1}(\beta_0) = [\delta_3, \delta_4] \geq [\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}] > [g_1, g_2]$ imply that $\alpha_0, \beta_0 \in U(\Lambda_1; [\alpha_1, \alpha_2])$.
Thus, $T^+_{\Lambda_1}(\alpha * \beta) \geq r\min\{T^+_{\Lambda_1}(\alpha), T^+_{\Lambda_1}(\beta)\}$for all $\alpha, \beta \in U$.

(ii) $I^+_{\Lambda_1}(\alpha_0) = [\delta_1, \delta_2] \leq [\max\{\delta_1, \delta_3\}, \max\{\delta_2, \delta_4\}] < [g_1, g_2]$
$I^+_{\Lambda_1}(\beta_0) = [\delta_3, \delta_4] \leq [\max\{\delta_1, \delta_3\}, \max\{\delta_2, \delta_4\}] < [g_1, g_2]$ imply that $\alpha_0, \beta_0 \in U(\Lambda_1; [\alpha_1, \alpha_2])$.
Thus $I^+_{\Lambda_1}(\alpha * \beta) \leq r\max\{I^+_{\Lambda_1}(\alpha), I^+_{\Lambda_1}(\beta)\}$ for all $\alpha, \beta \in U$.

(iii) $F^+_{\Lambda_1}(\alpha_0) = [\delta_1, \delta_2] \geq [\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}] > [g_1, g_2]$
$F^+_{\Lambda_1}(\beta_0) = [\delta_3, \delta_4] \geq [\min\{\delta_1, \delta_3\}, \min\{\delta_2, \delta_4\}] > [g_1, g_2]$ imply that $\alpha_0, \beta_0 \in U(\Lambda_1; [\alpha_1, \alpha_2])$.
Thus $F^+_{\Lambda_1}(\alpha * \beta) \geq r\min\{F^+_{\Lambda_1}(\alpha), F^+_{\Lambda_1}(\beta)\}$ for all $\alpha, \beta \in U$.

Which completes the proof. □

Theorem 3.13. Each $N\ INK$-subalg of U is an IV level $N\ INK$-subalg of an $IVN\ INK$-subalg of U.

Proof. Let V be a $N\ INK$-subalg of U, and Λ be an IVN set on U defined by

$$(i)\ T^+_{\Lambda}(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad \alpha \in V \\ [0, 0] & \text{otherwise} \end{cases}$$

$$(ii)\ I^+_{\Lambda}(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad \alpha \in V \\ [0, 0] & \text{otherwise} \end{cases}$$

$$(iii)\ F^+_{\Lambda}(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad \alpha \in V \\ [0, 0] & \text{otherwise} \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 < \alpha_2$. It is clear that $U(\Lambda; [\alpha_1, \alpha_2]) = V$. Let $\alpha, \beta \in U$. We examine the subsequent cases:

Case (i): If $\alpha,\ \beta \in V$, then $\alpha * \beta \in V$. Therefore,

(i) $T^+_{\Lambda}(\alpha * \beta) = [\alpha_1, \alpha_2] = r\min\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = r\min\{T^+_{\Lambda}(\alpha), T^+_{\Lambda}(\beta)\}$

(ii) $I^+_{\Lambda}(\alpha * \beta) = [\alpha_1, \alpha_2] = r\max\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = r\max\{I^+_{\Lambda}(\alpha), I^+_{\Lambda}(\beta)\}$

(iii) $F^+_{\Lambda}(\alpha * \beta) = [\alpha_1, \alpha_2] = r\min\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = r\min\{F^+_{\Lambda}(\alpha), F^+_{\Lambda}(\beta)\}$.

Case (ii): If $\alpha,\ \beta \notin V$, then

https://doi.org/10.54216/IJNS.240105
Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

(i) $T^+_\Lambda(\alpha) = [0,0] = T^+_\Lambda(\beta)$ and so, $T^+_\Lambda(\alpha * \beta) \geq [0,0] = r\min\{[0,0],[0,0]\} = r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}$.

(ii) $I^+_\Lambda(\alpha) = [0,0] = I^+_\Lambda(\beta)$ and so, $I^+_\Lambda(\alpha * \beta) \leq [0,0] = r\max\{[0,0],[0,0]\} = r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}$.

(iii) $F^+_\Lambda(\alpha) = [0,0] = F^+_\Lambda(\beta)$ and so, $F^+_\Lambda(\alpha * \beta) \geq [0,0] = r\min\{[0,0],[0,0]\} = r\min\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}$.

Case (iii): If $\alpha \in V$ and $\beta \notin V$, then

(i) $T^+_\Lambda(\alpha) = [\alpha_1,\alpha_2]$ and $T^+_\Lambda(\beta) = [0,0]$. Thus, $T^+_\Lambda(\alpha * \beta) \geq [0,0] = r\min\{[\alpha_1,\alpha_2],[0,0]\} = r\min\{T^+_\Lambda(\alpha),\ T^+_\Lambda(\beta)\}$.

(ii) $I^+_\Lambda(\alpha) = [\alpha_1,\alpha_2]$ and $I^+_\Lambda(\beta) = [0,0]$. Thus, $I^+_\Lambda(\alpha * \beta) \leq [0,0] = r\max\{[\alpha_1,\alpha_2],[0,0]\} = r\max\{I^+_\Lambda(\alpha),\ I^+_\Lambda(\beta)\}$.

(iii) $F^+_\Lambda(\alpha) = [\alpha_1,\alpha_2]$ and $F^+_\Lambda(\beta) = [0,0]$. Thus, $F^+_\Lambda(\alpha * \beta) \geq [0,0] = r\min\{[\alpha_1,\alpha_2],[0,0]\} = r\min\{F^+_\Lambda(\alpha),\ F^+_\Lambda(\beta)\}$.

Case (iv): If $\beta \in V$ and $\alpha \notin V$, then by the same arguments as in case (iii), we can conclude that

(i) $T^+_\Lambda(\alpha * \beta) \geq r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}$.

(ii) $I^+_\Lambda(\alpha * \beta) \leq r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}$.

(iii) $F^+_\Lambda(\alpha * \beta) \geq r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\}$.

Therefore Λ is an $IVN\ INK$-subalg of U. □

Theorem 3.14. Let V be a subset of U and Λ be an IVN set on U which is given in the proof of Theorem 3.12. If Λ is an $IVN\ INK$-subalg of U, then V is a $N\ INK$-subalg of U.

Proof. Let Λ be an $IVN\ INK$-subalg of U, and $\alpha,\ \beta \in V$. Then,

(i) $T^+_\Lambda(\alpha) = [\alpha_1,\alpha_2] = T^+_\Lambda(\beta)$, thus $T^+_\Lambda(\alpha * \beta) \geq r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\} = r\min\{[\alpha_1,\alpha_2],\ [\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2]$.

(ii) $I^+_\Lambda(\alpha) = [\alpha_1,\alpha_2] = I^+_\Lambda(\beta)$, thus $I^+_\Lambda(\alpha * \beta) \leq r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\} = r\max\{[\alpha_1,\alpha_2],\ [\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2]$.

(iii) $F^+_\Lambda(\alpha) = [\alpha_1,\alpha_2] = F^+_\Lambda(\beta)$, thus $F^+_\Lambda(\alpha * \beta) \geq r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\} = r\min\{[\alpha_1,\alpha_2],\ [\alpha_1,\alpha_2]\} = [\alpha_1,\alpha_2]$.

which implies that $\alpha * \beta \in V$. □

Theorem 3.15. If Λ is an $IVN\ INK$-subalg of U, then the set

$$(i)\ U_{T^+_\Lambda} := \{\alpha \in U | T^+_\Lambda(\alpha) = T^+_\Lambda(0)\}$$

$$(ii)\ U_{I^+_\Lambda} := \{\alpha \in U | I^+_\Lambda(\alpha) = I^+_\Lambda(0)\}$$

$$(iii)\ U_{F^+_\Lambda} := \{\alpha \in U | F^+_\Lambda(\alpha) = F^+_\Lambda(0)\}$$

is a $N\ INK$-subalg of U.

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

Proof. (i) Let $\alpha,\ \beta \in U_{T_\Lambda^+}$ then $T_\Lambda^+(\alpha) = T_\Lambda^+(0) = T_\Lambda^+(\beta)$, and so $T_\Lambda^+(\alpha * \beta) \geq r\min\{T_\Lambda^+(\alpha),\ T_\Lambda^+(\beta)\} = r\min\{T_\Lambda^+(0),\ T_\Lambda^+(0)\} = T_\Lambda^+(0)$, by Proposition 3.3 we get that $T_\Lambda^+(\alpha * \beta) = T_\Lambda^+(0)$ which means that $\alpha * \beta \in U_{T_\Lambda^+}$.

(ii) Let $\alpha,\ \beta \in U_{I_\Lambda^+}$ then $I_\Lambda^+(\alpha) = I_\Lambda^+(0) = I_\Lambda^+(\beta)$, and so $I_\Lambda^+(\alpha * \beta) \leq r\max\{I_\Lambda^+(\alpha),\ I_\Lambda^+(\beta)\} = r\max\{I_\Lambda^+(0), I_\Lambda^+(0)\} = I_\Lambda^+(0)$, by Proposition 3.3 we get that $I_\Lambda^+(\alpha * \beta) = I_\Lambda^+(0)$ which means that $\alpha * \beta \in U_{I_\Lambda^+}$.

(iii) Let $\alpha,\ \beta \in U_{F_\Lambda^+}$ then $F_\Lambda^+(\alpha) = F_\Lambda^+(0) = F_\Lambda^+(\beta)$, and so $F_\Lambda^+(\alpha * \beta) \geq r\min\{F_\Lambda^+(\alpha),\ F_\Lambda^+(\beta)\} = r\min\{F_\Lambda^+(0),\ F_\Lambda^+(0)\} = F_\Lambda^+(0)$, by Proposition 3.3 we get that $F_\Lambda^+(\alpha * \beta) = F_\Lambda^+(0)$ which means that $\alpha * \beta \in U_{F_\Lambda^+}$. □

Theorem 3.16. Let Λ be an $IVNS$ defined by:

$$(i)\ T_\Lambda^+(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad a \in \Lambda \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2],\ [\beta_1, \beta_2] \in \theta[0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$.

$$(ii)\ I_\Lambda^+(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad a \in \Lambda \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2],\ [\beta_1, \beta_2] \in \theta[0,1]$ with $[\alpha_1, \alpha_2] \leq [\beta_1, \beta_2]$.

$$(iii)\ F_\Lambda^+(\alpha) = \begin{cases} [\alpha_1, \alpha_2] & \text{if} \quad a \in \Lambda \\ [\beta_1, \beta_2], & \text{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2],\ [\beta_1, \beta_2] \in \theta[0,1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$. Then Λ is an an $IVN\ INK$-subalg if and only if Λ is a $N\ INK$-subalg of U. Moreover, in this case

(i) $U_{T_\Lambda^+} = \Lambda$,

(ii) $U_{I_\Lambda^+} = \Lambda$,

(iii) $U_{F_\Lambda^+} = \Lambda$.

Proof. Let Λ be an an $IVN\ INK$-subalg. Let $\alpha,\ \beta \in U$ be such that $\alpha * \beta \in \Lambda$. Then,

(i) $T_\Lambda^+(\alpha * \beta) \geq r\min\{T_\Lambda^+(\alpha), T_\Lambda^+(\beta)\} = r\min\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = [\alpha_1,\ \alpha_2]$ and so $\alpha * \beta \in \Lambda$.

(ii) $I_\Lambda^+(\alpha * \beta) \leq r\max\{I_\Lambda^+(\alpha), I_\Lambda^+(\beta)\} = r\max\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = [\alpha_1,\ \alpha_2]$ and so $\alpha * \beta \in \Lambda$.

(iii) $F_\Lambda^+(\alpha * \beta) \geq r\min\{F_\Lambda^+(\alpha), F_\Lambda^+(\beta)\} = r\min\{[\alpha_1, \alpha_2],\ [\alpha_1, \alpha_2]\} = [\alpha_1,\ \alpha_2]$ and so $\alpha * \beta \in \Lambda$.

Conversely, suppose that Λ is a $N\ INK$-subalg of U, let $\alpha, \beta \in U$.

(i) If $\alpha, \beta \in \Lambda$ then $\alpha * \beta \in \Lambda$, thus
$T_\Lambda^+(\alpha * \beta) = [\alpha_1, \alpha_2] = r\min\{T_\Lambda^+(\alpha), T_\Lambda^+(\beta)\}$.
$I_\Lambda^+(\alpha * \beta) = [\alpha_1, \alpha_2] = r\max\{I_\Lambda^+(\alpha), I_\Lambda^+(\beta)\}$.
$F_\Lambda^+(\alpha * \beta) = [\alpha_1, \alpha_2] = r\min\{F_\Lambda^+(\alpha), F_\Lambda^+(\beta)\}$.

https://doi.org/10.54216/IJNS.240105
Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

(ii) If $a \in \Lambda$ or $b \notin \Lambda$, then
$T^+_\Lambda(\alpha * \beta) \geq [\beta_1, \beta_2] = r\min\{T^+_\Lambda(\alpha), T^+_\Lambda(\beta)\}$.
$I^+_\Lambda(\alpha * \beta) \leq [\beta_1, \beta_2] = r\max\{I^+_\Lambda(\alpha), I^+_\Lambda(\beta)\}$.
$F^+_\Lambda(\alpha * \beta) \geq [\beta_1, \beta_2] = r\min\{F^+_\Lambda(\alpha), F^+_\Lambda(\beta)\}$.

This show that Λ is an $IVN\ INK$-subalg. Moreover, we have

(i) $U_{T^+_\Lambda} := \{\alpha \in U | T^+_\Lambda(\alpha) = T^+_\Lambda(0)\} = \{\alpha \in U | T^+_\Lambda(\alpha) = [\alpha_1, \alpha_2]\} = \Lambda$.

(ii) $U_{I^+_\Lambda} := \{\alpha \in U | I^+_\Lambda(\alpha) = I^+_\Lambda(0)\} = \{\alpha \in U | I^+_\Lambda(\alpha) = [\alpha_1, \alpha_2]\} = \Lambda$.

(iii) $U_{F^+_\Lambda} := \{\alpha \in U | F^+_\Lambda(\alpha) = F^+_\Lambda(0)\} = \{\alpha \in U | F^+_\Lambda(\alpha) = [\alpha_1, \alpha_2]\} = \Lambda$.

□

Definition 3.17. Let f be a mapping from the set U into a set V. Let $\Lambda_2 = \{T_{\Lambda_2}, I_{\Lambda_2}, F_{\Lambda_2}\}$ be an IVN set in V. Then the inverse image of Λ_2, denoted by $f^{-1}(\Lambda_2)$, is the IVN set in U with the membership function given by

(i) $T^+_{\Lambda_2} f^{-1}(\Lambda_2)(\alpha) = T^+_{\Lambda_2}(f(\alpha))$, for all $\alpha \in U$.

(ii) $I^+_{\Lambda_2} f^{-1}(\Lambda_2)(\alpha) = I^+_{\Lambda_2}(f(\alpha))$, for all $\alpha \in U$.

(iii) $F^+_{\Lambda_2} f^{-1}(\Lambda_2)(\alpha) = F^+_{\Lambda_2}(f(\alpha))$, for all $\alpha \in U$.

Lemma 3.18. Let f be a mapping from the set U into a set V. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be $IVFS$'s in U and V respectively. Then

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,

(ii) $f(m) = [f(m^L), f(m^U)]$.

Proposition 3.19. [6] Let f be a $N\ INK$-homomorphism from U into V and G be a $N\ INK$-subalg of V with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a $N\ INK$-subalg of U.

Proposition 3.20. [6] Let f be a $N\ INK$-homomorphism from U onto V and D be a $N\ INK$-subalg of U with the sup property. Then the image $f(D)$ of D is a $N\ INK$-subalg of V.

Proposition 3.21. Let f be a $\Lambda = \{T_\Lambda,\ I_\Lambda,\ F_\Lambda\}$ $N\ INK$ homomorphism from U into V and G be an $IVN\ INK$-subalg of V with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is an $IVN\ INK$-subalg of U.

Proof. Since,

(i) $T_{\Lambda_2} = [T^L_{\Lambda_2}, T^U_{\Lambda_2}]$

(ii) $I_{\Lambda_2} = [I^L_{\Lambda_2}, I^U_{\Lambda_2}]$

(iii) $F_{\Lambda_2} = [F^L_{\Lambda_2}, F^U_{\Lambda_2}]$

is an $IVN\ INK$-subalg of V, by Theorem 3.5, we get that $T^L_{\Lambda_2}$, $T^U_{\Lambda_2}$, $I^L_{\Lambda_2}$, $I^U_{\Lambda_2}$, $F^L_{\Lambda_2}$ and $F^U_{\Lambda_2}$ are $N\ INK$-subalg of V. By Proposition 3.20, $f^{-1}[T^L_{\Lambda_2}]$, $f^{-1}[T^U_{\Lambda_2}]$, $f^{-1}[I^L_{\Lambda_2}]$, $f^{-1}[I^U_{\Lambda_2}]$, $f^{-1}[F^L_{\Lambda_2}]$ and $f^{-1}[F^U_{\Lambda_2}]$ are $N\ INK$-subalg of U, by above Lemma 3.18 and Theorem 3.5, we can conclude that,

(i) $f^{-1}(\Lambda_2) = [f^{-1}[T^L_{\Lambda_2}], f^{-1}[T^U_{\Lambda_2}]]$,

https://doi.org/10.54216/IJNS.240105

Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024

(ii) $f^{-1}(\Lambda_2) = [f^{-1}[I^L_{\Lambda_2}], f^{-1}[I^U_{\Lambda_2}]]$,

(iii) $f^{-1}(\Lambda_2) = [f^{-1}[F^L_{\Lambda_2}], f^{-1}[F^U_{\Lambda_2}]]$.

are $IVN\ INK$-subalg of U. □

Definition 3.22. Let f be a mapping from the set U into a set V, and Λ_1 be an $IVNS$ in U with membership functions T_{Λ_1}, I_{Λ_1} and F_{Λ_1}. Then the image of Λ_1, denoted by $f(\Lambda_1)$, is the $IVNS$ in V with membership function defined by:

$$T^{+}_{f(\Lambda_1)}(\beta) = \begin{cases} rsup_{\gamma \in f^{-1}(\beta)} T^{+}_{\Lambda_1}(\gamma) & \text{if} \quad f^{-1}(\beta) \notin 0, \forall \beta \in B \\ [0,0], & \text{otherwise} \end{cases}$$

$$I^{+}_{f(\Lambda_1)}(\beta) = \begin{cases} rinf_{\gamma \in f^{-1}(\beta)} I^{+}_{\Lambda_1}(\gamma) & \text{if} \quad f^{-1}(\beta) \notin 0, \forall \beta \in B \\ [0,0], & \text{otherwise} \end{cases}$$

$$F^{+}_{f(\Lambda_1)}(\beta) = \begin{cases} rsup_{\gamma \in f^{-1}(\beta)} F^{+}_{\Lambda_1}(\gamma) & \text{if} \quad f^{-1}(\beta) \notin 0, \forall \beta \in B \\ [0,0], & \text{otherwise} \end{cases}$$

where $f^{-1}(\beta) = \{a | f(\alpha) = \beta\}$.

Theorem 3.23. Let f be a $N\ INK$-homomorphism from U onto V. If Λ_1 is an $IVN\ INK$-subalg of U with the sup property, then the image $f(\Lambda_1)$ of Λ_1 is an $IVN\ INK$-subalg of V.

Proof. Assume that Λ_1 is an $IVN\ INK$-subalg of U, then

(i) $\Lambda_1 = [T^L_{\Lambda_1},\ T^U_{\Lambda_1}]$

(ii) $\Lambda_1 = [I^L_{\Lambda_1},\ I^U_{\Lambda_1}]$

(iii) $\Lambda_1 = [F^L_{\Lambda_1},\ F^U_{\Lambda_1}]$

are $IVN\ INK$-subalg of U if and only if $T^L_{\Lambda_2}$, $T^U_{\Lambda_2}$, $I^L_{\Lambda_2}$, $I^U_{\Lambda_2}$, $F^L_{\Lambda_2}$ and $F^U_{\Lambda_2}$ are $N\ INK$-subalg of U. By Proposition 3.21 $f[T^L_{\Lambda_1}]$, $f[T^U_{\Lambda_1}]$, $f[I^L_{\Lambda_1}]$, $f[I^U_{\Lambda_1}]$, $f[F^L_{\Lambda_1}]$ and $f[F^U_{\Lambda_1}]$ are $N\ INK$-subalg of V, by Lemma 3.18 and Theorem 3.12 we can conclude that,

(i) $f(\Lambda_1) = [f(T^L_{\Lambda_1}),\ f(T^U_{\Lambda_1})]$,

(ii) $f(\Lambda_1) = [f(I^L_{\Lambda_1}), f(I^U_{\Lambda_1})]$,

(iii) $f(\Lambda_1) = [f(F^L_{\Lambda_1}), f(F^U_{\Lambda_1})]$.

is an $IVN\ INK$-subalg of V. □

4 Conclusions

In this paper, $IVN\ INK$-subalgebra is defined and studied some properties of it. Furthermore, several features of the homomorphism of $IVN\ INK$-subalgebra of INK-algebra are defined.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: August 10, 2023 Revised: December 11, 2023 Accepted: March 19, 2024