On The Minimal Sets and Stability Conditions For Compact Sets In I(X)-spaces

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Abstract
The set of all isometries on a metric space X with the usual composition of functions form a group and it is called the group of isometries and is denoted by *I(X)*. In this paper we study the generalization of the concepts of minimal sets, stability and attraction, from dynamic system into the topological transformation group *(I(X),X)*.We find that the collection of all minimal sets of *I(X)*-space is the collection of all the closures of orbits of X and we found some useful results about stability and attraction and we fixed the relationship among it's kinds.

Keywords: Compact set; Topology; Compact space; Minimal set

1. Introduction

If *(X,d)* and (Y,ρ) are metric spaces and f is a function from X onto Y , then f is called an isometry if $d(x,y) = \rho(f(x),f(y))$ for all points x and y of X. Every isometry is a one-to-one continuous open function. The composition of two isometries is again an isometry and the inverse of an isometry is also an isometry. Then the set of all isometries on a metric space *(X,d)* is a group and it is denoted by *I(X)*,[7].

This paper consists of three sections. In section one, we introduce some definitions, remarks, propositions, theorems of limit sets(see [1]) which are needed in the next sections. In section two we generalize the concepts of minimal sets from a dynamic system into I(X)-space. We find that a non empty limit set of a point is a minimal set if and only if it is closed, theorem(2.3), also we get that the closure of the orbit of any point of X is minimal , theorem(2.4), moreover the set of all minimal sets of X is the set of all closure of orbits of X, Cor.(2.5). In this section we also prove that the collection of closures of orbits of X forms a partition for X and then we have a quotient space of X, theorem(2.6), and we study some properties of this space, theorem(2.11). Moreover we study the relation between this space and the space of component of X, theorem(2.12). In section three we generalize the subject of stability and attraction from dynamics system into I(X)-spaces. We give a very useful characterization of the sets $\Lambda_W(M), \Lambda(M)$, theorem(3.4), and we find these sets are closed if I(X) is locally compact, theorem(3.5). Final we study the relationships among weak attractor, attractor and stable.

A topological transformation group is a triple (G,X,θ) where G is topological group, X is a topological space and: $\theta: G \times X \to X$ is a continuous function such that,

(i) $\theta(g,\theta(h,x)) = \theta(gh,x)$ for all $g,h \in G$ and $x \in X$.

(ii) $\theta(e,x) = x$, for all $x \in X$, where e is the identity element of G.

The map θ is called an action of G on X and the space X together with a given action θ of G is called a G-space (or, more precisely, a left G-space). We shall often use the notation $g.x$ for $\theta(g,x)$ $g.(h,x) = (gh).x$ for $\theta(g,\theta(h,x)) = \theta(gh,x)$. Similarly for $H \subseteq G$ and $A \subseteq X$ we put $HA = \{ga/a \in H,\ a \in A\}$ *for* $\theta(A,H)$. A set A is said to be invariant under G if $GA = A$, [4].

Let X be a G-space and $x \in X$ then the subspace $G(x) = \{g.x/g \in G\}$ is called the orbit (trajectory) of x under G . These subspaces form a partition on X and the sets of all orbits in X is denoted by X / G.

Let $\pi: X \to X/G$ denote the canonical map taking x into its orbit *G(x)*. Then *X/G* endowed with the quotient topology ($U \subseteq X/G$ is open if $\pi^{-1}(U)$ is open in X) is called the orbit space of X (with respect to G), For $x \in X$ the stabilizer subgroup G_x of G at x is the set $\{g \in G/gx = x\}$. A point $x \in X$ is called a critical (fixed) if *G(x)* *={x}*, where *G(x)* is the orbit of *x*,[4].

1.1 Theorem: Let *I(X)* be the group of isometrics of a metric space

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(X,d). If $\theta: I(X) \times X \to X$ is defined by $\theta(f,x) = f(x)$ for every $f \in I(X)$and $x \in X$, then $(I(X), X, \theta)$ is a topological transformation group with the pointwise convergence topology on *I(X)*.
Dydo [5] generalized the concepts of limit sets from a dynamic system into a *G*-space as follows.
1.2 Definition: Let *X* be a *G*-space. For any $x \in X$, define $\Lambda(x) = \{y \in X$
There exist a net $\{g_\alpha\}$ in *G* with $g_\alpha \to \infty$ such that $g_\alpha x \to y\}$, $\Lambda(x)$ is called the limit set of *x*.
The proof of the following proposition can be found in [2].
1.3 Proposition : Let *X* be a G-space and $x \in X$. Then,
(i) $\Lambda(x)$ is invariant under *G*.
(ii) If $x \notin \Lambda(x)$ then the stabilizer subgroup G_x of *G* at *x* is compact.
(iii) The orbit *G(x)* of *x* is closed iff $\Lambda(x) \subseteq G(x)$
(iv) $\overline{G(x)} = G(x) \cup \Lambda(x)$
(v) $\Lambda(x) = \Lambda(gx) = g\Lambda(x)$, for every $g \in G$.
1.4 Proposition[1]: Let *X* be *a G*-space and $x \in X$. If $x \in \Lambda(x)$ then $\Lambda(x)$ is closed.
1.5 Theorem[1]: Let *(X,d)* be an *I(X)*-space and $x \in X$. If $\Lambda(x) \neq \phi$, then $\Lambda(x)$ is closed iff $x \in \Lambda(x)$
1.6 Theorem[1]: Let *(X,d)* be an *I(X)*-space and $x \in X$. Then the following statements are equivalent.
(i) $\Lambda(x) \neq \phi$
(ii) $x \in \overline{\Lambda(x)}$
(iii) $\overline{\Lambda(x)} = \overline{G(x)}$, (where *G* = *I(X)* and *G(x)* is the orbit of *x*)
1.7 Proposition[1]: Let *(X,d)* be an *I(X)*- space such that *I(X)* is noncompact. If there exist $x \in X$ such that the closure of the orbit of *x* is compact, then,
(i) $\Lambda(x) \neq \phi$
(ii) $\overline{\Lambda(x)}$ is compact.
(iii) If $\Lambda(x)$ is closed, then $\Lambda(x)$ is compact.
Manoussos and Stranzalos give the necessary condition for the local compactness of *I(X)*,see the following theorem,[8],[9].
1.8 Theorem: Let *(X,d)* be a locally compact *I(X)*-space. If the space of the components of *X* is compact, then *I(X)* is locally compact.

2.Minimal Sets

In this section we study minimal sets in *I(X)*-spaces.
2.1 Definition ,[6],[10]: Let *X* be a *G*-space. A subset $M \subseteq X$is called minimal, if it is non-empty closed, and invariant, and no proper subset of *M* has these properties.
2.2 Proposition: Let *(X,d)* be an *I(X)*-space and $x \in X$ such that $\Lambda(x) \neq \phi$. If $\Lambda(x)$ is closed then $\Lambda(x)$ is minimal.
Proof: Since $\Lambda(x)$ is closed and invariant. So we have only to prove that no proper subset of $\Lambda(x)$ has these properties. Let $B \subseteq \Lambda(x)$ such that $B \neq \phi$ and *B* is closed and invariant. Let $z \in \Lambda(x)$, then there exists a net $\{g_\alpha\}$in *I(X)* with $g_\alpha \to \infty$ and $g_\alpha(x) \to z$. Since $B \neq \phi$, then there exists $y \in B$. But $B \subseteq \Lambda(x)$,then there exists *a* net $\{f_\alpha\}$ in *I(X)* with $f_\alpha \to \infty$ such that $f_\alpha(x) \to y$. Since f_α is an isometry, for every α, then $d(f_\alpha^{-1}(y), x) = d(y, f_\alpha(x))$. So $f_\alpha^{-1}(y) \to x$. But *B* is invariant and $y \in B$, then $f_\alpha^{-1}(y) \in B$, for every α. Also *B* is closed, then $x \in B$. Thus $g_\alpha(x) \in B$ (since *B* is invariant). But $g_\alpha(x) \to z$ and *B* is closed then $z \in B$. Hence $\Lambda(x) \subseteq B$ and this implies that $\Lambda(x) = B$.
2.3 Theorem: Let *(X,d)* be an *I(X)*-space and $x \in X$ such that $\Lambda(x) \neq \phi$. Then the following statements are equivalent:
(i) $\Lambda(x)$ is a minimal set.
(ii) $\Lambda(x)$ is a closed set.
(iii) $x \in \Lambda(x)$.

Proof:

i↔ii). By Definition (2.1) and Proposition(2.2).
ii↔iii). By Theorem (1.5).
The following theorem shows that in *I(X)*-space *X* the closure of the orbit of a point of *X* is minimal.
2.4 Theorem: Let *(X,d)* be an *I(X)*-space. Then the closure of the orbit of any point of *X* is minimal.
Proof: Let $x \in X$ and put *G* = *I(X)*.We will prove that the closure of orbit of *x* , $\overline{G(x)}$ is a minimal set. Let $B \subseteq \overline{G(x)}$ such that $B \neq \phi$.
and *B* is invariant and closed (see Definition (2.1)). Since $B \neq \phi$ then there exists $y \in B$ and since $B \subseteq \overline{G(x)}$, then there exists a net $\{f_\alpha\}$ in *I(X)* such that $f_\alpha(x) \to y$. Since f_α is isometry for every α, then $d(f_\alpha^{-1}(y), x) = d(y, f_\alpha(x))$, for every α. Thus $f_\alpha^{-1}(y) \to x$. Notice that $y \in B$ and *B* is invariant , then $\{f_\alpha^{-1}(y)\}$ is a net in *B*. But *B* is also closed, then $x \in B$. So $\overline{G(x)} \subseteq B$. This completes the proof.
2.5 Corollary: Let *(X,d)* be an *I(X)*- space .Then the collection of all minimal sets in *X* is the collection of all closures of orbits of elements of *X*.

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Proof: Let $B \neq \phi$ a minimal set. Thus *B*. Then there exists $x \in B$. Since *B* is invariant and closed then the closure orbit of *x*, $\overline{G(x)} \subseteq B$ (where *G=I(X))*. But by Theorem(2.4), $\overline{G(x)}$ is a minimal set. Thus $B = \overline{G(x)}$. So the collection $\{\overline{G(x)}/x \in X\}$ is all minimal sets of *X*.
We will give a useful partition of *I(X)*-space, in the following theorem.

2.6 Theorem: Let *(X,d)* be an *I(X)*- space. Then the collection of all closures of orbits of elements of *X* is a partition of *X*.

Proof: Put *G=I(X*). Let $x, y \in X$ such that $\overline{G(x)} \cap \overline{G(y)} \neq \phi$. Thus there exists $z \in \overline{G(x)} \cap \overline{G(y)}$. Since $z \in \overline{G(x)}$, then there exists a net $\{f_\alpha\}\}$ in *I(X)* such that $f_\alpha(x) \to z$. *Notice* that f_α is isometry for every α, then $d(f_\alpha^{-1}(y), x) = d(y, f_\alpha(x))$.
Then by $f_\alpha \to z$, we have $f_\alpha^{-1}(z) \to x$. Since $z \in \overline{G(y)}$ and $\overline{G(y)}$ is invariant and closed, then $x \in \overline{G(y)}$.
Thus $\overline{G(x)} \subseteq \overline{G(y)}$. So $\overline{G(x)} = \overline{G(y)}$. This completes the proof.

2.7 Definition: Let *(X,d)* be an *I(X)*-space, and let X^*denotes the collection whose elements are closures of orbits of elements of *X*. By Theorem (2.6), X^* is a partition of *X*, thus we can define the natural map $P: X \to X^*$ taking *x* into its closure of orbit $\overline{(I(X))(x)}$. Then X^* endowed with the quotient topology ($V \subseteq X^*$ is open if $P^{-1}(V)$ is open in *X)* is called the closure orbit space of *X*.

2.8 Proposition: Let *(X,d)* be an *I(X)*-space. If $\Lambda(x) = \phi$ for every $x \in X$, then the orbit space and the closure orbit space coincide.

Proof: Since in any I(X)-space $\overline{(I(X))(x)} = (I(X))(x) \cup \Lambda(x)$, for each $x \in X$, then we get the orbit space and the closure orbit space coincide.

2.9 Proposition: Let *(X,d)* be an *I(X)*-space. Then
(i) The closure orbit space X^* is a T_1-space.
(ii) If *A* is a finite subset of *X*, then *P(A)* is closed where $P: X \to X^*$ is the quotient map.

Proof:
(i) Put *G=I(X)*. Let $x \in X$. Now, $P^{-1}(P(\overline{G(x)}) = P^{-1}\{\overline{G(x)}\})$. Since the set of all closure orbits of *X* , X^* is a partition of *X*, then $P^{-1}(\{\overline{G(x)}\}) = \overline{G(x)}$. Thus $P^{-1}(\{\overline{G(x)}\})$ is closed, for every $x \in X$. Then $\{\overline{G(x)}\}$ is closed, for every $x \in X$, that is X^* is a T_1- space.
(ii) By (i).

The following proposition gives useful properties of minimal sets in *I(X)*-space.

2.10 Proposition*:* Let *(X,d)* be an *I(X)*- space and *M* be a minimal set. Then
(i) *M* is open if and only if $int(M) \neq \phi$, (where *int(M)* is the interior of *M*).
(ii) If $int(M) \neq \phi$, then *M* is a union of the components of elements of *M* in *X*.
(iii) If $int(M) \neq \phi$ and *I(X)* is connected, the *M* is a component of *X*.
(iv) If $int(M) \neq \phi$ and *X* is connected, then *M=X*.

Proof: i) $\to$). Let *M* be an open set. Since *M* is a minimal set, then $int(M) \neq \phi$. $\leftarrow$). Let $int(M) \neq \phi$, then there exists $a \in M$and an open set *B* in *X* such that $a \in B \subseteq M$. Since *M* is minimal and $a \in M$, then $\overline{(I(X))(a)} \subseteq M$. Thus by Theorem(2.4), $M = \overline{(I(X))(a)}$. Now, we want to prove that $M = int(M)$. Let $b \in M$ then $b \in \overline{(I(X))(a)}$. Thus there exists a net $\{f_\alpha\}$ in *I(X)* such that $f_\alpha(a) \to b$. Since f_α is an isometry, for every α, then $d(f_\alpha^{-1}(b), a) = d(b, f_\alpha(a))$.
Thus by $f_\alpha(a) \to b$, we have $f_\alpha^{-1}(b) \to a$. But *B* is an open nbhd of *a*, then there exists β such that $f_\alpha^{-1}(b) \in B$, for every$\alpha \geq \beta$. Then $b \in f_\beta(B)$. Since f_β is isometry, then $f_\beta(B)$ is open. But $B \subseteq M$ and *M* is invariant then$f_\beta(B) \subseteq M$. Thus $b \in int(M)$. So $M = int(M)$, i.e. *M* is open.
ii) Let $int(M) \neq \phi$. Let $a \in M$ and *C(a)* be a component of a. Then $C(a) \cap M \neq \phi$. Since $int(M) \neq \phi$, then by (i) M is open. But *M* is closed (Since *M* is minimal), then $C(a) \cap M^c = \phi$, otherwise *C(a)* is disconnected. Thus $C(a) \subseteq M$. So $M = \cup_{a \in A} C(a)$.
iii)Let $int(M) \neq \phi$ and *I(X)* is connected. Since $M \neq \phi$, then there exists $a \in M$.
By (ii), we have $C(a) \subseteq M$, where *C(a)* is a component of *a*.
Since $\overline{(I(X))(a)}$ is minimal (by Theorem (2.4))and *M* is a minimal set, then $M = \overline{(I(X))(a)}$ (since $a \in M$). Since *I(X)* is a connected space, then $I(X) \times \{a\}$ is connected.
Thus the orbit $\overline{(I(X))(a)}$ is connected. But *C(a)* is a component of a then $I(X)(a) \subseteq C(a)$. Thus $\overline{(I(X))(a)} \subseteq C(a)$ (since *C(a)* is closed). Then *M=C(a).*
iv).Since $int(M) \neq \phi$, then by (ii) , $C(a) \subseteq M$, for every component *C(a)* of $a \in M$. But *X* is connected , then *C(a)=X* . Thus *M=X.*

2.11 Theorem: Let *(X,d)* be an *I(X)*-space. If $\overline{(I(X))(a)} \neq \phi$ for every $x \in X$, then
(i) The quotient map $P: X \to X^*$ is open.
(ii) The closure orbit space X^* is a discrete space.

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Proof:
i). Let B be an open set of X. Note that $P^{-1}(P(B)) \bigcup_{x\in B}\overline{(I(X))(x)}$.By Theorem(2.4), $\overline{(I(X))(x)}$is a minimal set for every $x \in B$ and since $int(\overline{(I(X))(x)}) \neq \phi$, for every $x \in B$, then by Proposition (2.10) ,*i*, $\overline{(I(X))(x)}$ is an open set, for every $x \in B$. Thus $P^{-1}(P(B))$ is open .Then *P(B)* is open. So *P* is open.
ii). Since $int(\overline{(I(X))(x)}) \neq \phi$ and $\overline{(I(X))(x)}$ is a minimal set for every $x \in X$ (by Theorem (2.4)), then by Proposition (2.10),*i*, $\overline{(I(X))(x)}$ is open. It follows from (i), *P* is open, then $P(\overline{(I(X))(x)})$ is open. Thus $\{\overline{(I(X))(x)}\}$ is open in X^*, for every $x \in X$. So X^* is a discrete space.
Let *(X,d)* be an *I(X)*-space and let $\sum(X)$ denotes the collection of all components of *X*.
2.12 Theorem: Let *(X,d)* be an *I(X)*-space and $int(\overline{(I(X))(x)}) \neq \phi$, for every $x \in X$, then
(i) If *I(X)* is connected, then
(a) $X^* = \sum(X)$
(b) $\sum(X)$ is a discrete space.
(ii) If *X* is connected, then $X^* = \{X\}$.
Proof:
i)
a) By Theorem (2.4) and Proposition (2.10), iii.
b) By (a) and by Theorem (2.11),ii .
ii) By Theorem (2.4) and Proposition (2.10) , iv.
2.13Theorem: Let (X,d) be a locally compact I(X)-space such that $int(\overline{(I(X))(x)}) \neq \phi$, for every $x \in X$ and X^* is compact. If I(X) is connected ,then I(X) is a locally compact space.
Proof: By Theorem(1.8) and Theorem (2.12), i.

3. Stability and Attraction for compact sets

In this section we generalized the concepts of stability and attraction for compact sets from dynamic system into *I(X)*-space.
3.1 Definition: Let *(X,d)* be an *I(X)*-space and *M* be a non-empty compact subset of *X*. Define,
$\Lambda_w(M) = \{x \in X/\Lambda(x) \cap M \neq \phi$,
$\Lambda(M) = \{x \in X/\Lambda(x) \neq \phi$ and $\Lambda(x) \subseteq M\}$.
The sets $\Lambda_w(M)$, $\Lambda(M)$ are respectively called the region of weak attraction and attraction of the set *M*. Moreover, any point *x* in $\Lambda_w(M)$ or $\Lambda(M)$ respectively is said to be weakly attracted , attracted to *M*.
Notice that if *I(X)* is compact, then $\Lambda_w(M) = \Lambda(M) = \phi$, so we assume *I(X)* to be not compact in this section.
3.2 Example: Let *N* be the set of all positive integers and *(N,d)* be the discrete metric space, then $\Lambda_w(M) = N$, $\Lambda(M) = \phi$ for every non-empty compact subset *M* of *X*.
Solution: Since *N* is a discrete metric space then *M* is a non-empty finite set. First we want to calculate $\Lambda(x)$.
For every $x \in N$. Let $y \in N$ such that $y \neq x$. For every $n \in N$ such that $y \neq x+n$, define $f_n: N \to N$ as follows, $f_n(x) = y$ $f_n(y) = x+n$, $f_n(x+n) = x$ and $f_n(t) = t$, for every $t \in N$ distinct from *x*,*y* and *x*+*n*. Notice that $f_n \in I(N)$, for every $n \in N$, and $f_n(x) = y \to y$, $f_n \to \infty$, because $f_n(y) = x+n$, for every $n \in N$ (that is $f_n(y) \to \infty$). Thus $y \in \Lambda(x)$, for every $y \neq x$. Since *N* is a discrete space, then $\Lambda(x)$ is closed. Thus by Theorem (1.5), $x \in \Lambda(x)$. So $\Lambda(x) = N$. Since *M* is a non-empty finite set, then $\Lambda(x) \cap M \neq \phi$ for every $x \in N$. Thus $\Lambda_w(M) = N$ and since $\Lambda(x) = N \not\subset M$, for every $x \in N$ then $\Lambda(M) = \phi$.
3.3Example: Let $X = Y \cup Z$, where $Y = \{(0,y)/y \in R\}$ and $Z = \{(z,0)/z \geq 1 \text{ or } z \leq -1\}$. Let $d = \min\{1, d'\}$ where d' is the Euclidean metric, then $\Lambda_w(M) = \phi$ or $B \cup (-B)$and $\Lambda(M) = \phi$ or $B \cap (-B)$ where *B*=*M\Y* for every non-empty compact subset *M* of *X*.
Solution: First we will calculate $\Lambda(x)$, for every $x \in X$. Notice that $\Lambda(x) = \phi$ for every $x \in y$. Let $(0,z) \in Z$. Now, for every positive integer *n*, define $f_n: X \to X$, by $f_n((0,y)) = (0, y+n)$ and $f_n((x,0)) = (x,0) \in Z$. So $f_n \in I(X)$ for every positive integer *n*, and $f_n \to \infty$.
But $f_n\big((z,0)\big) = (z,0) \to (,0)$. Thus $(z,0) \in \Lambda(z,0)$. Also $(-z,0) \in \Lambda((z,0))$.
Thus $\Lambda(z,0) = \{(z,0), (-z,0)\}$. Now, let M be a non-empty compact subset of *X*.
If $M \subseteq Y$, then $\Lambda_w(M) = \phi$ and $\Lambda(M) = \phi$ (since $\Lambda((0,y)) = \phi$ and $\Lambda((z,0)) \cap M = \phi$).
If $M \cap Z \neq \phi$. Put $B = M/Y$, then $\Lambda((z,0)) \cap B \neq \phi$ and $\Lambda((-z,0)) \cap B \neq \phi$, for every $(z,0) \in B$.
Thus $\Lambda_w(M) = B \cup (-B)$. Notice that if $(z,0) \in B$ and $(-z,0) \notin$ B, then $\Lambda(-z,0) = \Lambda(z,0) \not\subset M$. Thus $\Lambda(M) = B \cap (-B)$.
The following theorem gives a useful characterization for weak attracted and attracted point.
3.4 Theorem: Let *(X,d)* be an *I(X)*-space and *M* a non-empty compact subset of *X*. Then
(i) A point *x* is weak attracted to *M* if and only if there exists a net $\{f_n\}$ in *I(X)* such that $f_n \to \infty$ and $d(f_n(x), M) \to 0$.
(ii) A point *x* is attracted to *M* if and if only if for every net $\{f_\alpha\}$ in *I(X)* with$f_\alpha \to \infty$, there exists a subnet $\{f_\beta\}$ of $\{f_\alpha\}$ such that $d(f_\beta(x), M) \to 0$.
Proof: *i)→).* Let $x \in \Lambda_w(M)$. Then $\Lambda(x) \cap M \neq \phi$. So there exists $y \in \Lambda(x) \cap M$.

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Thus there exists a net $\{f_\alpha\}$ in *I(X)* with $f_\alpha \to \infty$ and $f_q(x) \to y$. Then $d(f_\alpha(x), y) \to 0$.
Since $y \in M$, then $d(f_\alpha(x), M) \le d(f_\alpha(x), y)$, for every α. Hence $d(f_\alpha(x), M) \to 0$.
$\leftarrow$). Let $\{f_\alpha\}$ be a net in *I(X)* with $f_\alpha \to \infty$ such that $d(f_\alpha(x), M) \to 0$. For every α, put $t_\alpha = d(f_\alpha(x), m)$.Thus for every positive integer *n* there exists $y_n \in M$ such that $d(f_\alpha(x), y_n) < t_\alpha + \frac{1}{n}$. Since $\{y_n\}$ is a sequence in *M* and *M* is a compact set ,then there are $y_\alpha \in M$ and a subsequence $\{y_m\}$ of $\{y_n\}$ such that $y_m \to y_n$. Now,

$$d(f_\alpha(x), y_\alpha) \le d(f_\alpha(x), y_m) + d(y_m, y_\alpha)$$
$$< t_\alpha + \frac{1}{m} + d(y_m, y_\alpha)$$

Since $d(y_m, y_\alpha) \to 0$ and $\frac{1}{m} \to 0$ as $m \to \infty$, then $d(f_\alpha(x), y_\alpha) \le t_\alpha$.
Since $t_\alpha = d(f_\alpha(x), M) \le d(f_\alpha(x), y_\alpha)$. So $t_\alpha = d(f_\alpha(x), y_\alpha)$, for every α.
But we have a net $\{y_\alpha\}$ in *M*, then there exists $y \in M$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$ (since *M* is compact). Now,
$d(f_\beta(x), y) \le d(f_\beta(x), y_\beta) + d(y_\beta, y) = t_\beta + d(y_\beta, y)$. Since $t_\beta \to 0$ and $y_\beta \to y$, then $f_\beta(x) \to y$. But $f_\beta \to \infty$, then $y \in \Lambda(x)$. So $\Lambda(x) \cap M \ne \phi$ (Since $y \in M$). Then $x \in \Lambda_w(M)$.

ii) $\to$). Let $x \in \Lambda(x)$ and $\{f_\alpha\}$ be a net in *T(X)* such that $f_\alpha \to \infty$. Then $\Lambda(x) \ne \phi$ and $\Lambda(x) \subseteq M$.
Since $\Lambda(x) \ne \phi$, then by Theorem (1.6), $x \in \overline{\Lambda(x)}$. But $\overline{\Lambda(x)}$ is invariant then $\{f_\alpha(x)\}$ is a net in $\overline{\Lambda(x)}$. Since $\Lambda(x) \subseteq M$ and *M* is closed, then $\overline{\Lambda(x)} \subseteq M$, that is $\{f_\alpha(x)\}$is a net in *M*. Thus $d(f_\alpha(x), M) = 0$, for every α. This completes the proof.
$\leftarrow$). Since *I(X)* is non-compact, then there exists a net $\{f_\alpha\}$ in *I(X)* such that $f_\alpha \to \infty$. Thus there exists a subnet $\{f_\beta\}$ of $\{f_\alpha\}$ such that $d(f_\beta(x), M) \to 0$.
Then from the proof of (i),we have $\Lambda(x) \ne \phi$.
We will prove that $\Lambda(x) \subseteq M$, let $y \in \Lambda(x)$ then there exists a net $\{g_q\}$ in *I(X)* with $g_\alpha \to \infty$ and $g_\alpha(x) \to y$. So there exists a subnet $\{g_\beta\}$ of $\{g_\alpha\}$ such that $d(g_\beta(x), m) \to 0$.
It follows from the proof of (i), there exists $z \in M$ and a subnet $\{g_\gamma\}$ of $\{g_\beta\}$such that $g_\gamma \to z$. But $g_\gamma \to y$. Then *y=z* (since *X* is T_2-space).
Then $\Lambda(x) \subseteq M$, that is $x \in \Lambda(x)$.
We now give an important properties of $\Lambda_w(M)$ and $\Lambda(M)$.

3.5 Theorem: let *(X,d)* be an *I(X)*-space and *M* be a non-empty compact subset of *X* .Then

(i) $\Lambda(M) \subseteq \Lambda_w(M)$
(ii) $\Lambda_w(M)$and $\Lambda(M)$ are Invariant.
(iii) If $x \in \Lambda(M)$, then $\Lambda(x) \subseteq \Lambda(M)$.
(iv) If *I(X)* is locally compact then $\Lambda(M)$ and $\Lambda_w(M)$ are closed

Proof:

(i) Let $x \in \Lambda(M)$, then $\Lambda(x) \ne \phi$ and $\Lambda(x) \subseteq$ M, so $\Lambda(x) \cap M \ne \phi$. Thus $\Lambda(M) \subseteq \Lambda_w(M)$
(ii) It is clear that $\Lambda(x) = \Lambda(f(x))$, for every $x \in X$ and $f \in I(X)$, then $\Lambda_w(M)$ and $\Lambda(M)$ are invariant.
(iii) Let $x \in \Lambda(M)$, then $\Lambda(x) \ne \phi$ and $\Lambda(x) \subseteq$ M. We will prove that $\Lambda(x) \subseteq \Lambda(M)$, let $y \in \Lambda(x)$ and $z \in \Lambda(y)$, than there are two nets $\{f_\alpha\}$ and $\{g_\alpha\}$ with $f_\alpha \to \infty$, $g_\alpha \to \infty$, such that $f_\alpha(x) \to y$ and $g_\alpha(x) \to z$.
Now,
$d((g_\alpha o f_\alpha)(x), z) = d(f_\alpha(x), g_\alpha^{-1}(z))$ (since $g_\alpha$is an isometry)

$$\le d(f_\alpha(x), y) + d(y, g_\alpha^{-1}(z))$$
$$= d(f_\alpha(x), y) + d(g_\alpha(y), z)$$

Since $f_\alpha(x) \to y$ and $g_\alpha(y) \to z$, then we have $(g_\alpha o f_\alpha)(x) \to z$.
Now, if $g_\alpha o f_\alpha \to \infty$, then $z \in \Lambda(x)$ and if there exists $f \in I(X)$ such that $g_\alpha o f_\alpha \to f$ Then by Proposition(1.4), $z = f(x)$. Thus *z* belongs to the orbit of $x, (I(X))(x)$ so always $z \in \overline{(I(X))(x)}$. Since $\Lambda(x) \ne \phi$, then by Theorem (1.6), $z \in \overline{\Lambda(x)}$. Notice that $\Lambda(x) \subseteq$ M and *M* is closed, then $\overline{\Lambda(x)} \subseteq$ M.
Thus $\Lambda(y) \subseteq$ M, for every $y \in \Lambda(x)$. Then $\Lambda(x) \subseteq \Lambda(M)$.
(iv) Let *y* be a limit point of $\Lambda(M)$. First we show that $\Lambda(M) \subseteq$ M. Let $x \in \Lambda(M)$, then $\Lambda(x) \ne \phi$ and $\Lambda(x) \subseteq$ M. Since *M* is closed and $x \in \overline{\Lambda(x)}$, then $x \in M$. Thus $\Lambda(M) \subseteq$ M. But $\overline{\Lambda(M)}$ is invariant, then $\overline{(I(X))(y)} \subseteq \overline{\Lambda(M)}$. Thus $\Lambda(y) \subseteq$ M. We claim that $\Lambda(y) \ne \phi$, since *y* is a limit of $\Lambda(M)$, then there exists a sequence $\{y_n\}$ in $\Lambda(M)$ such that $y_n \to y$. So for every *n*, there exists $x_n \in X$ such that $y_n \in \Lambda(x_n) \subseteq$ M, therefore there exists a net $\{f_\alpha^n\}$ in $I(X)$ such that $f_\alpha^n \to \infty$ and $f_\alpha^n(x_n) \to y_n$. Since *I(X)* is locally compact then by [11,11D.d, page 77](for the proof see [1]), there exists a diagonal net $\{f_{\alpha_m}^m(x_m)\}$ such that $f_{\alpha_m}^m \to \infty$ and $f_{\alpha_m}^m(x_m) \to y$. But $\{x_m\}$ is a sequence in a compact set M, therefore there exists a subsequence $\{x_k\}$ of $\{x_m\}$ and $x \in M$ such that $x_k \to x$. Thus $x \in \Lambda(M)$. This completes the proof. In the same way we can to prove that $\Lambda_w(M)$ is closed.

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3.6 Corollary: Let *(X,d)* be an *I(X)*- space and *M* be a non-empty compact subset of *X*. If *I(X)* is locally compact then *(M)* is compact.

Proof: We will prove $\Lambda(M) \subseteq$ M, let $x \in \Lambda(M)$, then $\Lambda(x) \neq \phi$ and $\Lambda(x) \subseteq$ M. Since M is a compact subset of a Hausdorff space, then $\overline{\Lambda(x)} \subseteq M$.

But by Theorem (1.6), $x \in \overline{\Lambda(x)}$. Thus $\Lambda(M) \subseteq$ M. Then by Theorem (3.5) $\Lambda(M)$ is compact.

In general $\Lambda_w(M) \neq \Lambda(M)$ as shown in Examples (3.2), and (3.3). But the following theorem shows that $\Lambda_w(M) = \Lambda(M)$ if *M* is invariant.

3.7 Theorem: Let *(X,d)* be an *I(X)*-space and *M* be a non-empty compact subset of *X* .If M is invariant, then

(i) $\Lambda_w(M) = M$

(ii) $\Lambda(M) = \Lambda_w(M)$.

Proof:

(i) Let $x \in \Lambda_w(M)$. Then $\Lambda(x) \cap M \neq \phi$, thus there exists $y \in \Lambda(x) \cap M$. So there exists a net $\{f_\alpha\}$ in $I(X)$ with $f_\alpha \to \infty$ and $f_\alpha(x) \to y$. Since *M* is invariant and $y \in M$, then $\{f_\alpha^{-1}(y)\}$ is a net in *M*. But $f_\alpha(x) \to y$ and $d(f_\alpha^{-1}(y), x) = d(f_\alpha(x), y)$ (Since f_α is an isometry). Then $f_\alpha^{-1}(y) \to x$, so $x \in M$ (since *M* is closed). Thus $\Lambda_w(M) \subseteq$ M.

Now, we prove M $\subseteq \Lambda_w(M)$, let $x \in M$. Since *M* is invariant and closed, then the closure of orbit of *x* is a subset of *M*. Since *M* is compact then $\overline{(\mathrm{I(X)})(x)}$ is compact. So by Proposition (1.7), $\Lambda(x) \neq \phi$.

Now, $\phi \neq \Lambda(x) \subseteq \overline{(\mathrm{I(X)})(x)} \subseteq$ M, then $\Lambda(x) \cap M \neq \phi$, that is M $\subseteq \Lambda_w(M)$. Hence $\Lambda_w(M) = M$.

iii) First we will prove that M $\subseteq \Lambda(M)$. Let $x \in M$. Since *M* is invariant, then the closure orbit of *x* is a closed subset of *M* .But *M* is compact, then the closure orbit of *x* is compact .Thus by Proposition (1.7) , $\Lambda(x) \neq \phi$ and since $\Lambda(x) \subseteq M$, then $x \in \Lambda(M)$. Thus M $\subseteq \Lambda(M)$. So by (i) and Proposition (3.2.5),i, we have $\Lambda(M) = \Lambda_w(M)$.

The converse of Theorem (3.7),ii,is not true in general. In Example (3.3), if we take $M = \{(0,1), (1,0), (-1,0)\}$, then $\Lambda_w(M) = \Lambda(M)$. But *M* is not invariant.

3.8 Definitions,[3]: Let *(X,d)* be an *I(X)*- space .A non-empty compact subset *M* of *X* is said to be,

i) A weak attractor if $\Lambda_w(M)$ is a neighborhood of *M*.

ii) An attractor if $\Lambda(M)$ is a neighborhood of *M*.

iii) Stable if every neighborhood *U* of *M* contains an invariant neighborhood *V* of *M* and if it is not stable, it is called unstable.

3.9 Example: Let *(N,d)* be the discrete metric space where *N* is the set of all positive integers and *M* be a non-empty compact subset of N. Then,

i) *M* is a weak attractor.

ii) *M* is not attractor.

iii) *M* is unstable

Solution

i) It follows that from the solution of Example (3.2) $\Lambda_w(M) = N$ for every nonempty compact *M* of *N* and since $M \subseteq \Lambda_w(M)$ and $\Lambda_w(M)$ is open then *M* is a weak attractor.

ii) See the solution of Example (3.2) $\Lambda(M) = \phi$, for every *M*. Then $M \not\subset \Lambda(M)$ therefore M is not attractor.

iii) Notice that *M* is unstable .Since *N* is a discrete space ,then *M* is a finite set. So *M* is open in *N*.

Now, Put $n = \max\{k/k \in M\}$, define $f: N \to N$ by $f(n) = n + 1$, $f(n + 1) = n$ and $f(r) = r$ for every $r \in N$ distinct from *n* and $n + 1$. So $f \in I(N)$.

But $f(n) = n + 1 \notin M$ .Thus *M* is not invariant. Then *M* is open but not invariant.

Hence *M* is unstable.

3.10 Example: In Example (3.3), if we take $M = \{(1,0), (-1,0)\}$ then *M* is a weak attractor, attractor and stable.

Solution: First we will show that *M* is open, since $X = Y \cup Z$ where $Y = \{(0, y)/y \in R\}$, $Z = \{(z, 0)/z \geq 1 \; or \; z \leq -1\}$ and $d = \min\{1, d'\}$, where d' is the Euclidean metric, then $B\left((1,0), \frac{1}{2}\right) = \{(1,0)\}$ and $B\left((-1,0), \frac{1}{2}\right) = \{(-1,0)\}$, Thus *M* is open. See the solution of Example (3.3) $\Lambda\big((1,0)\big) = \{(1,0), (-1,0)\}$ and $\Lambda\big((-1,0)\big) = \{(1,0), (-1,0)\}$. Thus $\Lambda_w(M) = \Lambda(M) = \{(1,0), (-1,0)\}$. Then $\Lambda_w(M)$ and $\Lambda(M)$ are neighborhoods of *M*, thus *M* is a weak attractor and attractor.

It is clear that, for every $f \in I(X)$, then either $f\big((1,0)\big) = (1,0), f\big((-1,0)\big) = (-1,0)$ or $f\big((1,0)\big) = (-1,0), f\big((-1,0)\big) = (1,0)$, thus M is invariant and since M is open .Then M is stable .

Now we are ready to prove some results about the concepts that introduced.

3.11 Proposition: Let *(X,d)* be an *I(X)*-space and *M* be a non-empty compact subset of X. If *M* is a weak attractor or attractor ,then the corresponding sets $\Lambda_w(M)$ or $\Lambda(M)$ are open.

Proof: Let Y denote any one of the sets $\Lambda_w(M)$ or $\Lambda(M)$ since *Y* is a neighborhood of M, then there exists an open set *U* such that $M \subseteq U \subseteq Y$ thus $U \cap Y^c$. Since *U* is open then $U \cap \partial Y^c = \phi$ (where ∂Y^c is the boundary of Y^c). So $U \cap \partial Y = \phi$ (Since $\partial Y^c = \partial Y$), thus $M \cap \partial Y = \phi$.

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Let $Y = \Lambda(M)$, suppose that $Y \cap \partial Y \neq \phi$, then there exists $x \in Y \cap \partial Y$. So $\Lambda(M) \subseteq M$ and $\Lambda(x) \neq \phi$ (since $Y = \Lambda(M)$) .Then by Theorem (1.6) ,we have $x \in M$, a contradiction(since $M \cap \partial Y = \phi$) So $Y \cap \partial Y = \phi$, then Y is open. Also we want to prove that $\Lambda_w(M)$ is open. Suppose that $Y \cap \partial Y \neq \phi$ (where $Y = \Lambda_w(M)$) thus there exists $x \in Y \cap \partial Y$, so $\Lambda(x) \cap M \neq \phi$, that is there exists $y \in \Lambda(x) \cap M$. Then there exists a net $\{f_n\}$ in *I(X)* such that $f_\alpha \to \infty$ and $f_\alpha(x) \to y$. Since ∂Y is invariant and closed, then $y \in \partial Y$, a contradiction (since $M \cap \partial Y = \phi$). Hence $Y \cap \partial Y = \phi$ and thus Y is open.

3.12 Theorem: Let *(X,d)* be an *I(X)*-space and let *M* be a non-empty compact subset of *X*. Then *M* is an attractor if and only if *M* is invariant and open.

Proof: →). Let *M* be an attractor, then $\Lambda(M)$ is a neighborhood of *M* thus $M \subseteq \Lambda(M)$, and also $\Lambda(M) \subseteq M$. So $M = \Lambda(M)$. Hence by and Theorem (3.5),ii, and by Proposition (3.11)M is open and invariant.

←). Let *M* be open and invariant. Then by Theorem (3.5), $\Lambda(M) = M$. Thus *M* is an attractor.

3.13 Theorem: Let *(X,d)* be an *I(X)*-space and let *M* be a non-empty compact subset of *X* . If *M* is stable ,then

(i) *M* is invariant.

(ii) *If M* is a singleton *{x}*, then is a critical point.

Proof:

Let *D* be the intersection of all invariant neighborhoods of *M*. Since *X* is invariant then $D \neq \phi$ and $M \subseteq D$. Suppose that $D \not\subset M$, thus there exists $y \in D$ and *y M*. Since *(X,d)* is a metric space.

i) So $X\backslash\{y\}$is an open set and $M \subseteq X\backslash\{y\}$. But *M* is stable, then there exists an invariant neighborhood *U* of *M* such that $M \subseteq U \subseteq X\backslash\{y\}$. From the definition of *D*, we have $D \subseteq U$, then $y \in U$, a contradiction, (since $D \subseteq X\backslash\{y\}$). Thus must be *M=D* .So *M* is invariant.

Let *M={x}*, then by (i) , we have *{x}* is invariant, that is $f\{x\} \in \{x\}$ for every $f \in I(X)$. So *x* is a critical point.

In Example (3.9), *M* is open and unstable, this example gives a motivation to the following proposition.

3.14 Proposition: Let *(X,d)* be an *I(X)*-space and let *M* be a non-empty compact subset of *X*. If *M* is open ,then *M* is stable if and only if, is invariant

Proof:

→). By Theorem (3.13) .

←). Since M is open and invariant then every neighborhood of *M* contains an invariant neighborhood of *M*. Thus *M* is stable.

3.15 Corollary: Let *(X,d)* be an *I(X)*-space and *M* be a non-empty compact invariant subset of *X*. If $int\ (\overline{(I(X))(x)}) \neq \phi$ for every $x \in M$ then *M* is stable.

Proof: Since *M* is invariant and compact, then $M = \cup_{x \in M} \overline{I(X)(x)}$.Since $\overline{(I(X))(x)}$ has a non-empty interior for every $x \in M$, then by Theorem(2.4) and Proposition (2.10), $\overline{(I(X))(x)}$ is open, for every $x \in M$. Thus *M* is open ,then by Proposition (3.14) , *M* is stable.

We study now the relation between attractor and stability.

3.16 Theorem: Let *(X,d)* be an *I(X)*-space. If a subset *M* of *X* is attractor, then it is weak attractor .

Proof: By Theorem (3.12) and Theorem (3.7).

The converse of Theorem (3.16) is not true in general, see Example (3.9).

It follows from Theorem (3.7), the following Proposition.

3.17 Proposition: Let *(X,d)* be an *I(X)*-space and *M* be an invariant compact subset of *X* .Then *M* is attractor if and only if *M* is a weak attractor.

3.18 Theorem: Let *(X,d)* be an *I(X)*-space. If a compact subset *M* of *X* is an attractor, then *M* is stable.

Proof: By Theorem (3.12) and Proposition (3.14).

The converse of Theorem (3.18) is true if *M* is open, as shown by the following theorem.

3.19 Theorem: Let *(X,d)* be an *I(X)*-space and *M* be an open compact subset of *X*. If *M* is stable, then it is attractor .

Proof: By Theorem (3.13) ,i, and theorem(3.12).

3.20 Corollary: Let *(X,d)* be an *I(X)*-space and *M* be a compact open set. If *M* is stable then *M* is a weak attractor.

Proof: By Theorem (3.16) and Theorem (3.19).

In Example (3.9), there exists a weak attractor but it is unstable.

If we take *M* to be invariant, then the converse of corollary is true, see the following theorem

3.21 Theorem: Let *(X,d)* be an *I(X)*-space and *M* be an invariant compact subset of *X*. If M is a weak attractor, then *M* is stable.

Proof: By Theorem (3.7) and Theorem (3.18).

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