On The Algebraic Properties of *l*-Congruencies in Groups

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Abstract
This paper is dedicated to defining and studying the concept of congruencies in *l*-groups, where we prove the following main results:

1) If θ is a congruence relation on *l*-group G, then $\frac{G}{\theta}$ is *l*-group.
2) If $\frac{G}{\theta}$ is *l*-group, then for $\frac{x}{\theta},\frac{y}{\theta}\ belongs\ to \frac{G}{\theta}$ holds:
   $x, y\ are\ equivalent\ (mod\ \theta)$ if and only if $\frac{x\wedge y}{\theta},\frac{x\vee y}{\theta}$ are equivalent.

Also, we illustrate many examples to clarify the validity of our work.

Keywords: Group; l-Congruence; order relation; Lattice-ordered group.

1. Introduction and preliminaries

The concept of lattice-ordered group is an interesting algebraic concept that concerns groups and lattices.
Lattice-ordered group was studied by Dedekeind in [1], and then it was studied by levi [2].
Birkhoff dealt with *l*-groups, *l*-subgroups, and *l*-ideals [3].
In the literature, many researchers have studied the congruencies lattices on *l*-abelian groups,see [5-6].
In [7], we find the following definition: If $G \neq \phi$ is a non-empty set, with $+: G \times G \to G$, and an order relation (≤), G is called a partially ordered group (PO-group) if:

1- $(G, +)$ is a group.
2- $(G, \leq)$ is a partially ordered set.
3- $\forall\ x, y, z \in G$, then:

$$\begin{cases} x \leq y \Rightarrow x + z \leq y + z & (M_1) \\ x \leq y \Rightarrow z + x \leq z + y & (M_2) \end{cases}$$

Definition: [7]:
Let G be a PO-group, then if $(G, \leq)$ is a lattice, then G is called a lattice-ordered group or (*l*-group). We denote it by $(G, +, \leq, \vee, \wedge)$.

Definition: [8]:
Let A,B be two lattices, with $f: A \to B$, then:
1] f is called ordering mapping if:
$\forall\ x, y \in G; x \leq y \Rightarrow f(x) \leq f(y)$.
2] f is called anti-ordering mapping if:
$\forall\ x, y \in A; x \leq y \Rightarrow f(y) \geq f(x)$.

Definition: [8]:
Let G,H be two *l*-groups, $f: G \to H$ is a mapping, then:
1] f is called *l*-ordering homomorphism if it is a group homomorphism and a lattice ordering homomorphism, i.e.
$(H_1)\ f(a + b) = f(a) + f(b)$
$(H_2)\ f(a \vee b) = f(a) \vee f(b)$
$(H_3)\ f(a \wedge b) = f(a) \wedge f(b)\ \forall\ a, b \in G$
2] f is called *l*-anti ordering homomorphism if:
$(H_1)'\ f(a + b) = f(a) + f(b)$
$(H_2)'\ f(a \vee b) = f(a) \wedge f(b)$
$(H_3)'\ f(a \wedge b) = f(a) \vee f(b)\ \forall\ a, b \in G$

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2. Main discussion:

Definition:

Let $f: G \to H$ be l-homomorphism, then:

1] If f is a bijective mapping, and $f, f^{-1}$are ordering mappings, then G,H are called ordered isomorphic groups $G \cong H$.

2] If f is a bijection, and f, f^{-1} are anti-ordering mappings, then G,H are called anti-ordered isomorphic groups $G \cong H$.

Definition:

Let G be l-group, θ is called a congruence relation on G if:

1] θ is an equivalence relation on G.

2] If $a_1, b_1, a_2, b_2 \in G; \begin{cases} a_1 \equiv b_1 (mod\ \theta) \\ a_2 \equiv b_2 (mod\ \theta) \end{cases}$, then

$\begin{cases} a_1 + a_2 = b_1 + b_2 (mod\ \theta) \\ a_1 \vee a_2 = b_1 \vee b_2 (mod\ \theta) \\ a_1 \wedge a_2 = b_1 \wedge b_2 (mod\ \theta) \end{cases}$

For $a \in G$, we define: $\frac{a}{\theta} = \{x \in G;\ x \equiv a (mod\ \theta)\}$.

$\frac{G}{\theta}$ denotes to the all classes $\frac{a}{\theta}$ for $a \in G$.

Theorem:

Let G be l-group, θ be a congruence relation on G, we define ($\widetilde{\leq}$) on ($\frac{G}{\theta}$) by:

$\frac{x}{\theta} \widetilde{\leq} \frac{y}{\theta} \Leftrightarrow \exists\ a \in \frac{x}{\theta}, b \in \frac{y}{\theta}; a \leq b; x, y \in G$ (1)

Then ($\widetilde{\leq}$) is an order relation on $\frac{G}{\theta}$.

Proof:

For $a, b \in G$, then: $a \leq b \Longrightarrow \frac{a}{\theta} \widetilde{\leq} \frac{b}{\theta}$ (2)

$\forall\ \frac{x}{\theta} \in \frac{G}{\theta}$, then $x \in G$ and $x \leq x$, hence $\frac{x}{\theta} \widetilde{\leq} \frac{x}{\theta}$.

Let $\frac{x}{\theta}, \frac{y}{\theta}, \frac{z}{\theta} \in \frac{G}{\theta}; \frac{x}{\theta} \widetilde{\leq} \frac{y}{\theta}, \frac{y}{\theta} \widetilde{\leq} \frac{z}{\theta}$, hence:

$\frac{x}{\theta} \widetilde{\leq} \frac{y}{\theta} \Longrightarrow \exists\ a \in \frac{x}{\theta}, b \in \frac{y}{\theta}; a \leq b \Longrightarrow a \equiv x (mod\ \theta), b \equiv y (mod\ \theta) \Longrightarrow x \wedge y \equiv a \wedge b \equiv a (mod\ \theta)$.

Also, $\frac{y}{\theta} \widetilde{\leq} \frac{z}{\theta} \Longrightarrow \exists\ c \in \frac{y}{\theta}, d \in \frac{z}{\theta}; c \leq d \Longrightarrow c \equiv y (mod\ \theta), d \equiv z (mod\ \theta) \Longrightarrow y \wedge z \equiv c \wedge d \equiv d (mod\ \theta)$

Thus, $x \wedge y \leq y \vee z \Longrightarrow \frac{x \wedge y}{\theta} \widetilde{\leq} \frac{y \vee z}{\theta}$, thus:

$\frac{a}{\theta} \widetilde{\leq} \frac{d}{\theta} \Longrightarrow \frac{x}{\theta} \widetilde{\leq} \frac{z}{\theta}$.

Let $\frac{x}{\theta}, \frac{y}{\theta} \in \frac{G}{\theta}, \frac{y}{\theta} \widetilde{\leq} \frac{x}{\theta}, \frac{x}{\theta} \widetilde{\leq} \frac{y}{\theta}$, we have:

$\frac{x}{\theta} \widetilde{\leq} \frac{y}{\theta} \Longrightarrow \exists\ a \in \frac{x}{\theta}, b \in \frac{y}{\theta}; a \leq b \Longrightarrow a \equiv x (mod\ \theta), b \equiv y (mod\ \theta) \Longrightarrow x \wedge y \equiv a \wedge b = a \equiv x (mod\ \theta)$.

$\frac{y}{\theta} \widetilde{\leq} \frac{x}{\theta} \Longrightarrow \exists\ c \in \frac{y}{\theta}, d \in \frac{x}{\theta}; c \leq d \Longrightarrow c \equiv y (mod\ \theta), d \equiv x (mod\ \theta) \Longrightarrow x \wedge y \equiv c \wedge d = c \equiv y (mod\ \theta)$.

Hence, $x \equiv x \wedge y \equiv y (mod\ \theta)$, thus $\frac{x}{\theta} = \frac{y}{\theta}$.

Theorem:

Let G be an l-group, θ be a congruence relation on G.

We define $\widetilde{+}, \widetilde{\vee}, \widetilde{\wedge}$ on $\frac{G}{\theta}$ as follows:

$\begin{cases} \frac{a}{\theta} \widetilde{+} \frac{b}{\theta} = \frac{a+b}{\theta} \\ \frac{a}{\theta} \widetilde{\vee} \frac{b}{\theta} = \frac{a \vee b}{\theta} \\ \frac{a}{\theta} \widetilde{\wedge} \frac{b}{\theta} = \frac{a \wedge b}{\theta} \end{cases}$

$(\frac{G}{\theta}, \widetilde{\leq}, \widetilde{+}, \widetilde{\vee}, \widetilde{\wedge})$ is l-group.

Proof:

Assume that $\frac{x_1}{\theta}, \frac{x_2}{\theta}, \frac{y_1}{\theta}, \frac{y_2}{\theta} \in \frac{G}{\theta}$;

$$(\frac{x_1}{\theta}, \frac{x_2}{\theta}) = (\frac{y_1}{\theta}, \frac{y_2}{\theta}) \Rightarrow \begin{cases} \frac{x_1}{\theta} = \frac{y_1}{\theta} \\ \frac{x_2}{\theta} = \frac{y_2}{\theta} \end{cases}$$

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$$\Rightarrow \begin{cases} x_1 \equiv y_1 (mod\ \theta) \\ x_2 \equiv y_2 (mod\ \theta) \\ x_1 + x_2 \equiv y_1 + y_2 (mod\ \theta) \\ \dfrac{x_1 + x_2}{\theta} \equiv \dfrac{y_1 + y_2}{\theta} \end{cases}$$

$\Rightarrow \frac{x_1}{\theta} \tilde{+} \frac{x_2}{\theta} = \frac{y_1}{\theta} \tilde{+} \frac{y_2}{\theta}$, this means that $\tilde{+}$ is well defined.
By a similar argument, we prove the same for $\tilde{\wedge}, \tilde{\vee}$.
Now, $\forall\ \frac{a}{\theta}, \frac{b}{\theta}, \frac{c}{\theta} \in \frac{G}{\theta}$, then:

$$(\frac{a}{\theta} \tilde{+} \frac{b}{\theta}) \tilde{+} \frac{c}{\theta} = \frac{a+b}{\theta} \tilde{+} \frac{c}{\theta} = \frac{a+b+c}{\theta} = \frac{a}{\theta} \tilde{+} \frac{b+c}{\theta} = \frac{a}{\theta} \tilde{+} (\frac{b}{\theta} \tilde{+} \frac{c}{\theta})$$

$\forall\ \frac{a}{\theta} \in \frac{G}{\theta}$, then $\frac{a}{\theta} \tilde{+} \frac{e}{\theta} = \frac{a+e}{\theta} = \frac{a}{\theta}$

On the other hand, we have:
For $\frac{x}{\theta}, \frac{y}{\theta} \in \frac{G}{\theta}, x, y \in G, x \leq x \vee y, y \leq x \vee y$, hence

$$\begin{cases} \dfrac{x}{\theta} \lesssim \dfrac{x \vee y}{\theta} \\ \dfrac{y}{\theta} \lesssim \dfrac{x \vee y}{\theta} \end{cases}$$

Thus, $\frac{x \vee y}{\theta}$ is an upper bound of $\frac{x}{G}, \frac{y}{G}$ with respect to $\lesssim$.
Assume that $\frac{z}{G}$ is another upper bound of $(\frac{x}{G}, \frac{y}{G})$, then

$$\begin{cases} \dfrac{x}{\theta} \lesssim \dfrac{z}{\theta} \\ \dfrac{y}{\theta} \lesssim \dfrac{z}{\theta} \end{cases} \Rightarrow \begin{cases} \exists\ a \in \dfrac{x}{\theta}, b \in \dfrac{z}{\theta};\ a \leq b \\ \exists\ c \in \dfrac{y}{\theta}, d \in \dfrac{z}{\theta};\ c \leq d \end{cases}$$

$$\Rightarrow \begin{cases} a \equiv x\ (mod\ \theta), b \equiv z (mod\ \theta);\ a \leq b \\ c \equiv y\ (mod\ \theta), d \equiv z (mod\ \theta);\ c \leq d \end{cases}$$

$$\Rightarrow \begin{cases} x \vee y \equiv a \vee c\ (mod\ \theta), b \vee d \equiv z (mod\ \theta);\ a \vee c \leq b \vee d \\ a \vee c \in \dfrac{x \vee y}{\theta}, \dfrac{b \vee d}{\theta} \in \dfrac{z}{\theta};\ a \vee c \leq b \vee d \end{cases}$$

$\Rightarrow \frac{x \vee y}{\theta} \lesssim \frac{z}{\theta}$, which means that $\frac{x \vee y}{\theta}$ is the minimal upper bound of $\frac{x}{\theta}, \frac{y}{\theta}$, so that:

$$\frac{x}{\theta} \tilde{\vee} \frac{y}{\theta} = \frac{x \vee y}{\theta} = sup\ \{\frac{x}{\theta}, \frac{y}{\theta}\}$$

By a similar method, we get:

$$\frac{x}{\theta} \tilde{\wedge} \frac{y}{\theta} = \frac{x \wedge y}{\theta} = inf\ \{\frac{x}{\theta}, \frac{y}{\theta}\}$$

Finally, if $\frac{x}{\theta}, \frac{y}{\theta} \in \frac{G}{\theta}$, then:
$\frac{x}{\theta} \lesssim \frac{y}{\theta} \Longrightarrow \frac{x}{\theta} \tilde{+} \frac{z}{\theta} \lesssim \frac{y}{\theta} \tilde{+} \frac{z}{\theta}\ \forall\ \frac{z}{\theta} \in \frac{G}{\theta}$
Since $\frac{x}{\theta} \lesssim \frac{y}{\theta} \Longrightarrow \exists\ a \in \frac{x}{\theta}, b \in \frac{y}{\theta};\ a \leq b$

$$\Rightarrow \begin{cases} a \equiv x\ (mod\ \theta) \\ b \equiv y\ (mod\ \theta);\ a \leq b \end{cases}$$

For any $c \in \frac{z}{\theta}, c \in G$, and G is l-group, we get:

$$\begin{cases} a + c \leq b + c \\ \dfrac{a+c}{\theta} \lesssim \dfrac{b+c}{\theta} \\ \dfrac{a}{\theta} \tilde{+} \dfrac{c}{\theta} \lesssim \dfrac{b}{\theta} \tilde{+} \dfrac{c}{\theta} \\ \dfrac{x}{\theta} \tilde{+} \dfrac{z}{\theta} \lesssim \dfrac{y}{\theta} \tilde{+} \dfrac{z}{\theta} \end{cases}$$

By a similar argument, we get:
$\frac{x}{\theta} \lesssim \frac{y}{\theta} \Rightarrow \frac{z}{\theta} \tilde{+} \frac{x}{\theta} \lesssim \frac{z}{\theta} \tilde{+} \frac{y}{\theta}; \frac{z}{\theta} \in \frac{G}{\theta}$, thus
$(\frac{G}{\theta}, \lesssim, \tilde{+}, \tilde{\vee}, \tilde{\wedge})$ is l-group.

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Theorem:

Let G be l-group, and $(\frac{x}{\theta},\frac{y}{\theta}) \in \frac{G}{\theta}$ with $\frac{x}{\theta} \lesseqgtr \frac{y}{\theta}$, then $x \equiv y(mod\theta)$ if and only if $\frac{x\wedge y}{\theta} = \frac{x\vee y}{\theta}$

Proof:

Assume that $x \equiv y(mod\theta)$, then:

$$\begin{cases} x \wedge y \equiv x \wedge x \ \equiv x(mod\ \theta) \\ x \vee y \equiv x \vee x \ \equiv x(mod\ \theta) \end{cases}$$

Hence, $x \wedge y \equiv x \vee y(mod\ \theta)$, thus $\frac{x\wedge y}{\theta} = \frac{x\vee y}{\theta}$.

For the converse, assume that $\frac{x\wedge y}{\theta} = \frac{x\vee y}{\theta}$, hence:

$$\frac{x}{\theta} \tilde{\wedge} \frac{y}{\theta} \lesseqgtr \frac{x}{\theta} \leq \frac{x}{\theta} \tilde{\vee} \frac{y}{\theta} \Rightarrow \frac{x \wedge y}{\theta} \lesseqgtr \frac{x}{\theta} \lesseqgtr \frac{x \vee y}{\theta}$$

$$\Rightarrow \frac{x \wedge y}{\theta} \lesseqgtr \frac{x}{\theta} \lesseqgtr \frac{x \wedge y}{\theta} \Rightarrow \frac{x \wedge y}{\theta} = \frac{x}{\theta} \Rightarrow x \wedge y \equiv x(mod\ \theta)$$

By a similar argument, we find that $x \wedge y \equiv y(mod\ \theta)$, thus $x \equiv y(mod\ \theta)$

Theorem:

Let G be l-group, $\alpha: G \to \frac{G}{\theta}$; $\alpha(x) = \frac{x}{\theta}$ is an ordering surjective homomorphism.

Proof:

$\forall\ x.y \in G$, we have:

$$\Rightarrow \begin{cases} \alpha(x \vee y) = \frac{x \vee y}{\theta} = \frac{x}{\theta} \tilde{\vee} \frac{y}{\theta} = \alpha(x) \vee \alpha(y) \\ \alpha(x \wedge y) = \frac{x \wedge y}{\theta} = \frac{x}{\theta} \tilde{\wedge} \frac{y}{\theta} = \alpha(x) \wedge \alpha(y) \\ \alpha(x + y) = \frac{x + y}{\theta} = \frac{x}{\theta} \tilde{+} \frac{y}{\theta} = \alpha(x) + \alpha(y) \end{cases}$$

If $x \leq y$, then, $x \leq y \Rightarrow x = x \wedge y \Rightarrow \alpha(x) = \alpha(x \wedge y) \Rightarrow \frac{x}{\theta} = \frac{x\wedge y}{\theta} = \frac{x}{\theta} \tilde{\wedge} \frac{y}{\theta} \Rightarrow \frac{x}{\theta} \lesseqgtr \frac{y}{\theta} \Rightarrow \alpha(x) \lesseqgtr \alpha(y)$

Also, for $y = \frac{x}{\theta} \in \frac{G}{\theta}$, then $\alpha(x) = \frac{x}{\theta} = y$, hence α is surjective.

Theorem:

Let G_1, G_2 be two l-groups, then every anti-ordering homomorphism $f: G_1 \to G_2$ defines a congruence relation θ_f on G_1 by the following:

$$x\ \equiv y(mod\ \theta_f) \Leftrightarrow f(x) = f(y); x, y \in G_1$$

Proof:

$\forall\ x \in G_1$, then $f(x) = f(y)$, hence $x \equiv x(mod\ \theta_f)$

For $x, y \in G_1$, then $x\ \equiv y(mod\ \theta_f) \Longrightarrow f(x) = f(y) \Longrightarrow f(y) = f(x) \Longrightarrow y \equiv x(mod\ \theta_f)$.

For $x, y, z \in G_1$, if $\begin{cases} x\ \equiv y(mod\ \theta_f) \\ y \equiv z(mod\ \theta_f) \end{cases}$

We get: $\begin{cases} f(x) = f(y) \\ f(y) = f(z) \end{cases} \Longrightarrow f(x) = f(z) \Longrightarrow x \equiv z(mod\ \theta_f)$

Let $x, x_1, y, y_1 \in G_1$, then: $\begin{cases} x\ \equiv x_1(mod\ \theta_f) \\ y \equiv y_1(mod\ \theta_f) \end{cases}$

$$\Longrightarrow \begin{cases} f(x) = f(x_1) \\ f(y) = f(y_1) \end{cases} \Longrightarrow \begin{cases} f(x) \vee f(y) = f(x_1) \vee f(y_1) \\ \quad f(x \vee y) = f(x_1 \vee y_1) \end{cases}$$

$$\Longrightarrow x \vee y \equiv x_1 \vee y_1(mod\ \theta_f)$$

By a similar argument, we get $\begin{cases} x \wedge y \equiv x_1 \wedge y_1(mod\ \theta_f) \\ x + y \equiv x_1 + y_1(mod\ \theta_f) \end{cases}$

Theorem:

Let G_1, G_2 be two l-groups, $f: G_1 \to G_2$ be an anti-ordering homomorphism, then $\frac{G_1}{\theta_f} \cong f(G_1)$

Proof:

We define: $\Psi: \frac{G_1}{\theta_f} \to f(G_1); \Psi(\frac{x}{\theta_f}) = f(x)$.

If $\frac{x}{\theta_f} = \frac{y}{\theta_f} \Longrightarrow x \equiv y(mod\ \theta_f) \Longrightarrow f(x) = f(y) \Longrightarrow \Psi(\frac{x}{\theta_f}) = \Psi(\frac{y}{\theta_f})$.

$\forall\ \boldsymbol{x}, \boldsymbol{y} \in G_1$, then:

$\Psi(\frac{x}{\theta_f} \tilde{\vee} \frac{y}{\theta_f}) = \Psi(\frac{x\vee y}{\theta_f}) = \mathrm{f}(\mathrm{x} \vee \mathrm{y}) = \mathrm{f}(\mathrm{x}) \vee \mathrm{f}(\mathrm{y}) = \Psi(\frac{x}{\theta_f}) \vee \Psi(\frac{y}{\theta_f})$.

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By a similar method, we get $\Psi(\frac{x}{\theta_f}\tilde{\wedge}\frac{y}{\theta_f}) = \Psi(\frac{x}{\theta_f}) \wedge \Psi(\frac{y}{\theta_f})$.

$\Psi(\frac{x}{\theta_f}\tilde{+}\frac{y}{\theta_f}) = \Psi(\frac{x+y}{\theta_f}) = f(x+y) = f(x) + f(y) = \Psi(\frac{x}{\theta_f}) + \Psi(\frac{y}{\theta_f})$.

If $\Psi(\frac{x}{\theta_f}) = \Psi(\frac{y}{\theta_f}) \Rightarrow f(x) = f(y) \Rightarrow x \equiv y(mod\ \theta_f) \Rightarrow \frac{x}{\theta_f} = \frac{y}{\theta_f}$.

For any $\boldsymbol{y} \in f(G_1)$, there exists $\boldsymbol{x} \in G_1; f(y) = y$, thus $\frac{x}{\theta_f} \in \frac{G_1}{\theta_f}$, thus $\Psi(\frac{x}{\theta_f}) = f(x) = y$.

Ψ is ordering mapping: for $\boldsymbol{x}, \boldsymbol{y} \in G_1$, then:

$$\frac{x}{\theta_f} \lessapprox \frac{y}{\theta_f} \Rightarrow \frac{x}{\theta_f}\tilde{\vee}\frac{y}{\theta_f} = \frac{y}{\theta_f} \Rightarrow \frac{x \vee y}{\theta_f} = \frac{y}{\theta_f}$$

$$\Rightarrow \begin{cases} x \vee y \equiv y(\mathrm{mod}\ \theta_f) \\ f(x \vee y) = f(y) \\ f(x) \vee f(y) = f(y) \\ f(x) \leq f(y) \\ \Psi(\frac{x}{\theta_f}) \leq \Psi(\frac{y}{\theta_f}) \end{cases}$$

The mapping Ψ^{-1} is an ordering mapping, that is be cause:

If $y_1, y_2 \in f(G_1)$, there exists $x_1, x_2 \in f(G_1)$ such that:

$$\begin{cases} f(x_1) = y_1 \\ f(x_2) = y_2 \end{cases}$$

If $y_1 \leq y_2$, then:

$$y_1 \leq y_2 \Rightarrow f(x_1) \leq f(x_2) \Rightarrow f(x_1) \vee f(x_2) \leq f(x_2) \Rightarrow f(x_1 \vee x_2) = f(x_2) \Rightarrow \Psi(\frac{x_1 \vee x_2}{\theta_f}) = \Psi(\frac{x_2}{\theta_f}) \Rightarrow \frac{x_1 \vee x_2}{\theta_f}$$
$$= \frac{x_2}{\theta_f} \Rightarrow \frac{x_1}{\theta_f}\tilde{\vee}\frac{x_2}{\theta_f} = \frac{x_2}{\theta_f} \Rightarrow \frac{x_1}{\theta_f} \lessapprox \frac{x_2}{\theta_f} \Rightarrow \Psi^{-1}(f(x_1)) \leq \Psi^{-1}(f(x_2)) \Rightarrow \Psi^{-1}(y_1) \leq \Psi^{-1}(y_2)$$

Hence, $\frac{G_1}{\theta_f} \cong f(G_1)$.

3. Conclusion

In this paper, we have studied the concept of congruencies in *l*-groups, where we proved the following main results:

1) If θ is a congruence relation on *l*-group G, then $\frac{G}{\theta}$ is *l*-group.
2) If $\frac{G}{\theta}$ is *l*-group, then for $\frac{x}{\theta}, \frac{y}{\theta} \in \frac{G}{\theta}$ holds:
   $x \equiv y(mod\ \theta)$ if and only if $\frac{x \wedge y}{\theta} \equiv \frac{x \vee y}{\theta}$.

In the future, we aim that our results can be transmitted to rings, and semi-groups.

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