On Some Results About The n-Potent Fuzzy Groups and Anti-Fuzzy Groups

Walter C. Toapanta[1,*], Jorge Lenín A. Espinoza[2], Silvio M. Vivar[3], Tania L. Vizcaíno Cárdenas[2]

[1]Universidad Regional Autónoma de los Andes (UNIANDES), Ecuador
[2]Docente de la Universidad Técnica de Cotopaxi, Extensión Pujilí. Ecuador

Emails: ua.walterculque@uniandes.edu.ec; ui.jorgeacosta@uniandes.edu.ec; c.investigacionstd@uniandes.edu.ec; tania.vizcaino@utc.edu.ec

Abstract

In this paper, we have studied for the first time the concept of n-potent fuzzy groups and n-potent anti-fuzzy groups. Many related properties will be proved such as the intersection of n-potent fuzzy groups, the product of n-potent anti-fuzzy groups, and the factor groups formed by these structures.

Keywords: n-potent fuzzy group; n-potent anti-fuzzy group; intersection; factor fuzzy subgroup

1. Introduction and preliminaries

The algebra associated with fuzzy sets plays a major role in many studies in the field of theoretical mathematics, where we find that these sets have been widely used in the study of groups, rings, vector spaces, and also matrices [1-7].

On the other hand, many algebraic structures have appeared that are related to Fuzzy sets, which represent a generalization of them, such as neutrosophic and refined neutrosophic sets and their wide applications [8-12].

Fuzzy groups have been studied by many researchers, where a lot of their properties have been deduced such as subgroups, normal groups, solvability, and also their homomorphisms, see [2-4].

In this research paper, we focus our effort on taking advantage of the natural powers of group elements [13-14] and relating them to fuzzy groups, where we define for the first time the concept of n-potent fuzzy groups and n-potent anti-fuzzy groups. Many related properties will be proved such as the intersection of n-potent fuzzy groups, the product of n-potent anti fuzzy groups, and the factor groups formed by these structures.

2. Main discussion

Definition:
Let (G,μ) be a fuzzy group, we say that it is n-potent fuzzy group if and only if $\mu(x^n)=\mu(x)$ for all $x\in G$.
Example.
Consider $Z_5{}^* = \{1,2,3,4\}$ with $\mu: Z_5{}^* \to [0,1]$ such that:

$$\begin{cases}\mu(1)=0,\mu(3)=\dfrac{1}{3}\\ \mu(2)=\mu(4)=\dfrac{1}{3}\end{cases}$$

$(Z_5{}^*,\mu)$ is a fuzzy group that is because:

$$\mu(3^{-1})=\mu(2)=\mu(3)=\frac{1}{3},\mu(2^{-1})=\mu(2)=\mu(3)=\frac{1}{3},\mu(4^{-1})=\mu(4)=\frac{1}{4}$$

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$$\begin{cases} \mu(2,3) = \mu(1) \leq min(\mu(2), \mu(1)) \\ \mu(2,4) = \mu(3) = \frac{1}{3} \leq min(\mu(2), \mu(4)) \\ \mu(3,4) = \mu(2) = \frac{1}{3} \leq min(\mu(3), \mu(4)) \end{cases}$$

$(Z_5{}^*, \mu)$ is a 3-potent fuzzy group, that is because:

$$\begin{cases} \mu(3^3) = \mu(2) = \frac{1}{3} = \mu(3) \\ \mu(2^3) = \mu(3) = \frac{1}{3} = \mu(2) \\ \mu(4^3) = \mu(4) = \frac{1}{4} \end{cases}$$

Definition.
Let (G, μ) be a fuzzy group with a subgroup $H \leq G$, hence H is called n-potent fuzzy subgroup if and only if $\mu(x^n) = \mu(x)$ or all $x \in H$.

Theorem.
Let (G, μ) be a fuzzy group, H, K are two subgroups of G, hence:
1). If G is an n-potent fuzzy group, then H is an n-potent fuzzy subgroup.
2). If G is n-potent, then $H \cap K$ is n-potent.
3). If H, K are two n-potent fuzzy subgroups, then $H \cap K$ is n-potent.

Proof.
1). $\forall x \in H$, then $x \in G$, so that $\mu(x^n) = \mu(x)$, thus H is n-potent.
2). It holds by a similar argument.
3). Let $x \in H \cap K$, then $x \in H$ and $\mu(x^n) = \mu(x)$, this implies that $H \cap K$ is n-potent fuzzy subgroup.

Theorem.
Let (G, μ) be a fuzzy group, and H be a normal subgroup of G with $\acute{\mu}: G/H \to [0,1]$ such that:

$$\begin{cases} \acute{\mu}(xH) = \mu(x); x \notin H \\ \acute{\mu}(xH) = \mu(e); x \in H \end{cases}$$

If G is n-potent fuzzy group and $\mu(h) = \mu(e)$ for all $h \in H$ then G/H is n-potent fuzzy group if $x^n \notin H$ for all $x \notin H$.

Proof.
Let $x \notin H$ and $x^n \notin H$, then $\acute{\mu}(x^n H) = \mu(x^n) = \mu(x) = \acute{\mu}(xH)$.
If $x \in H$, then $\acute{\mu}(x^n H) = \mu(e) = \mu(x) = \acute{\mu}(xH)$.
Thus the proof is complete.

Theorem.
Let (G, μ) be a fuzzy group, and H, K are two subgroups of G, if H is n-potent and K is m-potent, then $H \cap K$ is an nm-potent fuzzy subgroup.

Proof.
Let $x \in H \cap K$, then $x \in H$ and $x \in K$, so that $\mu(x^{nm} H) = \mu((x^n)^m) = \mu(x^n) = \mu(x)$, thus $H \cap K$ is mn-potent subgroup.

Theorem.
Let G, T be two groups, with $\mu: G \to [0,1], \alpha: T \to [0,1]$, where $(G, \mu), (T, \alpha)$ are two fuzzy groups, then if G and T are two n-potent groups, then $G \times T$ is n-potent fuzzy group.

Proof.
Define $\beta: G \times T \to [0,1]$ such that:
$\beta(g, t) = min(\mu(g), \alpha(t))$, first we prove that $(G \times T, \beta)$ is fuzzy group.
Let $g_1, g_2 \in G, t_1, t_2 \in T$, then:
$\beta[(g_1, t_1)(g_2, t_2)] = \beta(g_1 g_2, t_1 t_2) = min(\mu(g_1 g_2), \alpha(t_1 t_2))$.
We have:

$$\begin{cases} \mu(g_1 g_2) \leq min(\mu(g_1), \mu(g_2)) \\ \alpha(t_1 t_2) \leq min(\alpha(t_1), \alpha(t_2)) \end{cases}$$

This implies that:

$$min(\mu(g_1 g_2), \alpha(t_1 t_2)) \leq min\left(min(\mu(g_1), \mu(g_2)), min(\alpha(t_1), \alpha(t_2))\right) = min(\beta(g_1, t_1), \beta(g_2, t_2))$$

On the other hand, we have:

$$\beta(g_1, t_1)^{-1} = \beta(g_1{}^{-1}, t_1{}^{-1}) = min(\mu(g_1{}^{-1}), \alpha(t_1{}^{-1})) = min(\mu(g_1), \alpha(t_1)) = \beta(g_1, t_1)$$

Also,

$$\beta(g_1, t_1)^n = \beta(g_1{}^n, t_1{}^n) = min(\mu(g_1{}^n), \alpha(t_1{}^n)) = min(\mu(g_1), \alpha(t_1)) = \beta(g_1, t_1)$$

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Which implies the proof.

Theorem.

Let $(G,\mu),(T,\alpha)$ be two fuzzy groups with $f: G \to T$ as a group homomorphism, for $\acute{\mu}: f(G) \to [0,1]$ with:

$$\begin{cases} \acute{\mu}(f(x)) = \mu(x); x \notin ker(f) \\ \acute{\mu}(f(x)) = \mu(e); x \in ker(f) \end{cases}$$

Then:

1). $(f(G),\acute{\mu})$ is a fuzzy group.

2). If $x^n \notin ker(f)$ for all $x \notin$ ker (f) and G is n-potent fuzzy group, then $f(G)$ is n-potent fuzzy group.

3). If H is n-potent fuzzy subgroup of G with $x^n \notin ker(f)$ for all $x \in H - \{e\}$, then $f(H)$ is n-potent fuzzy subgroup of $f(G)$.

Proof.

1). $\forall y_1, y_2 \in f(G)$, there exists $x_1, x_2 \in G$ such that:

$$\begin{cases} y_1 = f(x_1) \\ y_2 = f(x_2) \end{cases}$$

If $x_1, x_2 \notin ker(f)$, then $\acute{\mu}(y_1 y_2) = \acute{\mu}(f(x_1)f(x_2)) = \acute{\mu}(f(x_1 x_2)) = \mu(x_1 x_2) \leq min(\mu(x_1),\mu(x_2)) = min\left(\acute{\mu}(f(x_1)),\acute{\mu}(f(x_2))\right)$

If $x_1, x_2 \in ker(f)$, then $y_1 = y_2 = e_H$, hence:

$\acute{\mu}(y_1 y_2) = \acute{\mu}(e_H) \leq min(\acute{\mu}(y_1),\acute{\mu}(y_2))$

If $x_1 \in ker(f)$ and $x_2 \notin ker(f)$, then $x_1 x_2 \notin ker(f)$ and

$\acute{\mu}(y_1 y_2) = \acute{\mu}(f(x_1)f(x_2)) = \acute{\mu}(f(x_2)) = \mu(x_2) \leq min\left(\acute{\mu}(e_H),\acute{\mu}(f(x_2))\right)$

Also,

$\acute{\mu}(y_1^{-1}) = \acute{\mu}(f(x_1^{-1})) = \mu(x_1^{-1}) = \mu(x_1) = \mu(f(x_1)) = \acute{\mu}(y_1)$ for $x_1 \notin ker(f)$.

For $x_1 \in ker(f)$, $\acute{\mu}(y_1^{-1}) = \acute{\mu}(f(x_1^{-1})) = \acute{\mu}(e) = \mu(f(x_1)) = \mu(y_1)$

2). Let $y \in f(G)$, then there exists $x \in G$ such that $y = f(x)$.

$\acute{\mu}(y^n) = \acute{\mu}(f(x^n)) = \mu(x^n) = \mu(x) = \acute{\mu}(f(x)) = \acute{\mu}(y)$.

Thus $f(G)$ is n-potent fuzzy group.

3). Let $\acute{h} \in f(H)$, then there exists $h \in H$ such that $\acute{h} = f(h)$.

$\acute{\mu}(\acute{h}^n) = \acute{\mu}(f(h^n)) = \mu(h^n) = \mu(h) = \acute{\mu}\left(f(\acute{h})\right) = \acute{\mu}(\acute{h})$.

Thus $f(H)$ is an n-potent fuzzy subgroup.

Definition.

Let (G,μ) be an anti-fuzzy group, with $\mu: G \to [0,1]$ and:

$$\begin{cases} \mu(x^{-1}) = \mu(x) \\ \mu(x,y) \geq max(\mu(x),\mu(y)); \forall x,y \in G \end{cases}$$

We say that (G,μ) is an n-potent anti-fuzzy group if $\mu(x^n) = \mu(x)$ for all $x \in G$.

Example.

Consider $Z_5^* = \{1,2,3,4\}$ with $\mu: Z_5^* \to [0,1]$ such that:

$$\begin{cases} \mu(1) = \frac{1}{3} \\ \mu(2) = \mu(3) = \mu(4) = 0 \end{cases}$$

(Z_5^*,μ) is an anti-fuzzy group, that is because:

$\mu(3^{-1}) = \mu(2) = 0 = \mu(3), \mu(2^{-1}) = \mu(2) = \mu(3) = 0, \mu(4^{-1}) = \mu(4) = 0$

And $\begin{cases} \mu(2,3) = \mu(1) = \frac{1}{3} \geq max(\mu(2),\mu(3)) \\ \mu(2,4) = \mu(3) = 0 \geq max(\mu(2),\mu(4)) \\ \mu(3,4) = \mu(2) = 0 \geq max(\mu(3),\mu(4)) \end{cases}$

(Z_5^*,μ) is a 3-potent anti-fuzzy group, that is because:

$$\begin{cases} \mu(3^3) = \mu(2) = 0 = \mu(3) \\ \mu(2^3) = \mu(3) = 0 = \mu(2) \\ \mu(4^3) = \mu(4) = 0 \\ \mu(1^3) = \mu(1) = \frac{1}{3} \end{cases}$$

Definition.

Let (G,μ) be an anti-fuzzy group, with $H \leq G$ as a subgroup, hence H is called n-potent anti-fuzzy subgroup if and only if $\mu(x^n) = \mu(x)$ for all $x \in H$.

Theorem.

Let (G,μ) be an anti- fuzzy group, H, K are two subgroups of G, hence:

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1). If G is an n-potent anti- fuzzy group, then H is an n-potent anti- fuzzy subgroup.
2). If G is n-potent, then $H \cap K$ is n-potent.
3). If H, K are two n-potent anti- fuzzy subgroups, then $H \cap K$ is n-potent.

Proof.

1). $\forall x \in H$, then $x \in G$, so that $\mu(x^n) = \mu(x)$, thus H is n-potent.
2). It holds by a similar argument.
3). Let $x \in H \cap K$, then $x \in H$ and $\mu(x^n) = \mu(x)$, this implies that $H \cap K$ is n-potent anti-fuzzy subgroup.

Theorem.

Let (G, μ) be an anti- fuzzy group, and H be a normal subgroup of G with $\acute{\mu}: G/H \to [0,1]$ such that:

$$\begin{cases} \acute{\mu}(xH) = \mu(x); x \notin H \\ \acute{\mu}(xH) = \mu(e); x \in H \end{cases}$$

If G is an n-potent anti- fuzzy group and $\mu(h) = \mu(e)$ for all $h \in H$ then G/H is n-potent anti-fuzzy group if $x^n \notin H$ for all $x \notin H$.

Proof.

Let $x \notin H$ and $x^n \notin H$, then $\acute{\mu}(x^n H) = \mu(x^n) = \mu(x) = \acute{\mu}(xH)$.
If $x \in H$, then $\acute{\mu}(x^n H) = \mu(e) = \mu(x) = \acute{\mu}(xH)$.
Thus the proof is complete.

Theorem.

Let (G, μ) be an anti- fuzzy group, and H, K are two subgroups of G, if H is n-potent and K is m-potent, then $H \cap K$ is an nm-potent fuzzy subgroup.

Proof.

Let $x \in H \cap K$, then $x \in H$ and $x \in K$, so that $\mu(x^{nm} H) = \mu((x^n)^m) = \mu(x^n) = \mu(x)$, thus $H \cap K$ is mn-potent subgroup.

Theorem.

Let G, T be two groups, with $\mu: G \to [0,1], \alpha: T \to [0,1]$, where $(G, \mu), (T, \alpha)$ are two fuzzy groups, then if G and T are two n-potent groups, then $G \times T$ is n-potent anti-fuzzy group.

Proof.

Define $\beta: G \times T \to [0,1]$ such that:
$\beta(g, t) = max(\mu(g), \alpha(t))$, first we prove that $(G \times T, \beta)$ is fuzzy group.
Let $g_1, g_2 \in G, t_1, t_2 \in T$, then:
$\beta[(g_1, t_1)(g_2, t_2)] = \beta(g_1 g_2, t_1 t_2) = max(\mu(g_1 g_2), \alpha(t_1 t_2))$.
We have:

$$\begin{cases} \mu(g_1 g_2) \geq max(\mu(g_1), \mu(g_2)) \\ \alpha(t_1 t_2) \geq max(\alpha(t_1), \alpha(t_2)) \end{cases}$$

This implies that:

$$max(\mu(g_1 g_2), \alpha(t_1 t_2)) \geq max\left(max(\mu(g_1), \mu(g_2)), max(\alpha(t_1), \alpha(t_2))\right) = max(\beta(g_1, t_1), \beta(g_2, t_2))$$

On the other hand, we have:

$$\beta(g_1, t_1)^{-1} = \beta(g_1^{-1}, t_1^{-1}) = max(\mu(g_1^{-1}), \alpha(t_1^{-1})) = max(\mu(g_1), \alpha(t_1)) = \beta(g_1, t_1)$$

Also,

$$\beta(g_1, t_1)^n = \beta(g_1^n, t_1^n) = max(\mu(g_1^n), \alpha(t_1^n)) = max(\mu(g_1), \alpha(t_1)) = \beta(g_1, t_1)$$

Which implies the proof.

Theorem.

Let $(G, \mu), (T, \alpha)$ be two anti- fuzzy groups with $f: G \to T$ as a group homomorphism, for $\acute{\mu}: f(G) \to [0,1]$ with:

$$\begin{cases} \acute{\mu}(f(x)) = \mu(x); x \notin ker(f) \\ \acute{\mu}(f(x)) = \mu(e); x \in ker(f) \end{cases}$$

Then:

1). $(f(G), \acute{\mu})$ is anti- fuzzy group.
2). If $x^n \notin ker(f)$ for all $x \notin$ ker (f) and G is n-potent anti- fuzzy group, then $f(G)$ is n-potent anti-fuzzy group.
3). If H is n-potent anti- fuzzy subgroup of G with $x^n \notin ker(f)$ for all $x \in H - \{e\}$, then $f(H)$ is n-potent anti-fuzzy subgroup of $f(G)$.

Proof.

1). $\forall y_1, y_2 \in f(G)$, there exists $x_1, x_2 \in G$ such that:

$$\begin{cases} y_1 = f(x_1) \\ y_2 = f(x_2) \end{cases}$$

If $x_1, x_2 \notin ker(f)$, then $\acute{\mu}(y_1 y_2) = \acute{\mu}(f(x_1) f(x_2)) = \acute{\mu}(f(x_1 x_2)) = \mu(x_1 x_2) \geq max(\mu(x_1), \mu(x_2)) = max\left(\acute{\mu}(f(x_1)), \acute{\mu}(f(x_2))\right)$

If $x_1, x_2 \in ker(f)$, then $y_1 = y_2 = e_H$, hence:

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$\acute{\mu}(y_1y_2) = \acute{\mu}(e_H) \geq max(\acute{\mu}(y_1), \acute{\mu}(y_2))$
If $x_1 \in ker(f)$ and $x_2 \notin ker(f)$, then $x_1x_2 \notin ker(f)$ and
$\acute{\mu}(y_1y_2) = \acute{\mu}(f(x_1)f(x_2)) = \acute{\mu}(f(x_2)) = \mu(x_2) \geq max\left(\acute{\mu}(e_H), \acute{\mu}(f(x_2))\right)$
Also,
$\acute{\mu}({y_1}^{-1}) = \acute{\mu}(f({x_1}^{-1})) = \mu({x_1}^{-1}) = \mu(x_1) = \mu(f(x_1)) = \acute{\mu}(y_1)$ for $x_1 \notin ker(f)$.
For $x_1 \in ker(f)$, $\acute{\mu}({y_1}^{-1}) = \acute{\mu}(f({x_1}^{-1})) = \acute{\mu}(e) = \mu(f(x_1)) = \mu(y_1)$
2). Let $y \in f(G)$, then there exists $x \in G$ such that $y = f(x)$.
$\acute{\mu}(y^n) = \acute{\mu}(f(x^n)) = \mu(x^n) = \mu(x) = \acute{\mu}(f(x)) = \acute{\mu}(y)$.
Thus $f(G)$ is n-potent anti-fuzzy group.
3). Let $\acute{h} \in f(H)$, then there exists $h \in H$ such that $\acute{h} = f(h)$.
$\acute{\mu}(\acute{h}^n) = \acute{\mu}(f(h^n)) = \mu(h^n) = \mu(h) = \acute{\mu}\left(f(\acute{h})\right) = \acute{\mu}(\acute{h})$.
Thus $f(H)$ is an n-potent anti- fuzzy subgroup.

3. Conclusion

In this paper, we have defined for the first time the algebraic concept of n-potent fuzzy groups and n-potent anti-fuzzy groups, where many related properties are presented such as the intersection of n-potent fuzzy groups, the product of n-potent anti fuzzy groups, and the factor groups formed by these structures. We encourage other researcher to study the solvability and poly-cyclic of n-potent fuzzy/anti-fuzzy groups.

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