The Computation of the Roots for Equation $(ax + b)^n = c$

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Abstract

In this paper, we will describe a natural procedure formula that will lead us to find a solution for a class of polynomials with degree n associate with the equation $(ax + b)^n = c$.

Keywords: Exact solving; nth-degree of polynomial; New method

1. Introduction

Solving the polynomial equations of the third degree and above have been of interest for many mathematicians throughout the different eras of the development of mathematics. in this respect, Cardano formulas (1545) to factor cubic and quadratic to computation the roots of these polynomials that more complicated, in particular in cubic more than quadratic. Moreover, Cardano formulas produced only one out of three factors, while the quadratic formula produces both factors of any given quadratic polynomial. For this reason, it was generating an incentive for mathematicians to search for more simplified and generalizable methods to solving cubic and greater than the third degree. Author's [8] presented the new method to compute the three factors of an arbitrary cubic polynomial with real number coefficients and proved those roots over any cubic polynomial over $\mathbb{R}$. In an analogous manner, using ζ, and a solution (a, b) of the equation $X^3 + Y^3 = 1$ over $\mathbb{C}$. While author's [6] introduced a new approach to generalization Cardano formula for cubic equation. For many years, mathematicians have been working hard to find a method to solve a fifth-degree equation by using radicals, until P. Ruffini had presented a proof of non-existence of such solution in 1779. Next later in 1824, N. H. Abel introduced similar statement confirmed of Ruffini's statement there is no formula to solve a fifth-degree equation by using radicals. After that, E. Galois introduced Some properties of the equation (e.g. solvability in radicals) are translated to properties of its Galois group. Theorem (Abel and Ruffini)." A general algebraic equation of degree ≥ 5 cannot be solved in radicals. This means that there does not exist any formula which would express the roots of such equation as functions of the coefficients by means of the algebraic operations and roots of natural degrees, see author [9]", that he presented a proof of the non-solvability in radicals of a general algebraic equation of degree greater than four. This proof relies on the non-solvability of the monodromy group of a general algebraic function. The aim of this article to present the new approach to treatment with this issue by solving n-degree polynomials associate with equation $(ax + b)^n = c$.

2. New Approach to find The Fifth Roots for The Equation: $(ax + b)\ 5 = c\ (1)$. $n \leq 5$.

In this section, we introduced a new approach to find fifth roots solutions for the equation (1). Let $\mathbb{R}$ denote the real filed, and $\mathbb{R}[x]$ denote to the ring of polynomials over $\mathbb{R}$. Consider $\mathbb{C}$ is the complex filed and $\mathbb{C}[x]$ denote to the ring of polynomials over $\mathbb{C}$. If we consider a new form:

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$(ax + b)\,5 = c$ (1), Where $x \in \mathbb{C}$ and $a \in \mathbb{R}$, b and c are constants of complex numbers. This new method does depend on [3,5,7], But depending on a new formula of radicals and not use the Euler's form in Complex numbers which use the arguments of number and non-standard angles. In this method, if we expand the left side of equation (1) by binomial theorem we get a particular polynomial (or class of polynomials) of the form:

$$F_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 = c \ (2).$$

This equation becomes like:

$$F_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 - c = 0 \ (3).$$

Where $\delta = a_0 - c$. The equation (3) represent the class of polynomials of complex or real values according to constants $a, b\ an\ c$. To solve the polynomial of equation (3) and finding the roots we present the following method which called SHAD-method, where SH and AD are the first and second letters of the name authors.

Theorem 2.1. Consider the equation $(ax + b)^5 = c$ (1).where $x \in \mathbb{C}$ and $a \in \mathbb{R}$, b and c are constants of complex or real numbers, then the equation (1) has associate with the polynomial's equation of degree 5 with the form:

$$F_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 = c \ (2).$$ Equation (2) becomes,

$$F_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + \delta = 0 \ (3).$$ Where, $\delta = a_0 - c$.

Then the roots of polynomials of equation (3). Given by the following SHAD-radical formula:

$$x_{j\in J} = \frac{-a_4}{a_{5.}\,5} + \frac{\sqrt[5]{(a_4)^5-(a_5)^{5-1}.5^5.\delta)}}{a_{5.}\,5}\,.\,e^{2\pi i(j-1)/5} \qquad (4).$$

Where, $j \in J = \{1,2,3,4,5\}$ is index set of roots.

Proof. Consider the polynomials of fifth degree equation:

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + \delta = 0 \ (1).$$ Where, $\delta = a_0 - c$.

To show that the SHAD-Radical formula (4) is a solution of polynomials (1). Then the equation (4) can be written as:

$$x = \frac{-a_4}{a_{5.}\,5} + \frac{\sqrt[5]{(a_4)^5-(a_5)^{5-1}.5^5.\delta)}}{a_{5.}\,5}. \qquad (2).\text{ we get,}$$

$$x = \left(\frac{-a_4}{5.(\sqrt[5]{a_5}).(\sqrt[5]{a_5})^{5-1}} + \frac{\sqrt[5]{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}}{5.(\sqrt[5]{a_5}).(\sqrt[5]{a_5})^{5-1}}\right) \qquad (3).\text{ from (3), we have}$$

$$x = \left(\frac{1}{(\sqrt[5]{a_5})}\right)\left(\frac{-a_4}{5.(\sqrt[5]{a_5})^{5-1}} + \frac{\sqrt[5]{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}}{5.(\sqrt[5]{a_5})^{5-1}}\right) \qquad (4).\text{ so, we have}$$

$$x = \left\{\frac{1}{(\sqrt[5]{a_5})}\left(\frac{-a_4}{5.(\sqrt[5]{a_5})^{5-1}} + \sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)\right\} \qquad (5).$$

$$x = \frac{1}{(\sqrt[5]{a_5})^n}\left(\frac{-a_4}{5.(\sqrt[5]{a_5})^{5-1}} + \sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right) \qquad (6).$$

Where n the different power of polynomial of degree fifth, now put the equation (6) in equation (1) to get,

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$$\begin{pmatrix} \frac{a_5}{\left(\sqrt[5]{a_5}\right)^5}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^5 \\ + \\ \frac{a_4}{\left(\sqrt[5]{a_5}\right)^4}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^4 \\ + \\ \frac{a_3}{\left(\sqrt[5]{a_5}\right)^3}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3 \\ + \\ \frac{a_2}{\left(\sqrt[5]{a_5}\right)^2}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \\ + \\ \frac{a_1}{\left(\sqrt[5]{a_5}\right)^1}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ + \\ \delta \end{pmatrix} = 0 \qquad (7).$$

$$\begin{pmatrix} \binom{5}{0}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^0\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^5 \\ + \\ \binom{5}{1}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^4 \\ + \\ \binom{5}{2}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3 \\ + \\ \binom{5}{3}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^3\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \\ + \\ \binom{5}{4}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^4\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}+\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ + \\ \delta \end{pmatrix} = 0 \qquad (8).$$

To expand the powers in equation (8). We deduced that

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$$\binom{5}{0}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^0 \left(\begin{array}{c} \left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^5 \\ + \\ 5.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^4 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ + \\ 10.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^3 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \\ + \\ 10.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3 \\ + \\ 5.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^4 \\ + \\ \left(\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}\right) \end{array}\right) +$$

$$\binom{5}{1}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1 \left(\begin{array}{c} \left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^4 \\ + \\ 4.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^3 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ + \\ 6.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \\ + \\ 4.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1 \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3 \\ + \\ \left(\sqrt[5]{\frac{(a_4)^5-(5^5.{a_5}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^4 \end{array}\right) +$$

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$$\binom{5}{2}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2\left(\begin{array}{c}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^3\\ +\\ 3.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1\\ +\\ 3.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2\\ +\\ \left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3\end{array}\right)+$$

$$\binom{5}{3}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^3\left(\begin{array}{c}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^2\\ +\\ 2.\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1\\ +\\ \left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2\end{array}\right)+$$

$$\binom{5}{4}\left(\frac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^4\left(\begin{array}{c}\left(\frac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^1\\ +\\ \left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1\end{array}\right)+\delta=0 \qquad (9).$$

By multiplying the factors, we get the following equation:

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$$\left(\begin{array}{c} -\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{5} \\ + \\ 5.\left(\dfrac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{4}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{1} \\ - \\ 10.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{3}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{2} \\ + \\ 10.\left(\dfrac{-a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{2}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{3} \\ - \\ 5.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{1}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{4} \\ + \\ \left(\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}\right) \end{array}\right) +$$

$$\left(\begin{array}{c} 5.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{5} \\ - \\ 20.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{4}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{1} \\ + \\ 30.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{3}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{2} \\ - \\ 20.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{2}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{3} \\ + \\ 5.\left(\dfrac{a_4}{5.\left(\sqrt[5]{a_5}\right)^{5-1}}\right)^{1}\left(\sqrt[5]{\dfrac{(a_4)^5-\left(5^5.{a_5}^{5-1}.\delta\right)}{5^5.(a_5)^{5-1}}}\right)^{4} \end{array}\right) +$$

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$$\begin{pmatrix} -10.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^5 \\ + \\ 30.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^4\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ - \\ 30.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^3\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \\ + \\ 10.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^2\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^3 \end{pmatrix} +$$

$$\begin{pmatrix} 10.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^5 \\ - \\ 20.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^4\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \\ + \\ 10.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^3\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^2 \end{pmatrix} +$$

$$\begin{pmatrix} -5.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^5 \\ + \\ 5.\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^4\left(\sqrt[5]{\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}}\right)^1 \end{pmatrix} + \delta = 0 \qquad (10).$$

By adding the similarity terms in equation (10), we get the following terms only.

$$-\left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^5 + \left(\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}\right) + \delta = 0 \qquad (11).$$

But $\delta = a_0 - c = \left(\frac{a_4}{5.(\sqrt[5]{a_5})^{5-1}}\right)^5 - \left(\frac{(a_4)^5-(5^5.a_5{}^{5-1}.\delta)}{5^5.(a_5)^{5-1}}\right)$ (12).

It is clear that if we put the equation (12) and (11), this proved $x = \frac{-a_4}{a_{5.}5} + \frac{\sqrt[5]{(a_4)^5-(a_5)^{5-1}.5^5.\delta)}}{a_{5.}5}$ is a solution of polynomial of fifth degree $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + \delta = 0$.

Example 2.1. Let $n = 1, a = 4, b = 7$ and $c = 10i$ be the values of constants in equation (1) becomes. $(4x + 7) = 10i$, then the polynomial of degree 1 looks like: $a_1\, x^1 + a_0\, x^0 = c$ **iff** $4\, x^1 + 7\, x^0 = 10i$,

Where $a_1 = \binom{1}{0}(4) = 4, a_0 = \binom{1}{1}(7) = 7$ and $\delta = a_0 - c = 7 - 10i$, the solution given by

$$x = -\frac{a_0}{a_1.1} \pm \frac{\sqrt[1]{(a_0)^1 - (a_1)^{1-1}.1^1.\delta}}{a_1.1} e^{2\pi i(1-1)/_1}$$

$$x = -\frac{7}{4} + \frac{\sqrt[1]{(7)^1-(7)^0.1^1.(7-10i)}}{4} e^0$$

$$x = -\frac{7}{4} + \frac{10i}{4}$$

Example 2.2. Let $n = 2, a = 4, b = 7$ and $c = 10i$ be the values of constants in equation (1) becomes. $(4x + 7)^2 = 10i$, then the polynomial of degree 2 (or quadratic equation) looks like:

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$a_2\, x^2 + a_1\, x^1 + a_0 \; = c$ **iff** $16x^2 + 56\, x^1 + 49 \; = 10i$ **iff** $16x^2 + 56\, x^1 + 49 - 10i = 0$ Where $a_2 = \binom{2}{0}(4)^2 = 16, a_1 = \binom{2}{1}(4)^1 7 = 56, a_0 = \binom{2}{2}(7)^2 = 49$ and $\delta = a_0 - c = 49 - 10i$, the solution given by

$$x_j = -\frac{a_1}{a_2.2} + \frac{\sqrt[2]{(a_1)^2 - (a_2)^{2-1}.2^2.\delta}}{a_2.2} e^{2\pi i(J-1)/_2}$$

$$x_1 = -\frac{56}{16.2} + \frac{\sqrt[2]{(56)^2-(16)^{2-1}.2^2.(49-10i)}}{16.2} e^{2\pi i(1-1)/_2}$$

$$x_1 = -\frac{56}{32} + \frac{\sqrt[2]{(56)^2-64.(49-10i)}}{32} e^0$$

$$x_1 = -1.19098300563 + 0.55901699437i$$

$$x_2 = -\frac{56}{16.2} + \frac{\sqrt[2]{(56)^2-(16)^{2-1}.2^2.(49-10i)}}{16.2} e^{2\pi i(2-1)/_2}$$

$$x_2 = -2.30901699437 - 0.55901699437i$$

Example 2.3. Let $n = 3, a = 4, b = 7$ and $c = 10i$ be the values of constants in equation (1) becomes.

$(4x + 7)^3 = 10i$, then the polynomial of degree 3 (or cubic equation) looks like:

$a_3\, x^3 + a_2\, x^2 + a_1\, x^1 + a_0 \; = c$ **iff** $64x^3 + 336x^2 + 588\, x^1 + 343 \; = 10i$ **iff**

$64x^3 + 336x^2 + 588\, x^1 + 343 - 10i \; = 0$ Where $a_3 = \binom{3}{0}(4)^3 = 64$

$a_2 = \binom{2}{0}(4)^2(7)^1 = 336, a_1 = \binom{3}{2}(4)^1(7)^2 = 588\,, a_0 = \binom{3}{3}(7)^3 = 343,$ and $\delta = a_0 - c = 343 - 10i$, the solution given by

$$x_j = -\frac{a_2}{a_3.3} + \frac{\sqrt[3]{(a_2)^3 - (a_3)^{3-1}.3^3.\delta}}{a_3.3} e^{2\pi i(J-1)/_3}$$

$$x_1 = -\frac{336}{64.3} + \frac{\sqrt[3]{(336)^3-(64)^{3-1}.3^3.(343-10i)}}{64.3} e^{2\pi i(1-1)/_3}$$

$$x_1 = -\frac{336}{64.3} + \frac{\sqrt[3]{(336)^3-(64)^{3-1}.3^3.(343-10i)}}{64.3} e^{2\pi i(1-1)/_3}$$

$$x_1 = -1.28355120691 + 0.26930433625i$$

$$x_2 = -\frac{336}{64.3} + \frac{\sqrt[3]{(336)^3-(64)^{3-1}.3^3.(343-10i)}}{64.3} e^{2\pi i(2-1)/_3}$$

$$x_2 = -2.21644879309 + 0.26930433625i$$

$$x_3 = -\frac{336}{64.3} + \frac{\sqrt[3]{(336)^3-(64)^{3-1}.3^3.(343-10i)}}{64.3} e^{2\pi i(3-1)/_3}$$

$$x_3 = -1.75 - 0.53860867251i$$

Example 2.4. Let $n = 4, a = 2, b = 3i$ and $c = -2$ be the values of constants in equation (1) becomes.

$(2x + 3i)^4 = -2$, then the polynomial of degree 4 (or quartic equation) looks like:

$a_4\, x^4 + \; a_3\, x^3 + a_2\, x^2 + a_1\, x^1 + a_0 \; = c$ **iff** $16\, x^4 + 96i\, x^3 - 216\, x^2 - 216\, x^1 + 81 \; = -2$ **iff**

$16\, x^4 + 96i\, x^3 - 216\, x^2 - 216i\, x^1 + 83 \; = 0$ Where $a_4 = \binom{4}{0}(2)^4 = 16, a_3 = \binom{4}{1}(2)^3(3i)^1 = 96i.$

$a_2 = \binom{4}{0}(2)^2(3i)^2 = -216\,, a_1 = \binom{4}{3}(2)^1(3i)^3 = -216i\,, a_0 = \binom{4}{4}(3i)^4 = 81$,and

$\delta = a_0 - c = 83$, the solution given by

$$x_j = -\frac{a_3}{a_4.4} + \frac{\sqrt[4]{(a_3)^4-(a_4)^{4-1}.4^4.\delta}}{a_4.4} e^{2\pi i(J-1)/_4}$$

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$$x_1 = -\frac{96i}{16.4} + \frac{\sqrt[4]{(96i)^4-(16)^{4-1}.4^4.83}}{16.4} e^{2\pi i(1-1)/4}$$

$$x_1 = 0.42044820763 - 1.07955179237i.$$

$$x_2 = -\frac{96i}{16.4} + \frac{\sqrt[4]{(96i)^4-(16)^{4-1}.4^4.83}}{16.4} e^{2\pi i(2-1)/4}$$

$$x_2 = -0.42044820763 - 1.07955179237i.$$

$$x_3 = -\frac{96i}{16.4} + \frac{\sqrt[4]{(96i)^4-(16)^{4-1}.4^4.83}}{16.4} e^{2\pi i(3-1)/4}$$

$$x_3 = -0.42044820763 - 1.92044820763i.$$

$$x_4 = -\frac{96i}{16.4} + \frac{\sqrt[4]{(96i)^4-(16)^{4-1}.4^4.83}}{16.4} e^{2\pi i(4-1)/4}$$

$$x_4 = 0.42044820763 - 1.92044820763i.$$

Example 2.5. Let $n = 5, a = 2, b = 3$ and $c = 3$ be the values of constants in equation (1) becomes. $(2x + 3)^5 = 3$, then the polynomial of degree 5 (or fifth equation) looks like:

$$a_5\, x^5 + a_4\, x^4 + a_3\, x^3 + a_2\, x^2 + a_1\, x^1 + a_0 = c \textbf{ Iff}$$

$$32\, x^5 + 240x^4 + 720\, x^3 + 1080\, x^2 + 810\, x^1 + 243 = 3 \textbf{ Iff}$$

$$32\, x^5 + 240x^4 + 720\, x^3 + 1080\, x^2 + 810\, x^1 + 240 = 0$$

Where $a_5 = \binom{5}{0}(2)^5 = 32, a_4 = \binom{5}{1}(2)^4(3)^1 = 240,\ a_3 = \binom{5}{2}(2)^3(3)^2 = 720,$

$a_2 = \binom{5}{3}(2)^2(3)^3 = 1080\, , a_1 = \binom{5}{4}(2)^1(3)^4 = 810\, , a_0 = \binom{5}{5}(3)^5 = 243$,And

$\delta = a_0 - c = 240$, the solution given by

$$x_j = -\frac{a_4}{a_5.5} + \frac{\sqrt[5]{(a_4)^5 - (a_5)^{5-1}.5^5.\delta}}{a_5.5} e^{2\pi i(J-1)/5}$$

$$x_1 = -\frac{240}{32.5} + \frac{\sqrt[5]{(240)^5-(32)^{5-1}.5^5.240}}{32.5} e^{2\pi i(1-1)/5}$$

$$x_1 = -0.87713453019$$

$$x_2 = -\frac{240}{32.5} + \frac{\sqrt[5]{(240)^5 - (32)^{5-1}.5^5.240}}{32.5} e^{2\pi i(2-1)/5}$$

$$x_3 = -\frac{240}{32.5} + \frac{\sqrt[5]{(240)^5 - (32)^{5-1}.5^5.240}}{32.5} e^{2\pi i(3-1)/5}$$

$$x_3 = -2.00390875028 + 0.36611113732i.$$

$$x_4 = -\frac{240}{32.5} + \frac{\sqrt[5]{(240)^5 - (32)^{5-1}.5^5.240}}{32.5} e^{2\pi i(4-1)/5}$$

$$x_4 = -2.00390875028 - 0.36611113732i.$$

$$x_5 = -\frac{240}{32.5} + \frac{\sqrt[5]{(240)^5 - (32)^{5-1}.5^5.240}}{32.5} e^{2\pi i(5-1)/5}$$

$$x_2 = -1.30752398462 - 0.59238026384i.$$

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3. New Approach to find General Solution for Equation: $(ax+b)^n = c$ (1).

In this section, we introduced a new approach to find the general solution of equation (1). Let $\mathbb{R}$ denote the real field, and $\mathbb{R}[x]$ denote the ring of polynomials over $\mathbb{R}$. Consider $\mathbb{C}$ is the complex field and $\mathbb{C}[x]$ denote the ring of polynomials over $\mathbb{C}$. If we consider a new form $(ax+b)^n = c$ (1), where $x \in \mathbb{C}$ and $a \in \mathbb{R}, b$ and c are constants of complex numbers. This new method does depend on the matrix solution algorithm. But depending on a new formula of algorithm depending on radicals and Euler's form without using arguments of number and non-standard angles. In this method, if we expand the left side of equation (1) by binomial theorem we get a particular polynomials (or class of polynomials) of the form: $F_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_o = c$ (2). This equation becomes like:
$F_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0$ (3), where $\delta = a_0 - c$. Equation (3) represent the class of polynomials of complex or real values according to constants a, b *and* c. To solve the polynomial equation (3) to find the roots we present the following method which is called SHAD-method, where SH and AD are the first and second letters of the name authors.

Theorem. 3.1. Consider the formula of the equation $(ax+b)^n = c$ (1), where $x \in \mathbb{C}$, $a \in \mathbb{R}, b$ and c are constants of complex numbers. Then the equation $(ax+b)^n = c$ (1). Can be represented by the polynomial's equation of the form:

$F_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0$ (2), Where, $\delta = a_0 - c$. The equation (1) equivalent (2).

Proof. Let $(ax+b)^n = c$ (1), be a given equation**,** then by using Binomial theorem, we can expand the left side into:

$$(ax+b)^n = \binom{n}{0}(ax)^n b^0 + \binom{n}{1}(ax)^{n-1}b^1 + \binom{n}{2}(ax)^{n-2}b^2 + \cdots + (ax)^{n-n}\binom{n}{n}b^n \quad (1).$$

$$= (ax)^n b^0 + \frac{n!}{(n-1)!1!}(ax)^{n-1}b^1 + \frac{n!}{(n-2)!2!}(ax)^{n-2}b^2 + \cdots + b^n \quad (2).$$

$$= a^n x^n + na^{n-1}x^{n-1}b + \frac{n(n-1)a^{n-2}}{2!}x^{n-2}b^2 + \cdots + b^n \quad (3).$$

Consider $a_n = a^n, a_{n-1} = na^{n-1}b, a_{n-2} = {}^{n(n-1)}/_{2!}\, a^{n-2}b^2, \ldots, \alpha_0 = b^n$ (4).

Then we get from (4),

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c \quad (5).$$

And consequently,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \delta = a_0 - c \quad (6).$$

Conversely, to show that the equation (2) implies that the equation (1).

Consider the polynomials of degree n:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \delta = a_0 - c \quad (1).$$

By completing the term: $a_0 = \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n$, in the equating (1), we have,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n - \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n + \delta = 0, \quad (2).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n - \delta \quad (3).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_0 - \delta \quad (4).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_0 - (a_0 - c) \quad (5).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c \quad (6).$$

$$(a\,x+b)^n = c\blacksquare. \quad (7).$$

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Theorem 3.2. Consider the equation $(ax+b)^n = c$ (1), where $x \in \mathbb{C}$ and $a \in \mathbb{R}$, b and c are constants of complex numbers, by theorem 3.1 equation (1) has associate with the polynomials equation of degree n on the form: $a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \delta = a_0 - c$ (2). Then the roots of polynomials equation (2). Given by the following SHAD-radical formula:

$$x_{j\in J} = \frac{-a_{n-1}}{a_n.\, n} + \frac{\sqrt[n]{(a_{n-1})^n - (.a_n)^{n-1}.n^n.\delta)}}{a_n.\, n} \cdot e^{2\pi i(j-1)/n} \quad (3).$$

Where, $j \in J = \{1,2,3,\ldots,n\}$ is index set of roots.

Proof. Consider the equation: $(ax+b)^n = c$ (1).
then by theorem 3.1. The equation (1) can be written as:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \delta = a_0 - c \quad (2).$$

Assume that, $a_n = a^n$ and $a_0 = b^n$ (3).

By completing the term: $a_0 = \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n$, in the equating (2), we have,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n - \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n + \delta = 0, \quad (4).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n - \delta \quad (5).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \frac{(a_{n-1})^n}{\left((\sqrt[n]{a_n})^{n-1}\right)^n.n^n} - \delta \quad (6).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \frac{(a_{n-1})^n}{(a_n)^{n-1}.n^n} - \delta \quad (7).$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \left(\frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \frac{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}{(a_n)^{n-1}.n^n} \quad (8).$$

$$\left(\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n}\right)^n = \frac{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}{(a_n)^{n-1}.n^n} \quad (9).$$

By taking the n-root for both side we get, when we consider n is even, then we have**,**

$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n} = \pm\sqrt[n]{\left(\frac{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}{(a_n)^{n-1}.n^n}\right)} \quad (10).$$

$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n} = \pm\frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{\sqrt[n]{(a_n)^{n-1}.n^n}} \quad (11).$$

$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n} = \pm\frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{(\sqrt[n]{a_n})^{n-1}.n} \quad (12).$$

$$x = -\frac{a_{n-1}}{\sqrt[n]{a_n}.(\sqrt[n]{a_n})^{n-1}.n} \pm \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{\sqrt[n]{a_n}.(\sqrt[n]{a_n})^{n-1}.n} \quad (13).$$

$$x = -\frac{a_{n-1}}{a_n.n} \pm \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{a_n.n} \quad (14).$$

From equation (14), we get the two roots when $j = 1$ and $j = \frac{n}{2} + 1$

From other hand, when we consider n is odd, we have,

$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{(\sqrt[n]{a_n})^{n-1}.n} = +\sqrt[n]{\left(\frac{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}{(a_n)^{n-1}.n^n}\right)} \quad (15).$$

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$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{\left(\sqrt[n]{a_n}\right)^{n-1}.n} = + \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{\sqrt[n]{(a_n)^{n-1}.n^n}} \quad (16).$$

$$\sqrt[n]{a_n}\, x + \frac{a_{n-1}}{\left(\sqrt[n]{a_n}\right)^{n-1}.n} = + \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{\left(\sqrt[n]{a_n}\right)^{n-1}.n} \quad (17).$$

$$x = -\frac{a_{n-1}}{\sqrt[n]{a_n}.\left(\sqrt[n]{a_n}\right)^{n-1}.n} + \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{\sqrt[n]{a_n}.\left(\sqrt[n]{a_n}\right)^{n-1}.n} \quad (18).$$

$x = -\frac{a_{n-1}}{a_n.n} + \frac{\sqrt[n]{(a_{n-1})^n - (a_n)^{n-1}.n^n.\delta}}{a_n\ .n}$, when $j = 1$, we have only one root. (19).

From the equations (14) and (19), we deduced that the radicals:

$$x_{j\in J} = \frac{-a_{n-1}}{a_n.\, n} + \frac{\sqrt[n]{(a_{n-1})^n - (.a_n)^{n-1}.n^n.\delta)}}{a_n.\, n}\,.e^{2\pi i(j-1)/n} \quad (20).$$

Where, $j \in J = \{1,2,3, \dots, n\}$ is index set of roots ■.

Corollary 3.3. The formula $x_{j\in J} = \frac{-a_{n-1}}{a_n.\, n} + \frac{\sqrt[n]{(a_{n-1})^n - (.a_n)^{n-1}.n^n.\delta)}}{a_n.\, n}\,.e^{2\pi i(j-1)/n}$,Where $j \in J = \{1,2,3, \dots, n\}$ is index set of roots, it is represented the solution of polynomials:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \ \delta = a_0 - c\ .$$

Proof. Consider the equation: $(ax + b)^n = c$ (1).
then by theorem 3.1. The equation (1) can be written as:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + \delta = 0,\ \ \delta = a_0 - c \quad (2).$$

Assume that, $a_n = a^n$ and $a_0 = b^n$ (3).

By taking the $n - root$, we get, $\sqrt[n]{a_n} = a$ and $\sqrt[n]{a_0} = b$ (4).

We put $J = 1$ in formula of solution

$$x = \left(\frac{-a_{n-1}}{n.a_n} + \frac{\sqrt[n]{(a_{n-1})^n - (n^n.a_n{}^{n-1}.\delta)}}{n..a_n}\right) \quad (5).$$

Put $a_n = \left(\sqrt[n]{a_n}\right).\left(\sqrt[n]{a_n}\right)^{n-1}$ in equation (6).

We get,

$$x = \left(\frac{-a_{n-1}}{n.\left(\sqrt[n]{a_n}\right).\left(\sqrt[n]{a_n}\right)^{n-1}} + \frac{\sqrt[n]{(a_{n-1})^n - (n^n.a_n{}^{n-1}.\delta)}}{n..\left(\sqrt[n]{a_n}\right).\left(\sqrt[n]{a_n}\right)^{n-1}}\right) \quad (7).$$

$$x = \left\{\frac{1}{\left(\sqrt[n]{a_n}\right)}\left(\frac{-a_{n-1}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}} + \frac{\sqrt[n]{(a_{n-1})^n - (n^n.a_n{}^{n-1}.\delta)}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}}\right)\right\} \quad (8).$$

$$x = \left\{\frac{1}{\left(\sqrt[n]{a_n}\right)}\left(\frac{-a_{n-1}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}} + \sqrt[n]{\frac{(a_{n-1})^n - (n^n.a_n{}^{n-1}.\delta)}{n^n.(a_n)^{n-1}}}\right)\right\} \quad (9).$$

$$x = \left\{\frac{1}{\left(\sqrt[n]{a_n}\right)}\left(\frac{-a_{n-1}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}} + \sqrt[n]{\frac{(a_{n-1})^n}{n^n.(a_n)^{n-1}} - \frac{n^n.a_n{}^{n-1}.\delta}{n^n.(a_n)^{n-1}}}\right)\right\} \quad (10).$$

Since, $a_0 = \left(\frac{a_{n-1}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}}\right)^n$ (11).

Then $\sqrt[n]{a_0} = \frac{a_{n-1}}{n.\left(\sqrt[n]{a_n}\right)^{n-1}}$ (12).

Substitute Equations (11) and (12) in equation (10), we get,

$$x = \left\{\frac{1}{\left(\sqrt[n]{a_n}\right)}\left(-\sqrt[n]{a_0} + \sqrt[n]{a_0 - \delta}\right)\right\} \quad (13).$$

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But, $\delta = a_0 - c$, therefore, $c = a_0 - \delta$. So that (13) becomes,

$$x = \left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\} \quad (14).$$

Now, the equation (14) is a solution of equation (1) ■.

Corollary 3.4. The root $x = \left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\}$ is the solution of the equation $(ax + b)^n - c = 0$ (1).

Proof. To verify that $x = \left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\}$ is the solution of the equation (1), we have,

The left side: $(ax + b)^n - c = \left(a\left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\} + b\right)^n - c$

$$= \left(\sqrt[n]{a_n}\left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\} + \sqrt[n]{a_0}\right)^n - c$$

$$= \left(\sqrt[n]{a_n}\left\{\frac{1}{(\sqrt[n]{a_n})}\left(-\sqrt[n]{a_0} + \sqrt[n]{c}\right)\right\} + \sqrt[n]{a_0}\right)^n - c = 0 \text{ ■}.$$

Example 3.1. Let $n = 6, a = 3, b = -2$ and $c = 7$ be the values of constants in equation (1) becomes. $(3x + (-2))^6 = 7$, then the polynomial of degree 6 (or sixth equation) looks like:

$$a_6\, x^6 + a_5\, x^5 + a_4\, x^4 + a_3\, x^3 + a_2\, x^2 + a_1\, x^1 + a_0 = c \textbf{ Iff}$$

$$729\, x^6 - 2916x^5 + 4860 \quad x^4 -4320\, x^3 + 2160 \quad x^2 + -576\, x^1 + 64 = 7 \textbf{ Iff}$$

$$729\, x^6 - 2916x^5 + 4860 \quad x^4 -4320\, x^3 + 2160 \quad x^2 + -576\, x^1 + 57 = 0$$

Where $a_6 = \binom{6}{0}(3)^6 = 729, a_5 = \binom{6}{1}(3)^5(-2)^1 = -2916, a_4 = \binom{6}{2}(3)^4(-2)^2 = 4860,$ $a_3 = \binom{6}{3}(3)^3(-2)^3 = -4320, a_2 = \binom{6}{4}(3)^2(-2)^4 = 2160, a_1 = \binom{6}{5}(3)^1(-2)^5 = -576,$ $a_0 = \binom{6}{6}(-2)^6 = 64$, and $\delta = a_0 - c = 57$, the solution given by

$$x_j = -\frac{a_5}{a_6.6} + \frac{\sqrt[6]{(a_5)^6 - (a_6)^{6-1}.6^6.\delta}}{a_6.6} e^{2\pi i(J-1)/6}$$

$$x_1 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(J-1)/6}$$

$$x_1 = 1.12769585142$$

$$x_2 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(2-1)/6}$$

$$x_2 = 0.89718125904 + 0.39926298588i.$$

$$x_3 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(3-1)/6}$$

$$x_3 = 0.43615207429 + 0.39926298588i.$$

$$x_4 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(4-1)/6}$$

$$x_4 = 0.20563748191.$$

$$x_4 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(5-1)/6}$$

$$x_5 = 0.43615207429 - 0.39926298588i.$$

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$$x_6 = -\frac{(-2916)}{729.6} + \frac{\sqrt[6]{(-2916)^6 - (729)^{6-1}.6^6.57}}{729.6} e^{2\pi i(6-1)/6}$$

$$x_6 = 0.89718125904 - 0.39926298588i.$$

4. Conclusion

To solve the equation $(ax + b)^n = c$, after expanding the left side of equation by the Binomial theorem, the SHAS-formula is depending on the terms : $a_n x^n, a_{n-1}x^{n-1}, a_0$ and
$\delta = a_0 - c$, respectively for computing all roots. In fact, if we have: $(ax + b)^n.1 = c.1$
We no longer need the argument of the complex number in Euler's and Demoiver's ' formulas, thus:
$(ax + b)^n.1^n = c.1$, where $1 = e^{2\pi i(J-1)}$, therefore,
$(ax + b)^n.\left(e^{2\pi i(J-1)}\right)^n = c.e^{2\pi i(J-1)}$, by taking the nth-root for both sides, we get
$(ax + b) = \sqrt[n]{c.e^{2\pi i(J-1)}} = \sqrt[n]{c}.e^{2\pi i(j-1)/n}$, which is an equivalent to:
$x_{j\in J} = \frac{-a_{n-1}}{a_n.n} + \frac{\sqrt[n]{(a_{n-1})^n - (.a_n)^{n-1}.n^n.\delta)}}{a_n.n} .e^{2\pi i(j-1)/n}$Or we use the following theorem.
$x_{j\in J} = \frac{-\sqrt[n]{a_0}}{\sqrt[n]{a_n}} + \frac{\sqrt[n]{c}}{\sqrt[n]{a_n}}.e^{2\pi i(j-1)/n}$.

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