Infra bi-Topological space

Riad K. Al-Hamido
Department of Mathematics, College of Science, AlFurat University, Deir-ez-Zor, Syria
Email: riad-hamido1983@hotmail.com
Tel.: (+963988619783)

Abstract

In this paper, we introduce infra bi-topological structure which is a more general structure than infra-topological spaces. This new space make us enable to increate a new sub-classes of sets, called infra bi-open (bi-closed) sets, pairwise infra-open (closed) sets. also we define infra bi-closure, pairwise infra bi-interior and their basic properties are presented. The relations of these concepts with their counterparts in infra-topological space s are given and many examples are presented.

Keywords: Infra bi-topological spaces; pairwise infra-open (closed) sets; infra bi-open (bi-closed) sets.

1. Introduction

In 2015, Adel. M. AL-Odhari [1] have been discussed the concept of infra-Topological spaces as an extension of topological spaces. Adel's definition of the infra-Topological spaces is considered a revolution in the field of topology and opens the door for researchers to define new topological concepts in this space such that separation axioms, compactness, weak open set…etc.

Before this generalization of topological space, supra-topological space was defined, which made a big revolution in topology as well, and most of the previous topological concepts were studied in this space. This space was introduced A.S. Mashhour et al. [4] in 1983, as a generalization of the concept of topological space.

Mashhour et al. [11] generalized some topological notions such as interior and closure operators and continuity and separation axioms.

Al-Shami [12] has studied the classical topological notions such as limit points of a set, compactness, and separation axioms on the supra topological spaces.

Many paper generalized some topological notions to supra topological space such as the notions of supra α-open [13],supra preopen [14], supra b-open [15], supra β-open [16],supra R-open [17], and supra semiopen sets [18] have beenintroduced and their main properties have been discussed.,ese. Generalizations have beenutilized to define new versions of compactness andconnectedness, see, for example [19–24].

Recently, In 2018, R.Alhamido, in [10] studied several types of open and closed sets in a multi-topological spaces and introduced their basic properties. Also, introduced the concept of neutrosophic bi-topological space and neutrosophic crisp bi-topological space.

Kelly [2] in 1963 was first introduced The concept of bitopological spaces. This concept helps the authors to generalize the most results related to the topological spaces, which are known before. So, we find it necessary and important to construct a infra bi-Topological spaces.

The concept of supra bi-Topological spaces was introduced by R. Gowri, A.K.R. Rajayal [3] as an extension of supra topological spaces.

Since the concept of bi-topological space is fundamental and important concept in topology, many researchers have generalized it to the neutrosophic logic see[21-35].

In this paper, we presented infra bi-Topological spaces and studied some basic notions of this spaces, open

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022

(closed) set, closure, interior are defined in this new space. In addition, the theorems required for this structure are proved and their relations with infra-topological spaces are investigated.

2. Preliminaries

In this section, we discuss some basic definitions and properties of topology which are useful in sequel.

Definition 2.1. [1]

Let X be any arbitrary set. An Infra-topological space on X is a collection τ_i of subsets of X such that the following axioms are satisfying:

1. ϕ and X belong to τ_i.
2. The intersection of the elements of any subcollection of τ_i in X.

i.e, if $O_i \in \tau_i$, $1 \leq i \leq n$ then $\bigcap O_i \in \tau_i$.

the pair (X, τ_i) is called infra-topological space (ITS). Moreover, members of Γ are known as infra-open sets (IOS) and their complements are infra-closed sets(ICS).

Theorem 2.2. [1]

Let (X, τ_i) be a topological space (TS), then $(X, \tau_i)$is an infra-topological space (ITS),The converse of this theorem is not true.

Theorem 2.3. [1]

Let (X, τ_i) be infra-topological space, Then:

1. ϕ and X are infra-open set.
2. Any arbitrary intersections of infra-open sets are infra-open sets.
3. Finite union of infra-open sets may not be infra-open sets.

Theorem 2.4. [1]

Let (X, τ_i) be an infra-topological space (ITS), Then:

For a set N over X, the infra interior and the infra closure of N are defined as: i-int(N)=$\cup\{G : G \subseteq N , G \in \tau_i \}$ and i-cl (N)=$\cap\{F : N \subseteq F , F^c \in \tau_i \}$.

Definition 2.5. [2]

Let Γ_1, Γ_2be two topology on a nonempty set X, then (X, Γ_1, Γ_2) bi-Topological space (Bi-TS for short).

3. Infra Bi-topological Spaces:

In this section , We will introduce infra bi-Topological spaces. Moreover we will introduce new kinds of open and closed sets in infra bi-Topological spaces.

Definition 3.1.

Let $(X, \Gamma_1), (X, \Gamma_2)$ be two infra-topological space on a nonempty set X, then (X, Γ_1, Γ_2) infra bi-Topological space (IBi-TS for short).

Example 3.2.

Let X={e , f , g}. $\Gamma_1 = \{\emptyset, \{e\}, \{f\}, X\}$,

$\Gamma_2 = \{\emptyset, \{e\}, \{g\}, X\}$,Then (X , Γ_1) , (X , Γ_2) are infra-spaces therefore(X, Γ_1, Γ_2) is infra bi-topological space.

Definition 3.2.

Let (X, Γ_1, Γ_2) be an IBi-NTS, then :

The elements in $\Gamma_1 \cup \Gamma_2$ are said to be infra bi-open sets (IBi-OS for short). A set F is infra bi-closed (IBi-CS for short *)* if and only if its complement F^c is an infra bi-open set.

- the family of all infra bi-open sets is denoted by (IBi-OS*(X)*).
- the family of all infra bi-closed sets is denoted by (IBi-CS*(X)*).

Example 3.4.

In Example 3.2 the infra bi-open sets *(*IBi-OS*)* are :

IBi-OS(X) = $\Gamma_1 \cup \Gamma_2 = \{\emptyset, \{e\}, \{f\}, \{g\}, X\}$

Remark 3.5.

1) Every infra-open sets in (X , Γ_1) or (X , Γ_2) is infra bi-open set.
2) Every infra-closed sets in (X , Γ_1) or (X , Γ_2) is infra bi-closed set.

Proof:

1) Let A be an infra-open set in (X , Γ_1) or (X , Γ_2), then A$\in \Gamma_1 \cup \Gamma_2$. Therefore A is infra bi-open set.
2) Let A be an infra-closed set in (X , Γ_1) or (X , Γ_2). Therefore A is infra bi-closed set.
3)

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022

Remark 3.6.
Every infra bi-Topological space (X,Γ_1,Γ_2) induces two infra-topological space s as (X,Γ_1), (X,Γ_2) .
Remark 3.7.
If (X,Γ) infra-topological space then (X,Γ,Γ) infra bi-Topological space.
Remark 3.8.
Let (X,Γ_1,Γ_2) be a SBi-NTS, then :
The union of two infra bi-open (bi-closed) sets is not infra bi-open (bi-closed) set.
as the following example:
Example 3.9.
In Example 3.2 $\{e\},\{g\}$ are two infra bi-open sets but $\{e\}\cup\{g\} = \{e,g\}$ is not infra bi-open set.
Remark 3.10.
Let (X,Γ_1,Γ_2) be a IBi-CTS, then :
The intersection of two infra bi-open (bi-closed) sets is not infra bi-open (bi-closed) set.
as the following example:
Example 3.12.
Let X={e , f , g}. $\Gamma_1 = \{\emptyset,\{e\},\{f,g\},X\}$,
$\Gamma_2 = \{\emptyset,\{e,f\},X\}$,Then (X,Γ_1), (X,Γ_2) are infra-spaces therefore(X,Γ_1,Γ_2) is infra bi-topological space.
$N_1 = \{f,g\}, N_2 = \{e,f\}$ are two infra bi-open sets but $N_1 \cap N_2 = \{f\}$ is not infra bi-open set.
Definition 3.13.
Let (X,Γ_1,Γ_2) be an IBi-TS, A is set in X then :
The union of any infra bi-open sets ,contain in A is called infra bi-interior of A ($I^{Bi}\text{int}(A)$ for short).
$I^{Bi}\text{int}(A) = \cup\{B : B\subseteq A ; B \in \text{IBi}-OS(X)\}$
Theorem 3.14.
Let (X,Γ_1,Γ_2) be a IBi-TS, A ,B are subsets in X then :

1. $I^{Bi}\text{int}(A) \subseteq A$.
2. $I^{Bi}\text{int}(A)$is not infra bi-open set .
3. $A \subset B \Longrightarrow I^{Bi}\text{int}(A) \subset I^{Bi}\text{int}(B)$.

Proof :

1. Since $I^{Bi}\text{int}(A)$ is a union of any infra bi-open sets ,contains in A. Therefore $I^{Bi}\text{int}(A) \subseteq A$.
2. Follow from Remark 3.8.
3. Since $A \subset B$then $I^{Bi}\text{int}(A) = \cup\{N : N\subseteq A ; N \in \text{IBi}-OS(X)\}\subseteq$
$\cup\{N : N\subseteq B ; N \in \text{IBi}-OS(X)\} = I^{Bi}\text{int}(B)$.

Definition 3.15.
Let (X,Γ_1,Γ_2) be an IBi-TS, A is sub-set in X then :
The intersection of any infra bi-open sets ,contained A is called infra bi -closure of A ($I^{Bi}cl(A)$for short).

$$I^{Bi}cl(A) = \cap\{B : B\supseteq A ; B \in \text{IBi}-CS(X)\}$$

Theorem 3.16.
Let (X,Γ_1,Γ_2) be an IBi-TS, A is sub-set in X then :

1. $A\subseteq I^{Bi}cl(A)$.
2. $I^{Bi}cl(A)$ is not infra bi-closed set.

Proof :

1. Follow from the defintion of $I^{Bi}cl(A)$ as a intersection of any infra bi-closed set,contained in A.
2. Follow from Remark 3.10.

3. Pairwise Supra Open(Closed) Sets:
we introduced new concept of open and closed sets in infra bi-Topological space in this section, as pairwise infra-open(closed) sets. Also we introduced the basic properties of this new concept of open and closed sets in IBi-CTS .
Definition 4.1.
Let (X,Γ_1,Γ_2) be an IBi-TS, then :
A set A in X is said to be an infra pairwise open set in (X,Γ_1,Γ_2) if there exist a IBi-OS B in Γ_1 and a IBi-OS C in Γ_2 such that A=B∪C .
Definition 4.2.

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022

Let (X, Γ_1, Γ_2) be an IBi-TS, then :
A set A in X is said to be an infra pairwise closed set in (X, Γ_1, Γ_2)if its complement is a IBi-OS in (X, Γ_1, Γ_2) .Obviously, A set A in X is said to be an infra pairwise closed set in (X, Γ_1, Γ_2)if there exist a IBi-CS B in $(\Gamma_1)^c$ and a IBi-CS C in $(\Gamma_2)^c$ such that A=B∩C .
The family of all pairwise infra-open (closed) sets in (X, Γ_1, Γ_2) is denoted by $PIO(X, \Gamma_1, \Gamma_2)$ $[PIC(X, \Gamma_1, \Gamma_2)]$.

Remark 4.3.

1) Every infra-open sets in (X, Γ_1) or (X, Γ_2) is pairwise infra-open set.
2) Every infra-closed sets in (X, Γ_1) or (X, Γ_2) is pairwise infra-closed set.

Proof.

1) Let A be an infra-open sets in (X, Γ_1) or (X, Γ_2) then $A=A\cup\phi$ is pairwise infra-open set.
2) Let A be an infra-closed sets in (X, Γ_1) or (X, Γ_2) then $A=A\cap X$ is pairwise infra-closed set.

Remark 4.4.

Let (X, Γ_1, Γ_2) be an IBi-TS, then :

1) Every infra bi-open sets is pairwise infra-open set, but the converse is not true.
2) Every infra bi-closed sets is pairwise infra-closed set, but the converse is not true.

Proof.

1) Let A be an infra bi-open sets, then A is infra-open sets in (X, Γ_1) or (X, Γ_2). Therefore $A=A\cup\phi$ is pairwise infra-open set.
2) Let A be an infra bi-open sets, then A is infra-open sets in (X, Γ_1) or (X, Γ_2). Therefore $A=A\cup\phi$ is pairwise infra-open set.
   - The folooing example shows that converse of this theorem is not true.

Example 4.5.

Let X={e , f , g}. $\Gamma_1 = \{\emptyset, \{e\}, \{f\}, X\}$,
$\Gamma_2 = \{\emptyset, \{e\}, \{g\}, X\}$,Then (X, Γ_1) , (X, Γ_2) are infra-spaces therefore(X, Γ_1, Γ_2) is infra bi-topological space.
$\{f, g\}$ is pairwise infra-open sets in (X, Γ_1, Γ_2) ,but it is not infra bi-open set.

Definition 4.6.

Let (X, Γ_1, Γ_2) be an IBi-TS, and A is sub- set in X then :
The union of any pairwise infra-open sets ,contain in A is called pairwise infra interior of A ($PI^{Bi}int(A)$ for short).

$$PI^{Bi}int(A) = \cup \{B : B \subseteq A ; B \in PIO(X, \Gamma_1, \Gamma_2)\}$$

Theorem 4.7.

Let (X, Γ_1, Γ_2) be an IBi-TS, A ,B are sub-sets in X then :
$PIS^{Bi}int(A) \subseteq A$.

Proof :

Since $PI^{Bi}int(A) = \cup \{B : B \subseteq A ; B \in PIO(X, \Gamma_1, \Gamma_2)\}$,
therefore $PIS^{Bi}int(A) \subseteq A$.

Theorem 4.8.

Let (X, Γ_1, Γ_2) be an IBi-TS, A ,B are sub-sets in X then :
$A \subset B \Longrightarrow PI^{Bi}int(A) \subset PI^{Bi}int(B)$.

proof.

Since $A \subset B$ then $PI^{Bi}int(A) = \cup \{N : N \subseteq A ; N \in PIO(X, \Gamma_1, \Gamma_2)\} \subseteq$
$\cup \{N : N \subseteq B ; N \in PIO(X, \Gamma_1, \Gamma_2)\} = PI^{Bi}int(B)$.

Theorem 4.9. Let (X, Γ_1, Γ_2) be an IBi-TS, A ,B are sub-sets in X then :
$A \subseteq PI^{Bi}cl(A)$.

Proof :

Follow from the defintion of $PI^{Bi}cl(A)$ as a intersection of any pairwise infra-closed set,contained in A**.**

5. Conclusions

In this work, we have introduced infra bi-Topological space. Then we have introduced new open(closed) sets in infra bi-Topological space. Also we studied some of their basic properties and their relationship with each other .Finally, This paper is just a beginning of a new structure and we have studied a few ideas only, it will be necessary to carry out more theoretical research to establish a general framework for the practical application. In the future,

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022

using these notions, various classes of mappings on infra bi-Topological space, separation axioms on the infra bi-Topological spaces and many researchers can be studied.

6. References

a. M. AL-Odhari, On infra topological space, International Journal of Mathematical Archive, (2015), 6(11), 179-184.

[2] J. C. Kelly. Bitopological spaces. Proceedings of the London Mathematical Society,(1963), 3(1), 71-89.

[3] R. Gowri, A.K.R. Rajayal, On Supra Bi-Topological space4s, IOSR Journal of Mathematics (IOSR-JM) .No.5(113)(2017),55-58.

[4] S. Mashhour, A.A. Allam, F.S. Mahmoud and F.H. Khedr, On Supra topological spaces, Indian Jr.Pure and Appl.Math.No.4(14)(1983),502-510.

[5] L. A. Zadeh. Fuzzy Sets. Inform. Control, (1965),8, 338-353.

[6] F. Smarandache, "Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics" University of New Mexico, Gallup, NM 87301, USA (2002).

a. A Salama; F.Smarandache and Kroumov. "Neutrosophic crisp Sets and Neutrosophic crisp Topological Spaces" Neutrosophic Sets and Systems, Vlo.2, (2014), pp.25-30.

b. A. Salama and S. A. Alblowi. Neutrosophic set and neutrosophic topological space. ISORJ, Mathematics, Vol. (3), Issue 4, (2012), 31-35.

[7] Mashhour, S.; Allam, A.A.; Mahmoud F.S.; and Khedr, F.H.; On Supra topological spaces", Indian Jr.Pure and Appl.Math.No.4(14)(1983),502-510.

[8] R.K. Al-Hamido. "A study of multi-Topological Spaces", PhD Theses ,AlBaath university , Syria(2019).

a. S. Mashhour, A. A. Allam, F. S. Mahmoud, and F. H. Kheder, "On supra topological spaces," Indian Journal of Pure and Applied Mathematics, vol. 14, pp. 502–510, 1983.

[9] T. Al-Shami, "Some results related to supra topological spaces," Journal of Advanced Studies in Topology, vol. 7, no. 4, pp. 283–294, 2016.

[10] R. Devi, S. Sampathkumar, and M. Caldas, "On α-open sets and sα-continuous maps," General Mathematics, vol. 16, pp. 77–84, 2008.

[11] O. R. Sayed, "Supra pre-open sets and supra pre-continuous on topological spaces," Series Mathematics and Information, vol. 20, pp. 79–88, 2010.

[12] O. R. Sayed and T. Noiri, "On supra b-open sets and supra bcontinuity on topological spaces," European Journal of Pure and Applied Mathematics, vol. 3, pp. 295–302, 2010.

[13] S. Jafari and S. Tahiliani, "Supra β-open sets and supra β-continuity on topological spaces," Annales Universitatis Scientiarium Budapestinensis, vol. 56, pp. 1–9, 2013.

[14] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-Shami, "On supra R-open sets and some applications on topological spaces," Journal of Progressive Research in Mathematics, vol. 8, pp. 1237–1248, 2016.

[15] T. Al-Shami, "On supra semi open sets and some applications on topological spaces," Journal of Advanced Studies in Topology, vol. 8, no. 2, pp. 144–153, 2017. T. M. Al-Shami, "Utilizing supra α-open sets to generate new types of supra compact and supra lindel¨of spaces," Facta Universitatis, Series: Mathematics and Informatics, vol. 32, pp. 151–162, 2017.

[16] T. M. Al-Shami, "Supra semi-compactness via supra topological spaces," Journal of Taibah University for Science, vol. 12, no. 3, pp. 338–343, 2018.

[17] T. M. Al-Shami, B. A. Asaad, and M. A. El-Gayar, "Various types of supra pre-compact and supra pre-Lindel¨of spaces,"Missouri Journal of Mathematical Science, vol. 32, 2020.

[18] J. M. Mustafa, "Supra b-compact and supra b-Lindelof spaces," Journal of Mathematics and Applictions, vol. 36, pp. 79–83, 2013.

[19] G.Jayaparthasarathy; V.F.Little Flower; M.Arockia Dasan. "Neutrosophic Supra Topological Applications in Data Mining Process", Neutrosophic Sets and Systems, ,(2019) ,vol. 27, pp. 80-97.

a. B.AL-Nafee; R.K. Al-Hamido; F.Smarandache. "Separation Axioms In Neutrosophic Crisp Topological Spaces",Journal of newtheory, (2018),vol. 22, pp.

[20] R. Gowri, A.K.R. Rajayal, "On Supra Bi-topological spaces", IOSR Journal of Mathematics (IOSR-JM) .No.5(113)(2017),55-58.

[21] J. C. Kelly. "Bi-topological spaces". Proceedings of the London Mathematical Society, 3(1) ,(1963), 71-89.

[22] T.Y. Ozturk , A. Ozkan "Neutrosophic Bi-topological Spaces", Neutrosophic Sets and Systems, vol. 30,(2019) ,pp. 88-97.

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022

[23] R.K. Al-Hamido. "Neutrosophic Crisp Bi-Topological Spaces", Neutrosophic Sets and Systems, vol. 21, (2018), pp. 66-73.

[24] M. Parimala, M. Karthika ,F. Smarandache. "Introduction to neutrosophic minimal structure", Neutrosophic Sets and Systems, vol. 36, (2020) , pp. 376-388.

[25] R. K. Al-Hamido, On Neutrosophic Tri-Topological space, (2018). Journal of newtheory, vol. 23, pp. 13-21.

[26] R.K. Al-Hamido, Q. H. Imran, K. A. Alghurabi, T. Gharibah, "On Neutrosophic Crisp Semi Alpha Closed Sets", Neutrosophic Sets and Systems", vol. 21, 28-35, (2018).

[27] Q. H. Imran, F. Smarandache, R.K. Al-Hamido, R. Dhavasselan, "On Neutrosophic Semi Alpha open Sets", Neutrosophic Sets and Systems, vol. 18, 37-42, (2017).

[28] R.K. Al-Hamido, T. Gharibah, S. Jafari F.Smarandache, "On Neutrosophic Crisp Topology via N-Topology", Neutrosophic Sets and Systems, vol. 21, 96-109, (2018).

[29] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, (1999).

[30] F. Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA (2002).

a. A.Salma, S.A. Alblowi,. Neutrosophic set and neutrosophic topological spaces. IOSR J. Math., (2012) 3(4), 31–35.

Doi: https://doi.org/10.54216/PAMDA.010201
Received: August 18, 2022 Accepted: December 17, 2022