Analyzing on My Turf Game Over Some Finite Non-Abelian Groups

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Abstract

The aim of this paper is to solve the “ON MY TURF" game over some finite nonabelian groups. Also, it presents the following results:

1-) If G has odd order, and the set F contains the identity element, then the first player A has a winning strategy. If F does not contain the identity, then B has a winning strategy.

2-) If G has an even order with only one element of order two, there is a winning strategy related to set F.

3-) If G has an even order with only three elements of order two which generate a subgroup isomorphic to $Z_2 \times Z_2$, there is a winning strategy related to the set F.

Keywords: MT-Game; Finite group; Non-abelian group; Algebraic games.

1. Introduction

Game theory as a branch of applied mathematics has many applications to the real world and mathematical problems. Games played over groups can be very useful in group theoretic studies. Many games were defined over algebraic structures such as groups, rings, and neutrosophic structures. See [1,4,5,6].

The goal of analyzing a game is to determine if there exists a winning strategy for any player or not, which is the main goal of any game theoretic study [2].

In[1], the "ON MY TURF" game (denoted by MT-game) was defined to be played over finite groups. Authors studied this game over abelian finite groups and over the dihedral group. They determined possible winning strategies in previous cases.

In this work, we extend their work, where we study " MT-Game" over some nonabelian finite groups. We discuss the case of an odd order group with any subset F, and the case of an even order group with only one element or three elements of order two and with a subset F in all possible cases of F. Also, we determine the possible winning strategies in each case.

2. Preliminaries

Definition 2.1: [1]

Let G be a finite group and F is a subset of G, "ON MY TURF" game is played as follows:

The first player A picks an element x_i, then B picks an unchosen element y_i, and so on.

The game ends when there are no elements to choose from. If the group product of elements $(x_i.y_i \dots)$ in this order belongs to F then A wins, else B wins.

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Theorem 2.2: [1]

Let G be an abelian group with odd order then A has a winning strategy if and only if the identity element is in F.

Theorem 2.3 : [1]

Let G be an abelian group with even order then if G has a subgroup isomorphic to $Z_2 \times Z_2$, then A has a winning strategy if and only if the identity is in F.

Theorem 2.4 : [1]

If G is the dihedral group D_n and n mod 4 =2, then A has a winning strategy if and only if identity is in F.

If n mod 4 =0 then A has a winning strategy in MT_n(G, *) if and only if $r^{n/2}$ is in F.

3. Main discussion and results

Even order nonabelian groups :

Theorem 3.1:

Let G be an even order nonabelian finite group, suppose that G has one only element (a) with order two different forms of the identity. If set F does not contain the element (a), then the second player B has a winning strategy.

Proof:

We describe the winning strategy of player B as follows:

If A picks an element x, then B should pick its inverse x^{-1}. If A picks (a), then B should pick the identity e. If A picks e, then B should pick (a).

We will reach the final position and the group product x.x^{-1} ... $a.e$ ... $y.y^{-1} = a$. Hence, B will be the winner.

Example 3.2:

Let $H = \{x_1, x_2, \dots, x_{21}\}$ be a nonabelian group with order 21 with identity x_1, $K = Z_2$ the group of integers modulo 2. We have $G = H \times K$ as a nonabelian group with order 42. It has only one element of order two different from the identity, which is $a = (x_1, 1)$. Consider the subset F=$\{(x_1, 0), (x_2, 1)\}$. We remark that $a = (x_1, 1)$ does not belong to F.

We assume that player A begins with picking the identity $(x_1, 0)$, according to the previous theorem, B should take $a = (x_1, 1)$. Now for every choice of player A $(x_i, y_i); 2 \le i \le 21, y_i \in K$, B should pick its inverse. It is clear that the group product of chosen elements is equal to $a = (x_1, 1)$, thus B is the winner.

Theorem 3.3:

Let G be an even order nonabelian finite group, suppose that G has one only element with order two say (a). If set F contains (a), then player A has a winning strategy.

Proof:

The winning strategy of player A can be described as follows:

A begins with picking the element (a), if B picks an element x, then A should pick its inverse x^{-1}. If B picks the identity e before the last position, A should pick any element m. If B picks an element t which is different from m, A should pick t^{-1},

but if B picked m^{-1}, then A should pick any element s and so on.

In the final position, the group product of elements must be equal to (a), and A will be the winner.

Example 3.4:

Let $H = \{x_1, x_2, \dots, x_{21}\}$ be a nonabelian group with order 21 with identity x_1, $K = Z_2$ the group of integers modulo 2. We have $G = H \times K$ as a nonabelian group with order 42. It has only one element of order two different from the identity, which is $a = (x_1, 1)$. Consider the subset F=$\{(x_1, 1), (x_2, 1)\}$. We remark that $a = (x_1, 1)$ belongs to F.

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According to the previous theorem, A should begin with $a = (x_1, 1)$. For every choice of player B different from the identity $(x_1, 0)$, A should take its inverse.

If B picked $(x_1, 0)$, then A should pick any unchosen element. It is clear that the group product of chosen elements is equal to $a = (x_1, 1)$, thus A wins the game.

Theorem 3.5:

Let G be an even order nonabelian finite group, suppose that G has even ordered with only three elements {a, b, ab} of order two different forms of the identity e,

M= {a, b , ab , e} is a subgroup isomorphic to $Z_2 \times Z_2$. If set F contains the identity e, then player A has a winning strategy.

Proof:

A should begin with (a), if B picked an element x which is not contained in M, then A should pick its inverse x^{-1}, but if B picked an unchosen element from M then A should pick another element from M, if B picked the last possible element in M, then A should pick any unchosen element in G and so on. In the final position, the group product will be equal to e and A will be the winner.

Example 3.6:

Let $H = \{x_1, x_2, \dots, x_{21}\}$ be a nonabelian group with order 21 with identity x_1, $K = Z_2 \times Z_2$. We have $G = H \times K$ as a nonabelian group with order 84. It has only three elements of order two different from the identity. These elements are $a = (x_1, 1,0), b = (x_1, 0,1), ab = (x_1, 1,1); M = \{a, b, ab, e\}$ is a subgroup isomorphic to $Z_2 \times Z_2$. Consider the subset F=$\{(x_1, 0,0), (x_3, 1,0), (x_5, 1,1)\}$, we can see that the identity e=$(x_1, 0,0)$ is in F.

According to the previous theorem, player A should begin with $a = (x_1, 1,0)$. If B picked any element x which is not in M, A should take its inverse.

If B picked b, A should pick ab or e. If B picked ab, A should pick b or e. If B picked e, A should pick b or ab.

In the final position, we get e as a product of chosen elements, thus A is the winner.

Theorem 3.7:

Let G be an even order nonabelian finite group, suppose that G has even ordered with only three elements {a, b, ab} of order two different forms of the identity e,

M= {a, b , ab , e} is a subgroup isomorphic to $Z_2 \times Z_2$. If set F does not contain the identity e, then B has a winning strategy.

Proof:

If A picked any element from M, then B should pick an element from M, if A picked an element x which is not contained in M, then B should pick its inverse x^{-1}.

In the final position, the group product will be equal to e and B will be the winner.

Odd order finite group:

Theorem 3.8:

Let G be an odd-order finite non-abelian group. Then we have:

(a) If set F contains the identity e, then player A has a winning strategy.

(b) If the set F does not contain the identity e then B has a winning strategy.

Proof:

(a) A should begin with e, then for each picked element x by player B, A should pick its inverse x^{-1}. In the final position, we get e as a product of chosen elements, and A wins.

(b) If A picks an element x which is different from e then B should pick its inverse x^{-1}.

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If A picks the identity in any step, then B should pick any element y.

Now, if A picked y^{-1} then B should pick another element z, but if A picked an element m which is different from y, then B should pick its inverse m^{-1}, and so on.

In the final position, we have a group product that is equal to e, and B wins.

Example 3.9:

Let $G = \{x_1, x_2, \dots, x_{21}\}$ be a non abelian group with order 21 with identity x_1. Consider the subset F=$\{x_1, x_3\}$. It is clear that F contains the identity x_1.

According to Theorem 3.8, A should begin with x_1. Now if B picked any element x_i, A should pick its inverse x_i^{-1}. It is easy to see that A is the winner.

Conclusion

In this article, we have studied the algebraic game "On My Turf" over many non-abelian groups. Many interesting results were obtained through this direction, where we have determined a winning strategy in the following cases:

1-) If G has odd order, and the set F contains the identity element, then the first player A has a winning strategy. If F does not contain the identity, then B has a winning strategy.

2-) If G has an even order with only one element of order two, there is a winning strategy related to set F.

3-) If G has even ordered with only three elements of order two which generate a subgroup isomorphic to $Z_2 \times Z_2$, there is a winning strategy related to the set F.

As a future research direction, we aim to analyze it over more and more non-abelian finite groups with even orders.

References

[1] Babinkostova, L., Cosket, S., Kontradyuk, D., Navert, S., Potter, S., and Scheepers, M., "A study of games over finite groups", Publication at www.researchgate.net, July 2015, Boise State University, pp.(1-3).

[2] Al-Najjar, H., Dabash, M., and Barbara, H., "Create HIM, the Adjusted NIM Game, and Analyses Its Winning Strategy", PHD Thesis, University of Aleppo Press, 2013, pp.20-130.

[3] Haushi, M., "Algebraic Structures", Tishreen University Press, 2004, pp. 112-140.

[4] Selvakumari, K., and Lavanya, S., "Neutrosophic Fuzzy Soft Game", International Journal of Engineering and Technology, Vol. 7, 2018, pp. 667-669.

[5] Benesh, B.J., Ernest, D.C., and Sieben, N., "Impartial Avoidance Games for Generating Finite Groups", North-Western European Journal of Mathematics, 2016, pp.89-103.

[6] Ahmad Dar, Y., and Sharma, Y., "Algebraic Method in Game Theory", International Journal of Multidisciplinary Research and Development, Vol. 4, 2017, pp. 368-372.

[7] Bal, M., and Abobala, M., " On The Representation Of Winning Strategies Of Finite Games By Groups and Neutrosophic Groups", Journal Of Neutrosophic and Fuzzy Systems, 2022.

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