A Creative Approach on Bipolar Neutrosophic Nano$*$ Topology

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Abstract

In this article, Bipolar neutrosophic nano topology is introduced and some of its properties are studied. We found that, it is difficult to find a nano topology for every bipolar neutrosophic set. So we introduced a bipolar neutrosophic nano $-*$ topology, which is a generalization of bipolar neutrosophic nano topology and its fundamental features are investigated with appropriate examples. This approach is used for real life multi criteria decision making situations. The practical problems may be solved by finding CORE values through the criterion reduction.One can apply the basis of the bipolar neutrosophic nano $-*$ topology as the key factors of a particular decision making problem using topological reduction of attributes in complete or incomplete information system.

Keywords: Nano topology; Bipolar set, Fuzzy set; Neutrosophic set; Intuitionistic fuzzy set; Neutrosophic nano topology; Bipolar Neutrosophic set; Bipolar Neutrosophic nano topology; Bipolar Neutrosophic nano $-*$ topology.

1. Introduction

Nano topology is the smallest topology that is defined in terms of approximations and boundary region of a subset of a nonempty set[18], which is introduced by Lellis Thivagar M. et. al. In 2018, the concept of neutrosophic nano topology was introduced by Lellis Thivagar M., Saeid Jafari et.al.[17].

In this study, we introduce a structure of Bipolar neutrosophic nano topology which is a nano topology with respect to a bipolar neutrosophic set. When we introduce it we obtained that we cannot find nano topology for every bipolar neutrosophic set. So we introduce and extend the nano topology to a new version called Bipolar Neutrosophic Nano $-*$ Topology, which we can find as a topology for any bipolar neutrosophic set. We also studied its features with appropriate examples.

2. Preliminaries

Definition 2.1. [11] A bipolar neutrosophic set A in X is defined as an object of the form $A=\{x,T^+(x),I^+(x),F^+(x),T^-(x),I^-(x),F^-(x):x\in X\}$ where $T^+(x),I^+(x),F^+(x):X\to[0,1]$ and $T^-(x),I^-(x),F^-(x):X\to[-1,0]$. The positive membership degrees $T^+(x),I^+(x),F^+(x)$ denote the truth membership, indeterminate membership and false membership of an element $x\in X$ corresponding to a bipolar neutrosophic set A and the negative membership degrees $T^-(x),I^-(x),F^-(x)$ denote the truth membership, indeterminate membership and false membership of an element $x\in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

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Definition 2.2. [17] Let U be a nonempty set and R be an indiscernibility relation on U. Let F be a neutrosophic set in U with the membership function μ_F, indeterminacy function σ_F and the non-membership function ν_F. The neutrosophic nano lower, neutrosophic nano upper approximation and neutrosophic nano boundary of F in the approximation space (U,R) denoted by $\underline{N}(F)$, $\overline{N}(F)$ and $BN(F)$ are respectively defined as follows:

(i) $\underline{N}(F)=\left\{\left\langle x,\left(\mu_{\underline{R}(A)}(x),\sigma_{\underline{R}(A)}(x),\nu_{\underline{R}(A)}(x)\right)\right\rangle : y\in[x]_R, x\in U\right\}.$

(ii) $\overline{N}(F)=\left\{\left\langle x,\left(\mu_{\overline{R}(A)}(x),\sigma_{\overline{R}(A)}(x),_{\overline{R}(A)}(x)\right)\right\rangle : y\in[x]_R, x\in U\right\}.$

(iii) $BN(F)=\overline{N}(F)-\underline{N}(F).$

Where,

$$\mu_{\overline{R}(A)}(x)=\bigvee_{y\in[x]_R}\mu_A(y),\ \sigma_{\overline{R}(A)}(x)=\bigvee_{y\in[x]_R}\sigma_A(y),\ \nu_{\overline{R}(A)}(x)=\bigwedge_{y\in[x]_R}\nu_A(y),$$

$$\mu_{\underline{R}(A)}(x)=\bigwedge_{y\in[x]_R}\mu_A(y),\ \sigma_{\underline{R}(A)}(x)=\bigwedge_{y\in[x]_R}\sigma_A(y),\ \nu_{\underline{R}(A)}(x)=\bigvee_{y\in[x]_R}\nu_A(y).$$

3. Bipolar Neutrosophic Nano Topology

Definition 3.1. Let $\tilde{U}$ be a nonempty set and $\mathfrak{R}$ be an equivalence relation on $\tilde{U}$ which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a bipolar neutrosophic set (BNS) in $\tilde{U}$ with the positive degree of true membership η_Q^+, indeterminacy ψ_Q^+ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership η_Q^-, indeterminacy ψ_Q^- and the false membership function $\xi_Q{}^-$, where $\eta_Q^+,\psi_Q^+,\xi_Q^+:\tilde{U}\to[0,1]$, $\eta_Q^-,\psi_Q^-,\xi_Q^-:\tilde{U}\to[-1,0]$. Then the nether, higher and extremity estimations are respectively given as follows:

(i) $\underline{BN}(Q)=\left\{\left\langle q,\left(\eta^+_{\underline{\mathfrak{R}}(Q)}(q),\psi^+_{\underline{\mathfrak{R}}(Q)}(q),\xi^+_{\underline{\mathfrak{R}}(Q)}(q),\eta^-_{\underline{\mathfrak{R}}(Q)}(q),\psi^-_{\underline{\mathfrak{R}}(Q)}(q),\xi^-_{\underline{\mathfrak{R}}(Q)}(q)\right)\right\rangle : z\in[q]_{\mathfrak{R}}, q\in\tilde{U}\right\}.$

(ii) $\overline{BN}(Q)=\left\{\left\langle q,\left(\eta^+_{\overline{\mathfrak{R}}(Q)}(q),\psi^+_{\overline{\mathfrak{R}}(Q)}(q),\xi^+_{\overline{\mathfrak{R}}(Q)}(q),\eta^-_{\overline{\mathfrak{R}}(Q)}(q),\psi^-_{\overline{\mathfrak{R}}(Q)}(q),\xi^-_{\overline{\mathfrak{R}}(Q)}(q)\right)\right\rangle : z\in[q]_{\mathfrak{R}}, q\in\tilde{U}\right\}.$

(iii) $B_{BN}(Q)=\overline{BN}(Q)-\underline{BN}(Q).$

Where

$$\eta_{\underline{\mathfrak{R}}(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\eta_Q{}^+(z),\ \psi_{\underline{\mathfrak{R}}(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\psi_Q{}^+(z),\ \xi_{\underline{\mathfrak{R}}(Q)}{}^+(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\xi_Q{}^+(z),$$

$$\eta_{\underline{\mathfrak{R}}(Q)}{}^-(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\eta_Q{}^-(z),\ \psi_{\underline{\mathfrak{R}}(Q)}{}^-(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\psi_Q{}^-(z),\xi_{\underline{\mathfrak{R}}(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\xi_Q{}^-(z),$$

$$\eta_{\overline{\mathfrak{R}}(Q)}{}^+(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\eta_Q{}^+(z),\ \psi_{\overline{\mathfrak{R}}(Q)}{}^+(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\psi_Q{}^+(z),\xi_{\overline{\mathfrak{R}}(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\xi_Q{}^+(z),$$

$$\eta_{\overline{\mathfrak{R}}(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\eta_Q{}^-(z),\ \psi_{\overline{\mathfrak{R}}(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\mathfrak{R}}}\psi_Q{}^-(z),\xi_{\overline{\mathfrak{R}}(Q)}{}^-(q)=\bigvee_{z\in[q]_{\mathfrak{R}}}\xi_Q{}^-(z).$$

Definition 3.2. Let $\tilde{U}$ be a nonempty universe and K and H be the BNS's, where $K=\left\{\left\langle q,\left(\eta_K{}^+(q),\psi_K{}^+(q),\xi_K{}^+(q),\eta_K{}^-(q),\psi_K{}^-(q),\xi_K{}^-(q)\right)\right\rangle : q\in\tilde{U}\right\}$ and $H=\left\{\left\langle q,\left(\eta_H{}^+(q),\psi_H{}^+(q),\xi_H{}^+(q),\eta_H{}^-(q),\psi_H{}^-(q),\xi_H{}^-(q)\right)\right\rangle : q\in\tilde{U}\right\}.$

Then,

(i) the null bipolar neutrosophic nano set is given by $0_{BNN}=\left\{\left\langle q,(0,0,1,0,0,-1)\right\rangle : q\in\tilde{U}\right\}.$

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(ii) the absolute bipolar neutrosophic nano set is given by $1_{BNN} = \{\langle q, (1,1,0,-1,-1,0)\rangle : q \in \tilde{U}\}$.

(iii) $K \subseteq H$ iff $\eta_K{}^+(q) \le \eta_H{}^+(q), \psi_K{}^+(q) \le \psi_H{}^+(q), \ \xi_K{}^+(q) \ge \xi_H{}^+(q)$,
$\eta_K{}^-(q) \ge \eta_H{}^-(q), \ \psi_K{}^-(q) \ge \psi_H{}^-(q), \ \xi_K{}^-(q) \le \xi_H{}^-(q)$.

(iv) $K = H$ iff $K \subseteq H$ and $H \subseteq K$.

(v) $K^C = \{\langle q, (\xi_K{}^+(q), 1-\psi_K{}^+(q), \eta_K{}^+(q), \xi_K{}^-(q), 1-\psi_K{}^-(q), \eta_K{}^-(q))\rangle : q \in \tilde{U}\}$.

(vi) $$K \cap H = \left\{\left\langle q, \begin{pmatrix} \eta_K{}^+(q) \wedge \eta_H{}^+(q), \psi_K{}^+(q) \wedge \psi_H{}^+(q), \xi_K{}^+(q) \vee \xi_H{}^+(q), \\ \eta_K{}^-(q) \vee \eta_H{}^-(q), \psi_K{}^-(q) \vee \psi_H{}^-(q), \xi_K{}^-(q) \wedge \xi_H{}^-(q) \end{pmatrix}\right\rangle : q \in \tilde{U}\right\}.$$

(vii) $$K \cup H = \left\{\left\langle q, \begin{pmatrix} \eta_K{}^+(q) \vee \eta_H{}^+(q), \psi_K{}^+(q) \vee \psi_H{}^+(q), \xi_K{}^+(q) \wedge \xi_H{}^+(q), \\ \eta_K{}^-(q) \wedge \eta_H{}^-(q), \psi_K{}^-(q) \wedge \psi_H{}^-(q), \xi_K{}^-(q) \vee \xi_H{}^-(q) \end{pmatrix}\right\rangle : q \in \tilde{U}\right\}.$$

(viii) $$K - H = \left\{\left\langle q, \begin{pmatrix} \min\{\eta_K{}^+(q), \xi_H{}^+(q)\}, \min\{\psi_K{}^+(q), 1-\psi_H{}^+(q)\}, \max\{\xi_K{}^+(q), \eta_H{}^+(q)\}, \\ \max\{\eta_K{}^-(q), \xi_H{}^-(q), \max\{\psi_K{}^-(q), -1-\psi_H{}^-(q), \min\{\xi_K{}^-(q), \eta_H{}^-(q)\} \end{pmatrix}\right\rangle : q \in \tilde{U}\right\}.$$

Definition 3.3. Let $\tilde{U}$ be a nonempty set, $\Re$ be an equivalence relation on $\tilde{U}$ and let Q be a BNS. The collection $\tau_{\Re_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}, \underline{BN}(Q), \overline{BN}(Q), B_{BN}(Q)\}$ is known as bipolar neutrosophic nano topology (BNN_Q – topology), if it forms a topology. Then the space $(\tilde{U}, \tau_{\Re_{BNN}}(Q))$ is named as bipolar neutrosophic nano topological space. The elements of $\tau_{\Re_{BNN}}(Q)$ are known as bipolar neutrosophic nano open sets (BNN_Q – open).

Remark 3.4. We can deduce bipolar fuzzy nano topology and bipolar intuitionistic nano topology from the definition of bipolar neutrosophic nano topology. We consider only truth membership value for Bipolar fuzzy nano topology and both truth and false membership values for Bipolar intuitionistic nano topology.

Definition 3.5. Let $\tilde{U}$ be a nonempty universe and $\Re$ be an equivalence relation on $\tilde{U}$, which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a bipolar intuitionistic fuzzy set (BIS) in $\tilde{U}$ with the positive degree of true membership η_Q^+ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership η_Q^- and the false membership function $\xi_Q{}^-$, where $\eta_Q^+, \xi_Q^+ : \tilde{U} \to [0,1]$, $\eta_Q^-, \xi_Q^- : \tilde{U} \to [-1,0]$. Then the nether, higher and extremity estimations are respectively given as follows:

(i) $\underline{BI}(Q) = \{\langle q, (\eta^+_{\underline{\Re}(Q)}(q), \xi^+_{\underline{\Re}(Q)}(q), \eta^-_{\underline{\Re}(Q)}(q), \xi^-_{\underline{\Re}(Q)}(q))\rangle : z \in [q]_\Re, q \in \tilde{U}\}$.

(ii) $\overline{BI}(Q) = \{\langle q, (\eta^+_{\overline{\Re}(Q)}(q), \xi^+_{\overline{\Re}(Q)}(q), \eta^-_{\overline{\Re}(Q)}(q), \xi^-_{\overline{\Re}(Q)}(q))\rangle : z \in [q]_\Re, q \in \tilde{U}\}$.

(iii) $B_{BI}(Q) = \overline{BI}(Q) - \underline{BI}(Q)$.

Where,

$\eta_{\underline{\Re}(Q)}{}^+(q) = \bigwedge_{z\in[q]_\Re} \eta_Q{}^+(z), \xi_{\underline{\Re}(Q)}{}^+(q) = \bigvee_{z\in[q]_\Re} \xi_Q{}^+(z)$, $\eta_{\underline{\Re}(Q)}{}^-(q) = \bigvee_{z\in[q]_\Re} \eta_Q{}^-(z)$, $\xi_{\underline{\Re}(Q)}{}^-(q) = \bigwedge_{z\in[q]_\Re} \xi_Q{}^-(z)$,

$\eta_{\overline{\Re}(Q)}{}^+(q) = \bigvee_{z\in[q]_\Re} \eta_Q{}^+(z), \xi_{\overline{\Re}(Q)}{}^+(q) = \bigwedge_{z\in[q]_\Re} \xi_Q{}^+(z)$, $\eta_{\overline{\Re}(Q)}{}^-(q) = \bigwedge_{z\in[q]_\Re} \eta_Q{}^-(z)$, $\xi_{\overline{\Re}(Q)}{}^-(q) = \bigvee_{z\in[q]_\Re} \xi_Q{}^-(z)$.

The collection $\tau_{\Re_{BIN}}(Q) = \{0_{BIN}, 1_{BIN}, \underline{BI}(Q), \overline{BI}(Q), B_{BI}(Q)\}$, where $0_{BIN} = \{\langle q, (0,1,0,-1)\rangle : q \in \tilde{U}\}$ and 1_{BIN} = $\{\langle q, (1,0,-1,0)\rangle : q \in \tilde{U}\}$ is known as the bipolar intuitionistic nano (BIN_Q) topology, if it forms a topology. Then the space $(\tilde{U}, \tau_{\Re_{BIN}}(Q))$ is known as the bipolar intuitionistic nano topological space. The elements of $\tau_{\Re_{BIN}}(Q)$ are known as bipolar intuitionistic nano open sets.

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Definition 3.6. Let $\tilde{U}$ be a nonempty set and $\Re$ be an equivalence relation on $\tilde{U}$ which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a bipolar fuzzy set (BFS) in $\tilde{U}$ with the positive and negative degrees of true membership function η_Q^+ and η_Q^- respectively, where $\eta_Q^+ : \tilde{U} \to [0,1]$, $\eta_Q^- : \tilde{U} \to [-1,0]$. Then the nether, higher and extremity estimations are respectively given as follows:

(i) $\underline{BF}(Q) = \left\{\left\langle q, \left(\eta^+_{\underline{\Re}(Q)}(q), \eta^-_{\underline{\Re}(Q)}(q)\right)\right\rangle : z \in [q]_\Re, q \in \tilde{U}\right\}$.

(ii) $\overline{BF}(Q) = \left\{\left\langle q, \left(\eta^+_{\overline{\Re}(Q)}(q), \eta^-_{\overline{\Re}(Q)}(q)\right)\right\rangle : z \in [q]_\Re, q \in \tilde{U}\right\}$.

(iii) $B_{BF}(Q) = \overline{BF}(Q) - \underline{BF}(Q)$.

Where,

$\eta_{\underline{\Re}(Q)}{}^+(q) = \underset{z\in[q]_\Re}{\wedge} \eta_Q{}^+(z), \eta_{\underline{\Re}(Q)}{}^-(q) = \underset{z\in[q]_\Re}{\vee} \eta_Q{}^-(z)$, $\eta_{\overline{\Re}(Q)}{}^+(q) = \underset{z\in[q]_\Re}{\vee} \eta_Q{}^+(z)$, $\eta_{\overline{\Re}(Q)}{}^-(q) = \underset{z\in[q]_\Re}{\wedge} \eta_Q{}^-(z)$.

The collection $\tau_{\Re_{BFN}}(Q) = \{0_{BFN}, 1_{BFN}, \underline{BF}(Q), \overline{BF}(Q), B_{BF}(Q)\}$, where $0_{BFN} = \{\langle q, (0,0)\rangle : q \in \tilde{U}\}$ and $1_{BFN} = \{\langle q, (1,-1)\rangle : q \in \tilde{U}\}$ is known as the bipolar fuzzy nano (BFN_Q) topology, if it forms a topology. Then the space $(\tilde{U}, \tau_{\Re_{BFN}}(Q))$ is known as the bipolar fuzzy nano topological space. The elements of $\tau_{\Re_{BFN}}(Q)$ are known as the bipolar fuzzy nano open sets.

Example 3.7. Let $\tilde{U} = \{q_1, q_2, q_3\}$ and $\tilde{U}/\Re = \{\{q_1, q_3\}, \{q_2\}\}$.

Let $Q = \{\langle q_1, (.6,.5,.3,-.4,-.6,-.5)\rangle, \langle q_2, (.5,.7,.4,-.3,-.6,-.4)\rangle\} \subseteq \tilde{U}$.

Then $\overline{BN}(Q) = \{\langle q_1, (.6,.5,.3,-.4,-.6,-.5)\rangle, \langle q_2, (.5,.7,.4,-.3,-.6,-.4)\rangle\}$,

$\underline{BN}(Q) = \{\langle q_1, (.6,.5,.3,-.4,-.6,-.5)\rangle, \langle q_2, (.5,.7,.4,-.3,-.6,-.4)\rangle\}$,

$B_{BN}(Q) = \{\langle q_1, (.3,.5,.6,-.4,-.4,-.5)\rangle, \langle q_2, (.4,.3,.5,-.3,-.4,-.4)\rangle\}$.

The set $\tau_{\Re_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}, \underline{BN}(Q), \overline{BN}(Q), B_{BN}(Q)\}$ is the BNN_Q – topology on $\tilde{U}$.

Now let $Q = \{\langle q_1, (.6,.3,-.4,-.5)\rangle, \langle q_2, (.5,.4,-.3,-.4)\rangle\}$ be a BIS. Then

$\underline{BI}(Q) = \{\langle q_1, (.6,.3,-.4,-.5)\rangle, \langle q_2, (.5,.4,-.3,-.4)\rangle\}$,

$\overline{BI}(Q) = \{\langle q_1, (.6,.3,-.4,-.5)\rangle, \langle q_2, (.5,.4,-.3,-.4)\rangle\}$,

$B_{BI}(Q) = \{\langle q_1, (.3,.6,-.4,-.5)\rangle, \langle q_2, (.4,.5,-.3,-.4)\rangle\}$.

The set $\tau_{\Re_{BIN}}(Q) = \{0_{BIN}, 1_{BIN}, \underline{BI}(Q), \overline{BI}(Q), B_{BI}(Q)\}$ is the BIN_Q - topology on $\tilde{U}$.

Let $Q = \{\langle q_1, (.6,-.4)\rangle, \langle q_2, (.5,-.3)\rangle\}$ be a BFS. Then

$\underline{BF}(Q) = \{\langle q_1, (.6,-.4)\rangle, \langle q_2, (.5,-.3)\rangle\}$,

$\overline{BF}(Q) = \{\langle q_1, (.6,-.4)\rangle, \langle q_2, (.5,-.3)\rangle\}$,

$B_{BF}(Q) = \{\langle q_1, (.3,-.4)\rangle, \langle q_2, (.4,-.3)\rangle\}$.

The set $\tau_{\Re_{BFN}}(Q) = \{0_{BFN}, 1_{BFN}, \underline{BF}(Q), \overline{BF}(Q), B_{BF}(Q)\}$ is the BFN_Q – topology on $\tilde{U}$.

Definition 3.8. Let $\tilde{U}$ be a nonempty universe, $\Re$ be an equivalence relation on $\tilde{U}$ and Q be a BNS in $\tilde{U}$.

(i) If $\tau_{\Re_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}, \underline{BN}(Q), \overline{BN}(Q), B_{BN}(Q)\}$ is a BNN_Q – topology on $\tilde{U}$, then $[\tau_{\Re_{BNN}}(Q)]^C$ is named as the duel bipolar neutrosophic nano topology whose elements are $[D]^C$ for all $D \in \tau_{\Re_{BNN}}(Q)$. These elements are known as bipolar neutrosophic nano closed sets (BNN_Q – closed).

(ii) The collection $\tau_{\Re_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}\}$ is named as the indiscrete bipolar neutrosophic nano topology on $\tilde{U}$.

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(iii) The collection $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\underline{BN}(Q),\overline{BN}(Q),B_{BN}(Q)\}$ is named as the discrete bipolar neutrosophic nano topology on $\tilde{U}$.

Example 3.9. Let $\tilde{U}=\{q_1,q_2\}$ be a nonempty set and $\tilde{U}/\Re=\{\{q_1\},\{q_2\}\}$. Let $Q=0_{BNN}$ or $Q=1_{BNN}$. Then $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN}\}$ is the BNN_Q – topology on $\tilde{U}$.

Remark 3.10. Let $\tilde{U}$ be the nonempty set, $\Re$ be an equivalence relation on $\tilde{U}$ and Q be a BNS.

(i) If $\underline{BN}(Q)=\overline{BN}(Q)$, then $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\underline{BN}(Q),B_{BN}(Q)\}$ is a BNNT on $\tilde{U}$.

(ii) If $\underline{BN}(Q)=B_{BN}(Q)=\overline{BN}(Q)$, then $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\overline{BN}(Q)\}$ is a BNNT on $\tilde{U}$.

(iii) For $Q\neq 0_{BNN}$, $B_{BN}(Q)=\overline{BN}(Q)-\underline{BN}(Q)\neq 0_{BNN}$.

Example 3.11. In example 3.7, $\underline{BN}(Q)=\overline{BN}(Q)$ and the topology is $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\underline{BN}(Q),\ B_{BN}(Q)\}$.

Example 3.12. Let $\tilde{U}=\{q_1,q_2\}$ and $\tilde{U}/\Re=\{\{q_1\},\{q_2\}\}$.

Let $Q=\{\langle q_1,(.3,.3,.6,-.4,-.3,-.6)\rangle,\langle q_2,(.2,.2,.7,-.3,-.4,-.7)\rangle\}\subseteq\tilde{U}$. Then

$\overline{BN}(Q)=\{\langle q_1,(.3,.3,.6,-.4,-.3,-.6)\rangle,\langle q_2,(.2,.2,.7,-.3,-.4,-.7)\rangle\}$,

$\underline{BN}(Q)=\{\langle q_1,(.3,.3,.6,-.4,-.3,-.6)\rangle,\langle q_2,(.2,.2,.7,-.3,-.4,-.7)\rangle\}$,

$B_{BN}(Q)=\{\langle q_1,(.3,.3,.6,-.4,-.3,-.6)\rangle,\langle q_2,(.2,.2,.7,-.3,-.4,-.7)\rangle\}$.

Then the set $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\overline{BN}(Q)\}$ is the BNN_Q – topology on $\tilde{U}$.

Remark 3.13. For every bipolar neutrosophic set, the collection $\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\underline{BN}(Q),\overline{BN}(Q),B_{BN}(Q)\}$ need not be a topology.

Example 3.14. Let $\tilde{U}=\{q_1,q_2,q_3\}$ and $\tilde{U}/\Re=\{\{q_1,q_3\},\{q_2\}\}$.

Let $Q=\{\langle q_1,(.6,.5,.3,-.4,-.6,-.5)\rangle,\langle q_2,(.5,.7,.4,-.3,-.6,-.4)\rangle,\langle q_3,(.7,.8,.1,-.2,-.5,-.6)\rangle\}\subseteq\tilde{U}$.

$\overline{BN}(Q)=\{\langle q_1,(.7,.8,.1,-.4,-.6,-.5)\rangle,\langle q_2,(.5,.7,.4,-.3,-.6,-.4)\rangle,\langle q_3,(.7,.8,.1,-.4,-.6,-.5)\rangle\}$,

$\underline{BN}(Q)=\{\langle q_1,(.6,.5,.3,-.2,-.5,-.6)\rangle,\langle q_2,(.5,.7,.4,-.3,-.6,-.4)\rangle,\langle q_3,(.6,.5,.3,-.2,-.5,-.6)\rangle\}$,

$B_{BN}(Q)=\{\langle q_1,(.3,.2,.6,-.4,-.4,-.5)\rangle,\langle q_2,(.4,.3,.4,-.3,-.4,-.4)\rangle,\langle q_3,(.3,.2,.6,-.4,-.4,-.5)\rangle\}$.

$\tau_{\Re_{BNN}}(Q)=\{0_{BNN},1_{BNN},\underline{BN}(Q),\overline{BN}(Q),B_{BN}(Q)\}$ does not form a topology.

Remark 3.15. Also in the cases of bipolar intuitionistic fuzzy set and bipolar fuzzy set, $\tau_{\Re_{BIN}}(Q)=\{0_{BIN},1_{BIN},\underline{BI}(Q),\overline{BI}(Q),B_{BI}(Q)\}$ and $\tau_{\Re_{BFN}}(Q)=\{0_{BFN},1_{BFN},\underline{BF}(Q),\overline{BF}(Q),B_{BF}(Q)\}$ need not be topologies.

Example 3.16. Let $\tilde{U}=\{q_1,q_2,q_3\}$ and $\tilde{U}/\Re=\{\{q_1,q_3\},\{q_2\}\}$.

Let $Q=\{\langle q_1,(.6,.3,-.4,-.5)\rangle,\langle q_2,(.5,.4,-.3,-.4)\rangle,\langle q_3,(.7,.1,-.2,-.6)\rangle\}\subseteq\tilde{U}$.

$\overline{BI}(Q)=\{\langle q_1,(.7,.1,-.4,-.5)\rangle,\langle q_2,(.5,.4,-.3,-.4)\rangle,\langle q_3,(.7,.1,-.4,-.5)\rangle\}$,

$\underline{BI}(Q)=\{\langle q_1,(.6,.3,-.2,-.6)\rangle,\langle q_2,(.5,.4,-.3,-.4)\rangle,\langle q_3,(.6,.3,-.2,-.6)\rangle\}$,

$B_{BI}(Q)=\{\langle q_1,(.3,.6,-.4,-.5)\rangle,\langle q_2,(.4,.4,-.3,-.4)\rangle,\langle q_3,(.3,.6,-.4,-.5)\rangle\}$.

$\tau_{\Re_{BIN}}(Q)=\{0_{BIN},1_{BIN},\underline{BI}(Q),\overline{BI}(Q),B_{BI}(Q)\}$ does not form a topology.

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Also let $Q=\{\langle q_1,(.6,-.4)\rangle,\langle q_2,(.5,-.3)\rangle,\langle q_3,(.7,-.2)\rangle\}$.
$\overline{BF}(Q)=\{\langle q_1,(.7,-.4)\rangle,\langle q_2,(.5,-.3)\rangle,\langle q_3,(.7,-.4)\rangle\}$, $\underline{BF}(Q)=\{\langle q_1,(.6,-.2)\rangle,\langle q_2,(.5,-.3)\rangle,\langle q_3,(.6,-.2)\rangle\}$,
$B_{BF}(Q)=\{\langle q_1,(.2,-.4)\rangle,\langle q_2,(.4,-.3)\rangle,\langle q_3,(.3,-.4)\rangle\}$. $\tau_{\Re_{BFN}}(Q)=\{0_{BFN},1_{BFN},\underline{BF}(Q),\overline{BF}(Q),B_{BF}(Q)\}$ does not form a topology.

For every bipolar neutrosophic set, we cannot find a corresponding bipolar neutrosophic nano topology in $\tilde{U}$. Here is an attempt to define a topology which corresponds to any bipolar neutrosophic set in $\tilde{U}$ with respect to its extremity and estimations.

4. Bipolar Neutrosophic Nano $-*$ Topology

Definition 4.1. Let $\tilde{U}$ be a nonempty set and $\Re*$ be a relation on $\tilde{U}$, which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a BNS in $\tilde{U}$ with the positive degree of true membership $\eta_Q{}^+$, indeterminacy $\psi_Q{}^+$ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership $\eta_Q{}^-$, indeterminacy $\psi_Q{}^-$ and the false membership function $\xi_Q{}^-$, where $\eta_Q^+,\psi_Q^+,\xi_Q^+:\tilde{U}\to[0,1]$, $\eta_Q^-,\psi_Q^-,\xi_Q^-:\tilde{U}\to[-1,0]$. Then

(i) $\underline{BN}*(Q)=\{\langle q,(\eta^+_{\underline{\Re}*(Q)}(q),\psi^+_{\underline{\Re}*(Q)}(q),\xi^+_{\underline{\Re}*(Q)}(q),\eta^-_{\underline{\Re}*(Q)}(q),\psi^-_{\underline{\Re}*(Q)}(q),\xi^-_{\underline{\Re}*(Q)}(q))\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\}$ is the nether estimation of Q in respect of $\Re*$.

(ii) $\overline{BN}*(Q)=\{\langle q,(\eta^+_{\overline{\Re}*(Q)}(q),\psi^+_{\overline{\Re}*(Q)}(q),\xi^+_{\overline{\Re}*(Q)}(q),\eta^-_{\overline{\Re}*(Q)}(q),\psi^-_{\overline{\Re}*(Q)}(q),\xi^-_{\overline{\Re}*(Q)}(q))\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\}$ is the upper estimation of Q in respect of $\Re*$.

(iii) $B_{BN}*(Q)=\overline{BN}*(Q)-\underline{BN}*(Q)$ is the extremity of Q in respect of $\Re*$.

(iv) $BN_1*(Q)=\underline{BN}*(Q)\cup B_{BN}*(Q)$.

(v) $BN_2*(Q)=\underline{BN}*(Q)\cap B_{BN}*(Q)$.

Where,

$$\eta_{\underline{\Re}*(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^+(z),\ \psi_{\underline{\Re}*(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\Re*}}\psi_Q{}^+(z),\ \xi_{\underline{\Re}*(Q)}{}^+(q)=\bigvee_{z\in[q]_{\Re*}}\xi_Q{}^+(z),$$

$$\eta_{\underline{\Re}*(Q)}{}^-(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^-(z),\ \psi_{\underline{\Re}*(Q)}{}^-(q)=\bigvee_{z\in[q]_{\Re*}}\psi_Q{}^-(z),\ \xi_{\underline{\Re}*(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\Re*}}\xi_Q{}^-(z),$$

$$\eta_{\overline{\Re}*(Q)}{}^+(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^+(z),\ \psi_{\overline{\Re}*(Q)}{}^+(q)=\bigvee_{z\in[q]_{\Re*}}\psi_Q{}^+(z),\ \xi_{\overline{\Re}*(Q)}{}^+(q)=\bigwedge_{z\in[q]_{\Re*}}\xi_Q{}^+(z),$$

$$\eta_{\overline{\Re}*(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^-(z),\ \psi_{\overline{\Re}*(Q)}{}^-(q)=\bigwedge_{z\in[q]_{\Re*}}\psi_Q{}^-(z),\ \xi_{\overline{\Re}*(Q)}{}^-(q)=\bigvee_{z\in[q]_{\Re*}}\xi_Q{}^-(z).$$

Then the collection $\tau_{\Re_{BNN}}*(Q)=\{0_{BNN^*},1_{BNN^*},\underline{BN}*(Q),\overline{BN}*(Q),B_{BN}*(Q),BN_1*(Q),BN_2*(Q)\}$ is a topology which is named as a bipolar neutrosophic nano $-*$ topology (BNN_Q*-topology). The space $(\tilde{U},\tau_{\Re_{BNN}}*(Q))$ is a bipolar neutrosophic nano $-*$ topological space. The elements of $\tau_{\Re_{BNN}}*(Q)$ are bipolar neutrosophic nano $-*$ open sets (BNN_Q*- open). The complements of these elements are bipolar neutrosophic nano $-*$ closed sets (BNN_Q*- closed).

Remark 4.2. The null and absolute bipolar neutrosophic nano $-*$ open sets are given by $0_{BNN*}=\{\langle q,(0,0,1,0,0,-1)\rangle : q\in\tilde{U}\}$ and $1_{BNN*}=\{\langle q,(1,1,0,-1,-1,0)\rangle : q\in\tilde{U}\}$ respectively. Then

1. $0_{BNN*}\subset 1_{BNN*}$
2. $0_{BNN*}\bigcup 1_{BNN*}=1_{BNN*}$
3. $0_{BNN*}\bigcap 1_{BNN*}=0_{BNN*}$

Example 4.3. Let $\tilde{U}=\{q_1,q_2,q_3,q_4\}$ and $\tilde{U}/\Re=\{\{q_1,q_3\},\{q_2,q_4\}\}$.

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Let $Q=\left\{\begin{matrix}\langle q_1,(.8,.5,.2,-.6,-.5,-.3)\rangle,\langle q_2,(.7,.4,.3,-.4,-.6,-.6)\rangle,\\ \langle q_3,(.4,.6,.6,-.6,-.3,-.3)\rangle,\langle q_4,(.6,.5,.3,-.3,-.4,-.6)\rangle\end{matrix}\right\}\subseteq\tilde{U}$. Then

$$\overline{BN}*(Q)=\left\{\begin{matrix}\langle q_1,(.8,.6,.2,-.6,-.5,-.3)\rangle,\langle q_2,(.7,.5,.3,-.4,-.6,-.6)\rangle,\\ \langle q_3,(.8,.6,.2,-.6,-.5,-.3)\rangle,\langle q_4,(.7,.5,.3,-.4,-.6,-.6)\rangle\end{matrix}\right\},$$

$$\underline{BN}*(Q)=\left\{\begin{matrix}\langle q_1,(.4,.5,.6,-.6,-.3,-.3)\rangle,\langle q_2,(.6,.4,.3,-.3,-.4,-.6)\rangle,\\ \langle q_3,(.4,.5,.6,-.6,-.3,-.3)\rangle,\langle q_4,(.6,.4,.3,-.3,-.4,-.6)\rangle\end{matrix}\right\},$$

$$B_{BN}*(Q)=\left\{\begin{matrix}\langle q_1,(.6,.5,.4,-.3,-.5,-.6)\rangle,\langle q_2,(.3,.5,.6,-.4,-.6,-.6)\rangle,\\ \langle q_3,(.6,.5,.4,-.3,-.5,-.6)\rangle,\langle q_4,(.3,.5,.6,-.4,-.6,-.6)\rangle\end{matrix}\right\},$$

$$BN_1*(Q)=\left\{\begin{matrix}\langle q_1,(.6,.5,.4,-.6,-.5,-.3)\rangle,\langle q_2,(.6,.5,.3,-.4,-.6,-.6)\rangle,\\ \langle q_3,(.6,.5,.4,-.6,-.5,-.3)\rangle,\langle q_4,(.6,.5,.3,-.4,-.6,-.6)\rangle\end{matrix}\right\},$$

$$BN_2*(Q)=\left\{\begin{matrix}\langle q_1,(.4,.5,.6,-.3,-.3,-.6)\rangle,\langle q_2,(.3,.4,.6,-.3,-.4,-.6)\rangle,\\ \langle q_3,(.4,.5,.6,-.3,-.3,-.6)\rangle,\langle q_4,(.3,.4,.6,-.3,-.4,-.6)\rangle\end{matrix}\right\}.$$

The set $\tau_{\Re_{BNN}}*(Q)=\{0_{BNN*},1_{BNN*},\overline{BN}*(Q),\underline{BN}*(Q),B_{BN}*(Q),BN_1*(Q),BN_2*(Q)\}$ is the BNN_Q*-topology. The elements are BNN_Q*- open sets of Q in $\tilde{U}$.

BNN_Q*-closed sets of Q in $\tilde{U}$ are as follows:

$$(\overline{BN}*(Q))^C=\left\{\begin{matrix}\langle q_1,(.2,.4,.8,-.3,-.5,-.6)\rangle,\langle q_2,(.3,.5,.7,-.6,-.4,-.4)\rangle,\\ \langle q_3,(.2,.4,.8,-.3,-.5,-.6)\rangle,\langle q_4,(.3,.5,.7,-.6,-.4,-.4)\rangle\end{matrix}\right\},$$

$$(\underline{BN}*(Q))^C=\left\{\begin{matrix}\langle q_1,(.6,.5,.4,-.3,-.7,-.6)\rangle,\langle q_2,(.3,.6,.6,-.6,-.6,-.3)\rangle,\\ \langle q_3,(.6,.5,.4,-.3,-.7,-.6)\rangle,\langle q_4,(.3,.6,.6,-.6,-.6,-.3)\rangle\end{matrix}\right\},$$

$$(B_{BN}*(Q))^C=\left\{\begin{matrix}\langle q_1,(.4,.5,.6,-.6,-.5,-.3)\rangle,\langle q_2,(.6,.5,.3,-.6,-.4,-.4)\rangle,\\ \langle q_3,(.4,.5,.6,-.6,-.5,-.3)\rangle,\langle q_4,(.6,.5,.3,-.6,-.4,-.4)\rangle\end{matrix}\right\},$$

$$(BN_1*(Q))^C=\left\{\begin{matrix}\langle q_1,(.4,.5,.6,-.3,-.5,-.6)\rangle,\langle q_2,(.3,.5,.6,-.6,-.4,-.4)\rangle,\\ \langle q_3,(.4,.5,.6,-.3,-.5,-.6)\rangle,\langle q_4,(.3,.5,.6,-.6,-.4,-.4)\rangle\end{matrix}\right\},$$

$$(BN_2*(Q))^C=\left\{\begin{matrix}\langle q_1,(.6,.5,.4,-.6,-.7,-.3)\rangle,\langle q_2,(.6,.6,.3,-.6,-.6,-.3)\rangle,\\ \langle q_3,(.6,.5,.4,-.6,-.7,-.3)\rangle,\langle q_4,(.6,.6,.3,-.6,-.6,-.3)\rangle\end{matrix}\right\}.$$

Also $(\tau_{\Re_{BNN}}*(Q))^C=\left\{0_{BNN*},1_{BNN*},\left(\overline{BN}^*(Q)\right)^C,(\underline{BN}^*(Q))^C,(B_{BN}{}^*(Q))^C,(BN_1{}^*(Q))^C,(BN_2{}^*(Q))^C\right\}$.

Theorem 4.4. Every BNN_Q-open set is BNN_Q*-open.

The proof follows from the definition of BNN_Q*-open sets.

In the similar manner, we can deduce the following definitions.

Definition 4.5. Let $\tilde{U}$ be a nonempty set and $\Re*$ be a relation on $\tilde{U}$ which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a BIS in $\tilde{U}$ with the positive degree of true membership η_Q^+ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership η_Q^- and the false membership function $\xi_Q{}^-$, where $\eta_Q^+,\xi_Q^+:\tilde{U}\to[0,1]$, $\eta_Q^-,\xi_Q^-:\tilde{U}\to[-1,0]$. Then

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(i) $\underline{BI}*(Q)=\left\{\left\langle q,\left(\eta^{+}_{\underline{\Re}*(Q)}(q),\xi^{+}_{\underline{\Re}*(Q)}(q),\eta^{-}_{\underline{\Re}*(Q)}(q),\xi^{-}_{\underline{\Re}*(Q)}(q)\right)\right\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\right\}$ is the nether estimation of Q in respect of $\Re*$.

(ii) $\overline{BI}*(Q)=\left\{\left\langle q,\left(\eta^{+}_{\overline{\Re}*(Q)}(q),\xi^{+}_{\overline{\Re}*(Q)}(q),\eta^{-}_{\overline{\Re}*(Q)}(q),\xi^{-}_{\overline{\Re}*(Q)}(q)\right)\right\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\right\}$ is the higher estimation of Q in respect of $\Re*$.

(iii) $B_{BI}*(Q)=\overline{BI}*(Q)-\underline{BI}*(Q)$ is the extremity of Q in respect of $\Re*$.

(vi) $BI_1*(Q)=\underline{BI}*(Q)\cup B_{BI}*(Q)$.

(v) $BI_2*(Q)=\underline{BI}*(Q)\cap B_{BI}*(Q)$.

Where,

$\eta_{\underline{\Re}*(Q)}{}^{+}(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^{+}(z)$, $\xi_{\underline{\Re}*(Q)}{}^{+}(q)=\bigvee_{z\in[q]_{\Re*}}\xi_Q{}^{+}(z)$, $\eta_{\underline{\Re}*(Q)}{}^{-}(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^{-}(z)$, $\xi_{\underline{\Re}*(Q)}{}^{-}(q)=\bigwedge_{z\in[q]_{\Re*}}\xi_Q{}^{-}(z)$,

$\eta_{\overline{\Re}*(Q)}{}^{+}(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^{+}(z)$, $\xi_{\overline{\Re}*(Q)}{}^{+}(q)=\bigwedge_{z\in[q]_{\Re*}}\xi_Q{}^{+}(z)$, $\eta_{\overline{\Re}*(Q)}{}^{-}(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^{-}(z)$, $\xi_{\overline{\Re}*(Q)}{}^{-}(q)=\bigvee_{z\in[q]_{\Re*}}\xi_Q{}^{-}(z)$.

The collection $\tau_{\Re_{BIN}}*(Q)=\{0_{BIN*},1_{BIN*},\underline{BI}*(Q),\overline{BI}*(Q),B_{BI}*(Q),BI_1*(Q),BI_2*(Q)\}$, where $0_{BIN*}=\{\langle q,(0,1,0,-1)\rangle : q\in\tilde{U}\}$ and $1_{BIN*}=\{\langle q,(1,0,-1,0)\rangle : q\in\tilde{U}\}$ is a topology named as bipolar intuitionistic nano $-*$ topology and $(\tilde{U},\tau_{\Re_{BIN}}*(Q))$ is the bipolar intuitionistic nano $-*$ topological space. The elements of $\tau_{\Re_{BIN}}*(Q)$ are bipolar intuitionistic nano $-*$ open sets.

Definition 4.6. Let $\tilde{U}$ be a nonempty universe and $\Re*$ be a relation on $\tilde{U}$ which is imperceptible. Then $\tilde{U}$ is split into disjoint equivalence classes. Let Q be a BFS in $\tilde{U}$ with the positive and negative degrees of true membership functions η_Q^{+} and η_Q^{-}, where $\eta_Q^{+}:\tilde{U}\to[0,1]$, $\eta_Q^{-}:\tilde{U}\to[-1,0]$. Then

(i) $\underline{BF}*(Q)=\left\{\left\langle q,\left(\eta^{+}_{\underline{\Re}*(Q)}(q),\eta^{-}_{\underline{\Re}*(Q)}(q)\right)\right\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\right\}$ is the nether estimation of Q in respect of $\Re*$.

(ii) $\overline{BF}*(Q)=\left\{\left\langle q,\left(\eta^{+}_{\overline{\Re}*(Q)}(q),\eta^{-}_{\overline{\Re}*(Q)}(q)\right)\right\rangle : z\in[q]_{\Re*}, q\in\tilde{U}\right\}$ is the higher estimation of Q in respect of $\Re*$.

(iii) $B_{BF}*(Q)=\overline{BF}*(Q)-\underline{BF}*(Q)$ is the extremity of Q in respect of $\Re*$.

(vi) $BF_1*(Q)=\underline{BF}*(Q)\cup B_{BF}*(Q)$.

(v) $BF_2*(Q)=\underline{BF}*(Q)\cap B_{BF}*(Q)$.

Where, $\eta_{\underline{\Re}*(Q)}{}^{+}(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^{+}(z)$, $\eta_{\underline{\Re}*(Q)}{}^{-}(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^{-}(z)$,

$\eta_{\overline{\Re}*(Q)}{}^{+}(q)=\bigvee_{z\in[q]_{\Re*}}\eta_Q{}^{+}(z)$, $\eta_{\overline{\Re}*(Q)}{}^{-}(q)=\bigwedge_{z\in[q]_{\Re*}}\eta_Q{}^{-}(z)$.

The collection $\tau_{\Re_{BFN}}*(Q)=\{0_{BFN*},1_{BFN*},\underline{BF}*(Q),\overline{BF}*(Q),B_{BF}*(Q),BF_1*(Q),BF_2*(Q)\}$ where, $0_{BFN*}=\{\langle q,(0,0)\rangle : q\in\tilde{U}\}$ and $1_{BFN*}=\{\langle q,(1,-1)\rangle : q\in\tilde{U}\}$ is a topology named as bipolar fuzzy nano $-*$ topology and $(\tilde{U},\tau_{\Re_{BFN}}*(Q))$ is the bipolar fuzzy nano $-*$ topological space. The elements of $\tau_{\Re_{BFN}}*(Q)$ are bipolar fuzzy nano $-*$ open sets.

4.1 Bipolar Neutrosophic Nano * Interior and Closure

Definition 4.7. Let $(\tilde{U},\tau_{\Re_{BNN}}*(Q))$ be a $BNN_Q{}^{*}-$topological space in respect of a BNS $Q\subseteq\tilde{U}$ and let B be any BNS of $\tilde{U}$. Then

(i) the bipolar neutrosophic nano $-*$ interior of B is defined as the union of all bipolar neutrosophic nano $-*$ open sets in B and it is denoted by $BNN_Q*\mathrm{int}(B)$. It is the biggest bipolar neutrosophic nano $-*$ open subset of B.

(ii) the bipolar neutrosophic nano $-*$ closure of B is defined as the intersection of all bipolar neutrosophic nano $-*$ closed sets containing B and it is denoted by $BNN_Q*cl(B)$. It is the least bipolar neutrosophic nano $-*$ closed set having B.

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Example 4.8. Let $B=\left\{\begin{array}{l}\langle q_1,(.6,.6,.3,-.7,-.5,-.2)\rangle,\langle q_2,(.8,.5,.1,-.3,-.5,-.6)\rangle,\\ \langle q_3,(.5,.7,.3,-.7,-.6,-.2)\rangle,\langle q_4,(.6,.5,.3,-.5,-.5,-.4)\rangle\end{array}\right\}$ be a BNS in $\widetilde{U}$, in example 4.3, $BNN_Q*\text{int}(B)=\cup\{0_{BNN*},\underline{BN}*(Q),BN_2*(Q)\}=\underline{BN}*(Q)$ and $BNN_Q*cl(B)=\cap\{1_{BNN*}\}=1_{BNN*}$.

Theorem 4.9. $(\widetilde{U},\tau_{\Re_{BNN}}*(Q))$ is a $BNN_Q{}^*-$ topological space, where $Q\subseteq\widetilde{U}$ is a BNS. For any BNS P in $\widetilde{U}$,

(i) $BNN_Q*cl(1_{BNN*}-P)=BNN_Q*cl(P^C)=[BNN_Q*\text{int}(P)]^C=1_{BNN*}-BNN_Q*\text{int}(P)$.

(ii) $BNN_Q*\text{int}(1_{BNN*}-P)=BNN_Q*\text{int}(P^C)=[BNN_Q*cl(P)]^C=1_{BNN*}-BNN_Q*cl(P)$.

Let $P=\left\{\langle q,(\eta_P^+(q),\psi_P^+(q),\xi_P^+(q),\eta_P^-(q),\psi_P^-(q),\xi_P^-(q))\rangle\right\}$.

Suppose that the family of $BNN_Q{}^*-$ open sets G_i contained in Q is indexed by the family

$\left\{\langle q,(\eta_{G_i}^+(q),\psi_{G_i}^+(q),\xi_{G_i}^+(q),\eta_{G_i}^-(q),\psi_{G_i}^-(q),\xi_{G_i}^-(q))\rangle : i\in J\right\}$.

$BNN_Q*\text{int}(P)=\langle q,(\vee\,\eta_{G_i}^+(q),\vee\psi_{G_i}^+(q),\wedge\xi_{G_i}^+(q),\wedge\eta_{G_i}^-(q),\wedge\psi_{G_i}^-(q),\vee\xi_{G_i}^-(q))\rangle$.

$(BNN_A*\text{int}(P))^C=\langle q,(\wedge\,\xi_{G_i}^+(q),1-\vee\psi_{G_i}^+(q),\vee\eta_{G_i}^+(q),\vee\xi_{G_i}^-(q),-1-\wedge\psi_{G_i}^-(q),\wedge\eta_{G_i}^-(q))\rangle$ …….. (1)

We obtain that $\left\{\langle q,(\xi_{G_i}^+(q),1-\psi_{G_i}^+(q),\eta_{G_i}^+(q),\xi_{G_i}^-(q),-1-\psi_{G_i}^-(q),\eta_{G_i}^-(q))\rangle : i\in J\right\}$ is the family of $BNN_Q{}^*-$closed sets containing P^C.

$BNN_Q*cl(P^C)=\langle q,(\wedge\,\xi_{G_i}^+(q),\wedge(1-\psi_{G_i}^+(q)),\vee\eta_{G_i}^+(q),\vee\xi_{G_i}^-(q),\vee(-1-\psi_{G_i}^-(q)),\wedge\eta_{G_i}^-(q))\rangle$.

$BNN_Q*cl(P^C)=\langle q,(\wedge\,\xi_{G_i}^+(q),1-\vee\psi_{G_i}^+(q),\vee\eta_{G_i}^+(q),\vee\xi_{G_i}^-(q),-1-\wedge\psi_{G_i}^-(q),\wedge\eta_{G_i}^-(q))\rangle$ ……… (2)

From (1) and (2) and the result that $1_{BNN*}-P=P^C$, (i) follows.

Similarly we can prove (ii).

Remark 4.10. Taking complements of equations of (i) and (ii) of Theorem 4.9, we have

(i) $1_{BNN*}-BNN_Q*cl(1_{BNN*}-P)=(BNN_Q*cl(P^C))^C=BNN_Q*\text{int}(P)$.

(ii) $1_{BNN*}-BNN_Q*\text{int}(1_{BNN*}-P)=(BNN_Q*\text{int}(P^C))^C=BNN_Q*cl(P)$.

Example 4.11. Let $\widetilde{U}=\{q_1,q_2,q_3,q_4\}$, $\widetilde{U}/\Re=\{\{q_1,q_2\},\{q_3\},\{q_4\}\}$ and let us consider a BNS Q of $\widetilde{U}$, where

$Q=\{\langle q_1,(.2,.5,.6,-.7,-.5,-.2)\rangle,\langle q_2,(.3,.4,.6,-.6,-.4,-.3)\rangle,\langle q_3,(.4,.5,.5,-.5,-.5,-.4)\rangle\}$.

We have $BNN_Q{}^*-$open sets are as follows:

$\overline{BN}^*(Q)=\{\langle q_1,(.3,.5,.6,-.7,-.5,-.2)\rangle,\langle q_2,(.3,.5,.6,-.7,-.5,-.2)\rangle,\langle q_3,(.4,.5,.5,-.5,-.5,-.4)\rangle\}$,

$\underline{BN}*(Q)=\{\langle q_1,(.2,.4,.6,-.6,-.4,-.3)\rangle,\langle q_2,(.2,.4,.6,-.6,-.4,-.3)\rangle,\langle q_3,(.4,.5,.5,-.5,-.5,-.4)\rangle\}$,

$B_{BN}*(Q)=\{\langle q_1,(.3,.5,.6,-.3,-.5,-.6)\rangle,\langle q_2,(.3,.5,.6,-.3,-.5,-.6)\rangle,\langle q_3,(.4,.5,.5,-.4,-.5,-.5)\rangle\}$,

$BN_1*(Q)=\{\langle q_1,(.3,.5,.6,-.6,-.5,-.3)\rangle,\langle q_2,(.3,.5,.6,-.6,-.5,-.3)\rangle,\langle q_3,(.4,.5,.5,-.5,-.5,-.4)\rangle\}$,

$BN_2*(Q)=\{\langle q_1,(.2,.4,.6,-.3,-.4,-.6)\rangle,\langle q_2,(.2,.4,.6,-.3,-.4,-.6)\rangle,\langle q_3,(.4,.5,.5,-.4,-.5,-.5)\rangle\}$.

$BNN_Q{}^*-$closed sets are as follows:

$\left(\overline{BN}^*(Q)\right)^C=\{\langle q_1,(.6,.5,.3,-.2,-.5,-.7)\rangle,\langle q_2,(.6,.5,.3,-.2,-.5,-.7)\rangle,\langle q_3,(.5,.5,.4,-.4,-.5,-.5)\rangle\}$,

$(\underline{BN}*(Q))^C=\{\langle q_1,(.6,.6,.2,-.3,-.6,-.6)\rangle,\langle q_2,(.6,.6,.2,-.3,-.6,-.6)\rangle,\langle q_3,(.5,.5,.4,-.4,-.5,-.5)\rangle\}$,

$(B_{BN}*(Q))^C=\{\langle q_1,(.6,.5,.3,-.6,-.5,-.3)\rangle,\langle q_2,(.6,.5,.3,-.6,-.5,-.3)\rangle,\langle q_3,(.5,.5,.4,-.5,-.5,-.4)\rangle\}$,

$(BN_1*(Q))^C=\{\langle q_1,(.6,.5,.3,-.3,-.5,-.6)\rangle,\langle q_2,(.6,.5,.3,-.3,-.5,-.6)\rangle,\langle q_3,(.5,.5,.4,-.4,-.5,-.5)\rangle\}$,

$(BN_2*(Q))^C=\{\langle q_1,(.6,.6,.2,-.6,-.6,-.3)\rangle,\langle q_2,(.6,.6,.2,-.6,-.6,-.3)\rangle,\langle q_3,(.5,.5,.4,-.5,-.5,-.4)\rangle\}$.

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0_{BNN*} and 1_{BNN*} are both $BNN_Q{}^*-$ open and $BNN_Q{}^*-$closed.

Let $K = \{\langle q_1,(.7,.6,.3,-.3,-.2,-.7)\rangle, \langle q_2,(.7,.7,.2,-.3,-.3,-.6)\rangle, \langle q_3,(.6,.6,.3,-.4,-.4,-.6)\rangle\}$.

$1_{BNN*} - K = \{\langle q_1,(.3,.4,.7,-.7,-.8,-.3)\rangle, \langle q_2,(.2,.3,.7,-.6,-.7,-.3)\rangle, \langle q_3,(.3,.4,.6,-.6,-.6,-.4)\rangle\}$.

Here $BNN_Q{}^*\operatorname{int}(K) = 0_{BNN*}$ and $BNN_Q{}^*cl(1_{BNN*} - K) = 1_{BNN*} = 1_{BNN*} - BNN_Q{}^*\operatorname{int}(K)$,

$BNN_Q{}^*cl(K) = 1_{BNN*}$ and $BNN_Q{}^*\operatorname{int}(1_{BNN*} - K) = 0_{BNN*} = 1_{BNN*} - BNN_Q{}^*cl(K)$.

Theorem 4.12. For $(\tilde{U}, \tau_{R_{BNN*}}(Q))$, $Q \subseteq \tilde{U}$ is a BNS, let K and H be bipolar neutrosophic subsets of $\tilde{U}$. Then

(i) $K \subseteq BNN_Q{}^*cl(K)$.

(ii) K is $BNN_Q{}^*-$ closed $\Leftrightarrow$ $BNN_Q{}^*cl(K) = K$.

(iii) $BNN_Q{}^*cl(0_{BNN^*}) = 0_{BNN^*}$ and $BNN_Q{}^*cl(1_{BNN^*}) = 1_{BNN^*}$.

(iv) $K \subseteq H \Rightarrow BNN_Q{}^*cl(K) \subseteq BNN_Q{}^*cl(H)$.

(v) $BNN_Q{}^*cl(K \cup H) = BNN_Q{}^*cl(K) \cup BNN_Q{}^*cl(H)$.

(vi) $BNN_Q{}^*cl(K \cap H) \subseteq BNN_Q{}^*cl(K) \cap BNN_Q{}^*cl(H)$.

(vii) $BNN_Q{}^*cl(BNN_Q{}^*cl(K)) = BNN_Q{}^*cl(K)$.

Proof:

(i) By the definition of $BNN_Q{}^*-$ closure, $K \subseteq BNN_Q{}^*cl(K)$.

(ii) K is $BNN_Q{}^*-$closed $\Leftrightarrow$ the least $BNN_Q{}^*-$closed set having itself is K.

$\Leftrightarrow$ $BNN_Q{}^*cl(K) = K$.

(iii) Since 0_{BNN^*} and 1_{BNN^*} are $BNN_Q{}^*-$closed, then by (ii), $BNN_Q{}^*cl(0_{BNN^*}) = 0_{BNN^*}$ and $BNN_Q{}^*cl(1_{BNN^*}) = 1_{BNN^*}$.

(iv) Let $K \subseteq H$. Since $H \subseteq BNN_Q{}^*cl(H)$, then $K \subseteq BNN_Q{}^*cl(H)$. That is, $BNN_Q{}^*cl(H)$ is a $BNN_Q{}^*-$closed set containing K. But $BNN_Q{}^*cl(K)$ is the smallest $BNN_Q{}^*-$ closed set containing K. Hence $BNN_Q{}^*cl(K) \subseteq BNN_Q{}^*cl(H)$.

(v) $K \subseteq K \cup H$ and $H \subseteq K \cup H$ $\Leftrightarrow$ $BNN_Q{}^*cl(K) \subseteq BNN_Q{}^*cl(K \cup H)$ and $BNN_Q{}^*cl(H) \subseteq BNN_Q{}^*cl(K \cup H)$.

$\Leftrightarrow$ $BNN_Q{}^*cl(K) \cup BNN_Q{}^*cl(H) \subseteq BNN_Q{}^*cl(K \cup H)$.

Now, since $K \cup H \subseteq BNN_Q{}^*cl(K) \cup BNN_Q{}^*cl(H)$, and since $BNN_Q{}^*cl(K \cup H)$ is the smallest $BNN_Q{}^*-$closed set containing $K \cup H$, $BNN_Q{}^*cl(K \cup H) \subseteq BNN_Q{}^*cl(K) \cup BNN_Q{}^*cl(H)$.

Thus $BNN_Q{}^*cl(K \cup H) = BNN_Q{}^*cl(K) \cup BNN_Q{}^*cl(H)$.

(vi) Since $K \cap H \subseteq K$ and $K \cap H \subseteq H$,

$\Leftrightarrow$ $BNN_Q{}^*cl(K \cap H) \subseteq BNN_Q{}^*cl(K)$ and $BNN_Q{}^*cl(K \cap H) \subseteq BNN_Q{}^*cl(H)$.

$\Leftrightarrow$ $BNN_Q{}^*cl(K \cap H) \subseteq BNN_Q{}^*cl(K) \cap BNN_Q{}^*cl(H)$.

(vii) Since $BNN_Q{}^*cl(K)$ is $BNN_Q{}^*-$closed, $BNN_Q{}^*cl(BNN_Q{}^*cl(K)) = BNN_Q{}^*cl(K)$.

Example 4.13. Consider example 4.11.

(i) $BNN_Q{}^*cl(K) = 1_{BNN*}$, then $P \subseteq BNN_Q{}^*cl(K)$.

(ii) For elements belongs to $(\tau_{\mathfrak{R}_{BNN}}*(Q))^C$, $BNN_K{}^*cl(K) = K$.

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(iii) Let $K = \{\langle q_1,(.7,.6,.3,-.3,-.2,-.7)\rangle,\langle q_2,(.7,.7,.2,-.3,-.3,-.6)\rangle,\langle q_3,(.6,.6,.3,-.4,-.4,-.6)\rangle\}$,
$H = \{\langle q_1,(.8,.7,.1,-.5,-.4,-.4)\rangle,\langle q_2,(.9,.8,.1,-.4,-.4,-.5)\rangle,\langle q_3,(.7,.7,.2,-.5,-.5,-.4)\rangle\}$.
Here $K \subseteq H$ and $BNN_Q{}^*cl(K) = 1_{BNN*} = BNN_Q{}^*cl(H)$.
Let $K_1 = \{\langle q_1,(.2,.2,.7,-.3,-.5,-.7)\rangle,\langle q_2,(.3,.3,.6,-.2,-.4,-.8)\rangle,\langle q_3,(.2,.3,.6,-.3,-.4,-.6)\rangle\}$,
$H_1 = \{\langle q_1,(.3,.4,.6,-.4,-.5,-.6)\rangle,\langle q_2,(.4,.5,.5,-.3,-.5,-.6)\rangle,\langle q_3,(.4,.4,.4,-.5,-.5,-.5)\rangle\}$.
$BNN_Q * cl(K_1) = \cap\{(\underline{BN} *(Q))^C,(B_{BN} *(Q))^C,(BN_1 *(Q))^C,(BN_2 *(Q))^C,1_{BNN*}\} = (BN_1 *(Q))^C$.
$BNN_Q * cl(Q_1) = \cap\{(B_{BN} *(Q))^C,(BN_1 *(Q))^C,1_{BNN*}\} = (B_{BN} *(Q))^C$.
$K_1 \subseteq H_1 \Rightarrow BNN_Q * cl(K_1) = (BN_1 *(Q))^C \subseteq (B_{BN} *(Q))^C = BNN_Q * cl(H_1)$.

(iv) $K_1 \cup H_1 = \{\langle q_1,(.3,.4,.6,-.4,-.5,-.6)\rangle,\langle q_2,(.4,.5,.5,-.3,-.5,-.6)\rangle,\langle q_3,(.4,.4,.4,-.5,-.5,-.5)\rangle\}$.
$BNN_Q{}^*cl(K_1 \cup H_1) = (B_{BN} *(Q))^C = BNN_Q{}^*cl(K_1) \cup BNN_Q{}^*cl(H_1)$.

(v) $K_2 = \{\langle q_1,(.3,.5,.5,-.7,-.5,-.2)\rangle,\langle q_2,(.3,.4,.6,-.7,-.4,-.2)\rangle,\langle q_3,(.5,.5,.4,-.6,-.6,-.3)\rangle\}$,
$H_2 = \{\langle q_1,(.8,.7,.1,-.5,-.4,-.4)\rangle,\langle q_2,(.9,.8,.1,-.4,-.4,-.5)\rangle,\langle q_3,(.7,.7,.2,-.5,-.5,-.4)\rangle\}$.
$K_2 \cap H_2 = \{\langle q_1,(.3,.5,.5,-.5,-.4,-.4)\rangle,\langle q_2,(.3,.4,.6,-.4,-.4,-.5)\rangle,\langle q_3,(.5,.5,.4,-.5,-.5,-.4)\rangle\}$.
$BNN_Q * cl(K_2) = 1_{BNN*} = BNN_Q * cl(H_2)$.
$BNN_Q * cl(K_2 \cap H_2) = \cap\{(B_{BN} *(Q))^C,(BN_2 *(Q))^C,1_{BNN*}\} = (B_{BN} *(Q))^C$.
$BNN_Q * cl(K_2 \cap H_2) \subseteq BNN_Q * cl(K_2) \cap BNN_Q * cl(H_2)$.

Theorem 4.14. For $(\tilde{U},\tau_{\Re_{BNN}} *(Q))$, $Q \subseteq \tilde{U}$ is a BNS, let K and H be bipolar neutrosophic subsets of $\tilde{U}$. Then

(i) $BNN_Q * \text{int}(K) \subseteq K$.

(ii) K is $BNN_Q{}^*-$ open $\Leftrightarrow$ $BNN_Q * \text{int}(K) = K$.

(iii) $BNN_Q * \text{int}(0_{BNN*}) = 0_{BNN*}$ and $BNN_Q * \text{int}(1_{BNN*}) = 1_{BNN*}$.

(iv) $K \subseteq H \Rightarrow BNN_Q * \text{int}(K) \subseteq BNN_Q * \text{int}(H)$.

(v) $BNN_Q * \text{int}(K) \cup BNN_Q * \text{int}(H) \subseteq BNN_Q * \text{int}(K \cup H)$.

(vi) $BNN_Q * \text{int}(K \cap H) = BNN_Q * \text{int}(K) \cap BNN_Q * \text{int}(H)$.

(vii) $BNN_Q * \text{int}(BNN_Q * \text{int}(K)) = BNN_Q * \text{int}(K)$.

Proof:

(i) By the definition of $BNN_Q{}^*-$open, $BNN_Q * \text{int}(K) \subseteq K$.

(ii) K is $BNN_Q{}^*-$open $\Leftrightarrow$ K is the biggest $BNN_Q{}^*-$open set $\subseteq$ K.
$\Leftrightarrow$ $BNN_Q{}^*\text{int}(K) = K$.

(iii) Since 0_{BNN^*} and 1_{BNN^*} are $BNN_Q{}^*-$open, then by (ii), $BNN_Q{}^*\text{int}(0_{BNN^*}) = 0_{BNN^*}$ and $BNN_Q{}^*\text{int}(1_{BNN^*}) = 1_{BNN^*}$.

(iv) Let $K \subseteq H$. Since $BNN_Q{}^*\text{int}(K) \subseteq K$, then $BNN_Q{}^*\text{int}(K) \subseteq H$. That is, $BNN_Q{}^*\text{int}(K)$ is a $BNN_Q{}^*-$open set contained in H. But $BNN_Q{}^*\text{int}(H)$ is the largest $BNN_Q{}^*-$ open set contained in H. Hence $BNN_Q{}^*\text{int}(K) \subseteq BNN_Q{}^*\text{int}(H)$.

(v) $K \subseteq K \cup H$ and $H \subseteq K \cup H$ $\Leftrightarrow$ $BNN_Q{}^*\text{int}(K) \subseteq BNN_Q{}^*\text{int}(K \cup H)$ and $BNN_Q{}^*\text{int}(H) \subseteq BNN_Q{}^*\text{int}(K \cup H)$

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$\Leftrightarrow BN_Q \operatorname{int}(K) \cup BN_Q \operatorname{int}(H) \subseteq BN_Q \operatorname{int}(K \cup H)$.

(vi) $K \cap H \subseteq K$ and $K \cap H \subseteq H \Leftrightarrow BNN_Q^* \operatorname{int}(K \cap H) \subseteq BNN_Q^* \operatorname{int}(K)$ and $BNN_Q^* \operatorname{int}(K \cap H) \subseteq BNN_Q^* \operatorname{int}(H)$

$\Leftrightarrow BNN_Q^* \operatorname{int}(K \cap H) \subseteq BNN_Q^* \operatorname{int}(K) \cap BNN_Q^* \operatorname{int}(H)$.

Now, since $BNN_Q^* \operatorname{int}(K) \cap BNN_Q^* \operatorname{int}(H) \subseteq K \cap H$, and since $BNN_Q^* \operatorname{int}(K \cap H)$ is the largest $BNN_Q^* -$ open set contained in $K \cap H$, $BNN_Q^* \operatorname{int}(K) \cap BNN_Q^* \operatorname{int}(H) \subseteq BNN_Q^* \operatorname{int}(K \cap H)$.

Thus $BNN_Q^* \operatorname{int}(K \cap H) = BNN_Q^* \operatorname{int}(K) \cap BNN_Q^* \operatorname{int}(H)$.

(vii) Since $BNN_Q^* \operatorname{int}(K)$ is $BNN_Q^* -$open, $BNN_Q^* \operatorname{int}(BNN_Q^* \operatorname{int}(K)) = BNN_Q^* \operatorname{int}(K)$.

Example 4.15. Consider example 4.11,

(i) Let $K = \{\langle q_1,(.7,.6,.3,-.3,-.2,-.7)\rangle, \langle q_2,(.7,.7,.2,-.3,-.3,-.6)\rangle, \langle q_3,(.6,.6,.3,-.4,-.4,-.6)\rangle\}$.

$H = \{\langle q_1,(.8,.7,.1,-.5,-.4,-.4)\rangle, \langle q_2,(.9,.8,.1,-.4,-.4,-.5)\rangle, \langle q_3,(.7,.7,.2,-.5,-.5,-.4)\rangle\}$.

Here $K \subseteq H$ and $BNN_Q * \operatorname{int}(K) = 0_{BNN*}$, $BNN_Q * \operatorname{int}(H) = BN_2 * (Q)$. Hence $BNN_Q^* \operatorname{int}(K) \subseteq BNN_Q^* \operatorname{int}(H)$.

(ii) Let $K_1 = \{\langle q_1,(.4,.5,.6,-.6,-.6,-.3)\rangle, \langle q_2,(.3,.6,.6,-.4,-.5,-.3)\rangle, \langle q_3,(.4,.4,.5,-.5,-.3,-.2)\rangle\}$.

$H_1 = \{\langle q_1,(.3,.5,.5,-.7,-.5,-.2)\rangle, \langle q_2,(.3,.4,.6,-.7,-.4,-.2)\rangle, \langle q_3,(.5,.5,.4,-.6,-.6,-.3)\rangle\}$.

$K_1 \cup H_1 = \{\langle q_1,(.4,.5,.5,-.7,-.6,-.2)\rangle, \langle q_2,(.3,.6,.6,-.7,-.5,-.2)\rangle, \langle q_3,(.5,.5,.4,-.6,-.6,-.2)\rangle\}$.

$BNN_Q * \operatorname{int}(K_1) = 0_{BNN*}$, $BNN_Q * \operatorname{int}(H_1) = \underline{BN} * (Q)$ and $BNN_Q * \operatorname{int}(K_1 \cup H_1) = \overline{BN} * (Q)$.

Hence $BNN_Q * \operatorname{int}(K_1) \cup BNN_Q * \operatorname{int}(H_1) \subseteq BNN_Q * \operatorname{int}(K_1 \cup H_1)$.

(iii) $K_1 \cap H_1 = \{\langle q_1,(.3,.5,.6,-.6,-.5,-.3)\rangle, \langle q_2,(.3,.4,.6,-.4,-.4,-.3)\rangle, \langle q_3,(.4,.4,.5,-.5,-.3,-.3)\rangle\}$.

$BNN_Q * \operatorname{int}(K_1) \cap BNN_Q * \operatorname{int}(H_1) = 0_{BNN*} \cap \underline{BN} * (Q) = 0_{BNN*} = BNN_Q * \operatorname{int}(K_1 \cap H_1)$.

5. Conclusion

This study discovers a new nano topology for every bipolar neutrosophic set. We have introduced two topologies called Bipolar Neutrosophic Nano Topology and Bipolar Neutrosophic Nano $-*$ Topology and deduced these topologies to fuzzy sets and intuitionistic fuzzy sets. We studied the structural properties of the topologies. This paper can be further developed into several possible theories and applications. The concept can be used for real life decision making problems, where the situations of indeterminacy occur. The practical problems may be solved by finding CORE values through the criterion reduction.

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