On Finite NeutroGroups of Type-NG[1,2,4]

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◇ In commemoration of the 60th birthday of the author

Abstract

The NeutroGroups as alternatives to the classical groups are of different types with different algebraic properties. In this paper, we are going to study a class of NeutroGroups of type-NG[1,2,4]. In this class of NeutroGroups, the closure law, the axiom of associativity and existence of inverse are taking to be either partially true or partially false for some elements; while the existence of identity element and axiom of commutativity are taking to be totally true for all the elements. Several examples of NeutroGroups of type-NG[1,2,4] are presented along with their basic properties. It is shown that Lagrange's theorem holds for some NeutroSubgroups of a NeutroGroup and failed to hold for some NeutroSubgroups of the same NeutroGroup. It is also shown that the union of two NeutroSubgroups of a NeutroGroup can be a NeutroSubgroup even if one is not contained in the other; and that the intersection of two NeutroSubgroups may not be a NeutroSubgroup. The concepts of NeutroQuotientGroups and NeutroGroupHomomorphisms are presented and studied. It is shown that the fundamental homomorphism theorem of the classical groups is holding in the class of NeutroGroups of type-NG[1,2,4].

Keywords: NeutroGroup; AntiGroup; NeutroSubgroup; NeutroQuotientGroup; NeutroGroupHomomorphism.

1 Introduction and Preliminaries

In any classical algebraic structure $(X, *)$, the law of composition of the elements of X otherwise called a binary operation $*$ is well defined for all the elements of X that is, $x * y \in X \ \ \forall x, y \in X$; and axioms like associativity, commutativity, distributivity, etc. defined on X with respect to $*$ are totally true for all the elements of X. The compositions of elements of X this way are restrictive and do not reflect the reality. It does not give room for compositions that are either partially defined, partially undefined (indeterminate), and partially outerdefined or totally outerdefined with respect to $*$. However in the domain of knowledge, science and reality, the law of composition and axioms defined on X may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to $*$. In an attempt to model the reality by allowing the law of composition on X to be either partially defined, partially undefined (indeterminate), and partially outerdefined or totally outerdefined, Smarandache [8] in 2019 introduced the notions of NetroDefined and AntiDefined laws, as well as the notions of NeutroAxiom and AntiAxiom inspired by his work in [9], which has given birth to new fields of research called NeutroStructures and AntiStructures. For any classical algebraic law or axiom defined on X, there correspond neutrosophic triplets $<$ Law, NeutroLaw, AntiLaw $>$ and $<$ Axiom, NeutroAxiom, AntiAxiom $>$ respectively. Smarandache in [7] studied NeutroAlgebras and AntiAlgebras and in [6], he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. Rezaei and Smarandache [5] studied Neutro-BE-algebras and Anti-BE-algebras and fundamentally they showed that any classical algebra S with n operations (laws and axioms) where $n \geq 1$ will have $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras. Agboola et al. in [1] studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and in [2], he studied NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he studied NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. He showed that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not

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hold in the class of NeutroGroups. In [3], Agboola studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). He presented Several results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms. He showed that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings. Motivated and inspired by the work of Rezaei and Smarandache in [5], the work on NeutroGroups presented in [2] is revisited and the present work is devoted to the study of a class of NeutroGroups of type-NG[1,2,4]. In this class of NeutroGroups, the closure law, the axiom of associativity and existence of inverse are taking to be either partially true or partially false for some elements; while the existence of identity element and axiom of commutativity are taking to be totally true for all the elements. Several examples of NeutroGroups of type-NG[1,2,4] are presented along with their basic properties. It is shown that Lagrange's theorem holds for some NeutroSubgroups of a NeutroGroup and failed to hold for some NeutroSubgroups of the same NeutroGroup. It is also shown that the union of two NeutroSubgroups of a NeutroGroup can be a NeutroSubgroup even if one is not contained in the other; and that the intersection of two NeutroSubgroups may not be a NeutroSubgroup. The concepts of NeutroQuotientGroups and NeutroGroupHomomorphisms are presented and studied. It is shown that the fundamental homomorphism theorem of the classical groups is holding in the class of NeutroGroups of type-NG[1,2,4].

Definition 1.1. [6]

(i) A classical operation is an operation well defined for all the set's elements.

(ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).

(v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

(vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom.

(viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 1.2. *[5] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

2 Main Results

Definition 2.1. [Classical group][4]
Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a classical group if the following conditions hold:

(G1) $x * y \in G \; \forall x, y \in G$ [closure law].

(G2) $x * (y * z) = (x * y) * z \; \forall x, y, z \in G$ [axiom of associativity].

(G3) There exists $e \in G$ such that $x * e = e * x = x \; \forall x \in G$ [axiom of existence of neutral element].

(G4) There exists $y \in G$ such that $x * y = y * x = e \; \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G.

If in addition $\forall x, y \in G$, we have

(G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.2. [NeutroSophication of the law and axioms of the classical group]

(NG1) There exist at least three duplets $(x, y), (u, v), (p, q), \in G$ such that $x * y \in G$ (degree of truth T) and $[u * v =$ outer-defined/indeterminate (degree of indeterminacy I) or $p * q \notin G]$ (degree of falsehood F) [NeutroClosureLaw].

(NG2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in G$ such that $x*(y*z) = (x*y)*z$ (degree of truth T) and $[[p * (q * r)]$or$[(p * q) * r] =$ outer-defined/indeterminate (degree of indeterminacy I) or $u*(v*w) \neq (u*v)*w]$ (degree of, falsehood F) [NeutroAxiom of associativity (NeutroAssociativity)].

(NG3) There exists an element $e \in G$ such that $x * e = e * x = x$ (degree of truth T) and $[[x * e]$or$[e * x] =$ outer-defined/indeterminate (degree of indeterminacy I) or $x * e \neq x \neq e * x]$ (degree of falsehood F) for at least one $x \in G$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NG4) There exists an element $u \in G$ such that $x * u = u * x = e$ (degree of truth T) and $[[x * u]$or$[u * x)] =$ outer-defined/indeterminate (degree of indeterminacy I) or $x * u \neq e \neq u * x]$ for at least one $x \in G$ (degree of falsehood F) [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where e is a NeutroNeutralElement in G.

(NG5) There exist at least three duplets $(x, y), (u, v), (p, q) \in G$ such that $x * y = y * x$ (degree of truth T) and $[[u * v]$or$[v * u] =$ outer-defined/indeterminate (degree of indeterminacy I) or $p * q \neq q * p]$ (degree of falsehood F) [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.3. [AntiSophication of the law and axioms of the classical group]

(AG1) For all the duplets $(x, y) \in G$, $x * y \notin G$ [AntiClosureLaw].

(AG2) For all the triplets $(x, y, z) \in G$, $x*(y*z) \neq (x*y)*z$ [AntiAxiom of associativity (AntiAssociativity)].

(AG3) There does not exist an element $e \in G$ such that $x * e = e * x = x \; \forall x \in G$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AG4) There does not exist $u \in G$ such that $x * u = u * x = e \; \forall x \in G$ [AntiAxiom of existence of inverse element (AntiInverseElement)] where e is an AntiNeutralElement in G.

(AG5) For all the duplets $(x, y) \in G$, $x * y \neq y * x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.4. [NeutroGroup]
A NeutroGroup NG is an alternative to the classical group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ with no AntiLaw or AntiAxiom.

Definition 2.5. [AntiGroup]
An AntiGroup AG is an alternative to the classical group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$.

Definition 2.6. [NeutroAbelianGroup]
A NeutroAbelianGroup NG is an alternative to the classical abelian group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ and $NG5$ with no AntiLaw or AntiAxiom.

Definition 2.7. [AntiAbelianGroup]
An AntiAbelianGroup AG is an alternative to the classical abelian group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$ and $AG5$.

Proposition 2.8. *Let $(G, *)$ be a finite or infinite classical non abelian group. Then:*

(i) there are 15 types of NeutroNonAbelianGroups,

(ii) there are 65 types of AntiNonAbelianGroups.

Proof. It follows from Theorem 1.2. □

Proposition 2.9. *Let $(G, *)$ be a finite or infinite classical abelian group. Then:*

(i) there are 31 types of NeutroAbelianGroups,

(ii) there are 211 types of AntiAbelianGroups.

Proof. It follows from Theorem 1.2. □

Remark 2.10. It is evident from Theorem 2.8 and Theorem 2.9 that there are many types of NeutroGroups and NeutroAbelianGroups. The type of NeutroGroups studied by Agboola in [2] is that for which $G1, G2, G3, G4$ and $G5$ are either partially true or partially false.

Definition 2.11. Let $(NG, *)$ be a NeutroGroup. NG is said to be finite of order n if the cardinality of NG is n that is $o(NG) = n$. Otherwise, NG is called an infinite NeutroGroup and we write $o(NG) = \infty$.

Definition 2.12. Let $(AG, *)$ be an AntiGroup. AG is said to be finite of order n if the cardinality of AG is n that is $o(AG) = n$. Otherwise, AG is called an infinite AntiGroup and we write $o(AG) = \infty$.

Example 2.13. Let $NG = \mathbb{N} = \{1, 2, 3, 4 \cdots, \}$. Then $(NG, .)$ is a finite NeutroGroup where $''.''$ is the binary operation of ordinary multiplication.

Example 2.14. Let $AG = \mathbb{Q}^*_+$ be the set of all irrational positive numbers and consider algebraic structure $(AG, *)$ where $*$ is ordinary multiplication of numbers. It is clear that $*$ is a total AntiLaw defined on AG. The binary operation $*$ is totally associative for all the triplets (x, y, z) with $x, y, z \in AG$. There is no neutral element(s) for all the elements AG and hence no element of AG has an inverse. Finally, the operation $*$ is commutative for all the duplets (x, y) with $x, y \in AG$. Hence by Definition 2.5, $(AG, *)$ is an infinite AntiGroup.

Example 2.15. Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $AG = \{a, b, c, \}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on AG as shown in the Cayley table below:

$*$	a	b	c
a	d	c	b
b	c	e	a
c	b	a	f

It is clear from the table that except for the compositions $a * a = d, b * b = e, c * c = f$ that are outer-defined with the degree of falsity 33%, the rest compositions are inner-defined with 66.7% degree of truth. This shows that $G1$ is partially true and partially false so that $*$ is a NeutroLaw. Also, $G2$ is partial true and partially false. There are $3^3 = 27$ possible triplets out of which only 6 can verify associativity of $*$. Hence degree of associativity of $*$ is 22.2% while the degree of non-associativity is 77.8% so that $*$ is NeutroAssociative. However, $G3$ and $G4$ are totally false for all the elements of AG which shows that $AG3$ and $AG4$ are satisfied. Lastly, $G5$ is partially true with the degree of truth 50% and partially false with 50% degree of falsity which shows that $*$ is NeutroCommutative. Hence by Definition 2.5, $(AG, *)$ is a finite AntiGroup.

3 Certain Types of NeutroGroups

In this section, we are going to study certain types of NeutroGroups $(NG, *)$. The NeutroGroups will be named according to which of $NG1 - NG5$ is(are) satisfied. In the sequel, $x * y$ will be written as $xy\ \forall x, y \in NG$.

Example 3.1. Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $NG = \{e, a, b, c\}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on NG as shown in the Cayley table below:

$*$	e	a	b	c
e	e	a	b	c
a	a	b	a	b
b	b	c	f	c
c	c	d	c	e

It is clear from the table that $G1, G2, G3, G4$ and $G5$ are partially true and partially false with respect to $*$ as shown below:

(i) **NeutroClosureLaw (NG1):** Except for the compositions $b * b = f, c * a = d$ which are false with 12.5% degree of falsity, all other compositions are true with 87.5% degree of truth.

(ii) **NeutroAssociativity (NG2):**

$$\begin{aligned} a*(b*c) &= (a*b)*c = b. \\ a*(a*a) &= a, \quad \text{but, } (a*a)*a = c \neq a. \end{aligned}$$

(iii) **NeutroNeutralElement (NG3):**

$$\begin{aligned} N_e &= N_a = N_b = e \quad \text{but} \\ N_c &= e \text{ or } b. \end{aligned}$$

(iv) **NeutroInverseElement (NG4):**

$$\begin{aligned} I_e &= e, \\ I_a & \quad \text{does not exist,} \\ I_b & \quad \text{does not exist,} \\ I_c &= e. \end{aligned}$$

(v) **NeutroCommutativity (NG5):**

$$\begin{aligned} b*c &= c*b = c. \\ a*b &= a \quad \text{but } b*a = c \neq a. \end{aligned}$$

We have just shown that $(NG,*)$ is a finite NeutroAbelianGroup. This is an example of the class of NeutroGroups studied by Agboola in [2]. This class of NeutroGroups are referred to as of type-NG[1,2,3,4,5].

Example 3.2. Let $\mathbb{U} = \{a,b,c,d,e,f\}$ be a universe of discourse and let $NG = \{a,b,c,e\}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on NG as shown in the Cayley table below:

$*$	a	b	c	e
a	e	c	f	a
b	c	e	d	b
c	d	a	e	c
e	a	b	c	e

.

It is clear from the table that $G3$ and $G4$ are totally true for all the elements of NG. However, $G1$, $G2$ and $G5$ are partially true and partially false with respect to $*$ as shown below:

(i) **NeutroClosureLaw (NG1):** Except for the compositions $a*c = f, b*c = d, c*a = d$ which are outer-defined with 18.75% degree of falsity, all other compositions are inner-defined with 81.25% degree of truth.

(ii) **NeutroAssociativity (NG2):**

$$\begin{aligned} c*(b*b) &= (c*b)*b = c. \\ a*(b*c) &= a*d = \text{outer-defined}, (a*b)*c = e. \end{aligned}$$

(iii) **NeutroCommutativity (NG5):**

$$\begin{aligned} a*b &= b*a = c. \\ a*c &= f \quad \text{but } c*a = d \neq f. \end{aligned}$$

We have just shown that $(NG,*)$ is a finite NeutroAbelianGroup of type-NG[1,2,5].

Example 3.3. Let $NG = \mathbb{Z}$ and let $*$ be a binary operation defined on NG by

$$x*y = x + xy \quad x,y \in \mathbb{Z}.$$

It is clear that only $G1$ is totally true for all $x,y \in NG$ but $G2, G3, G4$ and $G5$ are partially true and partially false with respect to $*$ as shown below:

(i) **NeutroAssociativity (NG2):**

$$\begin{aligned} x*(y*z) &= x+xy+xyz \\ (x*y)*z &= x+xy+xz+xyz \quad \text{by equating, we have} \\ x+xy+xyz &= x+xy+xz+xyz \\ \Rightarrow \quad xz &= 0 \quad \text{from which we obtain} \\ x &= 0 \quad \text{or} \quad z=0. \end{aligned}$$

This shows that only the triplets $(0,y,0),(0,y,z),(x,y,0)$ can verify associativity of $*$.

(ii) **NeutroNeutralElement (NG3):**
It is clear that only the element $0 \in NG$ has 0 as its neutral element and no neutral(s) for other elements of NG.

(iii) **NeutroInverseElement (NG4):**
Again, only the element $0 \in NG$ has 0 as the inverse element and no inverse(s) for other elements of NG.

(iv) **NeutroCommutativity (NG5):**
Only the duplet $(0,0)$ can verify the commutativity of $*$ and not any other duplet(s) (x,y). Hence, $(NG,*)$ is an infinite NeutroAbelianGroup of type-NG[2,3,4,5].

Definition 3.4. Let $(NG,*)$ be a NeutroGroup. A nonempty subset NH of NG is called a NeutroSubgroup of NG if $(NH,*)$ is also a NeutroGroup of the same type as NG. If $(NH,*)$ is a NeutroGroup of a type different from that of NG, then NH will be called a QuasiNeutroSubgroup of NG.

Example 3.5. Let $(NG,*)$ be the NeutroGroup of Example 3.2 and let $NH_1=\{a,c,e\}$ and $NH_2=\{b,c,e\}$ be two subsets of NG. Let $*$ be defined on NH_1 and NH_2 as shown in the Cayley tables below:

NH_1:

$*$	a	c	e
a	e	f	a
c	d	e	c
e	a	c	e

NH_2:

$*$	b	c	e
b	e	d	b
c	a	e	c
e	b	c	e

It can easily be shown that $(NH_1,*)$ and $(NH_2,*)$ are NeutroGroups of type-NG[1,2,5] and therefore NH_1 and NH_2 are NeutroSubgroups of NG. We note that $o(NG)=4, o(NH_1)=3=o(NH_2)$. Since 3 does not divide 4, it follows that Lagranges' theorem does not hold. Now consider the following:

$$\begin{aligned} NH_1 \cup NH_2 &= \{a,b,c,e\} = NG. \\ NH_1 \cap NH_2 &= \{c,e\}. \end{aligned}$$

These show that $NH_1 \cup NH_2$ is a NeutroSubgroup of NG but $NH_1 \cap NH_2$ is not a Neutrosubgroup of NG. It is however observed that $NH_1 \cap NH_2$ is a group as can be seen in the Cayley table below:

$NH_1 \cap NH_2$:

$*$	c	e
c	e	c
e	c	e

Definition 3.6. Let $(NG,*)$ be a NeutroGroup and let $a \in NG$ be a fixed element.

(i) The center of NG denoted by $Z(NG)$ is a set defined by

$$Z(NG) = \{x \in NG : xg = gx \quad \text{for at least one} \quad g \in NG\}.$$

(ii) The centralizer of $a \in G$ denoted by NC_a is a set defined by

$$NC_a = \{g \in NG : ga = ag\}.$$

Example 3.7. Let $(NG,*)$ be the NeutroGroup of Example 3.2. Then:

(i) $Z(NG)=\{a,b,c,e\}=NG$. This shows that $Z(NG)$ is a NeutroSubgroup of NG.

(ii) $NC_a=\{a,b,e\}$, $NC_b=\{a,b,e\}$, $NC_c=\{c,e\}$ and $NC_e=\{a,b,c,e\}$. We have that NC_a and NC_b are not NeutroSubgroups of NG, NC_e is a group and NC_c is a NeutroSubgroup of NG.

4 Characterization of Finite NeutroGroups of type-NG[1,2,4]

In this section, we are going to study finite NeutroGroups of type-NG[1,2,4] that is NeutroGroups $(NG, *)$ where $G3$ and $G5$ are totally true for all the elements of NG and where $G1$, $G2$ and $G4$ are either partially true or partially false for some elements of NG.

Example 4.1. Let $NG = \{1, 2, 3, 4\} \subseteq \mathbb{Z}_5$ and let $*$ be a binary operation on NG defined as

$$x * y = x + y + 4 \quad \forall x, y \in NG.$$

Then $(NG, *)$ is a finite NeutroGroup of type-NG[1,2,4] as can be seen in the Cayley table

$*$	1	2	3	4
1	1	2	3	4
2	2	3	4	0
3	3	4	0	1
4	4	0	1	2

.

Example 4.2. Let $NG = \{1, 2, 3\} \subseteq \mathbb{Z}_4$ and let $\bullet$ be a binary operation defined on NG as shown in the Cayley table

$\star$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

.

Then $(NG, \bullet)$ is a finite NeutroGroup of type-NG[1,2,4].

Example 4.3. Let $NK = \{1, 3, 5\} \subseteq \mathbb{Z}_8$ and let $\circ$ be a binary operation define on NK as shown in the Cayley table

$\circ$	1	3	5
1	1	3	5
3	3	1	7
5	5	7	1

.

Then $(NK, \circ)$ is a finite NeutroGroup of type-NG[1,2,4].

Example 4.4. Let $NG = \{1, 2, 3, 4, 5\} \subseteq \mathbb{Z}_{10}$ and let $*$ be a binary operation define on NG as shown in the Cayley table

$*$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	0
3	3	6	9	2	5
4	4	8	2	6	0
5	5	0	5	0	5

.

Then $(NG, *)$ is a finite NeutroGroup of type-NG[1,2,4].

Example 4.5. Let $(NG, \bullet)$ and $(NK, \circ)$ be NeutroGroups of Examples 4.2 and 4.3 respectively. Then

$$\begin{aligned} NG \times NG &= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}, \\ NK \times NK &= \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}, \\ NG \times NK &= \{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(3,1),(3,3),(3,5)\}. \end{aligned}$$

It can easily be shown that $(NG \times NG, \bullet)$, $(NK \times NK, \circ)$ and $(NG \times NK, \square)$ are NeutroGroups of type-NG[1,2,4].

Proposition 4.6. *Let $(NG, \bullet)$ and $(NK, \circ)$ be any two NeutroGroups of the same type-NG[1,2,4]. Then $(NG \times NG, \bullet)$, $(NK \times NK, \circ)$ and $(NG \times NK, \square)$ are NeutroGroups of type-NG[1,2,4].*

Proof. The proof is easy and so omitted. □

Proposition 4.7. *Let $(NG, *)$ be a finite NeutroGroup of type-NG[1,2,4]. The classical laws of indices do not hold.*

Proof. Since $*$ is a NeutroLaw and $*$ is NeutroAssociative over NG, the result follows. □

Corollary 4.8. *No NeutroGroup $(NG, *)$ of type-NG[1,2,4] can be cyclic that is, generated by an element $x \in NG$.*

Proposition 4.9. *Let $(NG, *)$ be a finite NeutroGroup of type-NG[1,2,4]. If x and y are invertible elements of NG, then*

(i) $\left(x^{-1}\right)^{-1} = x$.

(ii) $(xy)^{-1} = x^{-1}y^{-1}$.

Proof. Obvious. □

Example 4.10. Let $(NG, *)$ be the NeutroGroup of Example 4.1 and consider the following subsets of NG.

$$NH_1 = \{1, 2\}, NH_2 = \{1, 3\}, NH_3 = \{1, 4\}, NH_4 = \{1, 2, 3\}, NH_5 = \{1, 2, 4\}, NH_6 = \{1, 3, 4\}.$$

It can be shown that $(NH_i, *)$, $i = 1, 2, 3, 4, 5, 6$ are NeutroSubgroups of NG. Next consider the following:

$$\begin{aligned}
NH_1 \cup NH_2 &= NH_1 \cup NH_4 = \{1, 2, 3\} \quad \text{[NeutroSubgroups of NG]}.\\
NH_1 \cup NH_3 &= NH_1 \cup NH_5 = NH_3 \cup NH_5 = \{1, 2, 4\} \quad \text{[NeutroSubgroups of NG]}.\\
NH_2 \cup NH_3 &= NH_2 \cup NH_6 = \{1, 3, 4\} \quad \text{[NeutroSubgroups of NG]}.\\
NH_3 \cup NH_4 &= NH_5 \cup NH_6 = \{1, 2, 3, 4\} \quad \text{[trivial NeutroSubgroups of NG]}.\\
NH_1 \cap NH_4 &= NH_1 \cap NH_5 = NH_4 \cap NH_5 = \{1, 2\} \quad \text{[NeutroSubgroups of NG]}.\\
NH_2 \cap NH_4 &= NH_2 \cap NH_6 = NH_4 \cap NH_6 = \{1, 3\} \quad \text{NeutroSubgroups of NG]}.\\
NH_3 \cap NH_5 &= NH_3 \cap NH_6 = \{1, 4\} \quad \text{[NeutroSubgroups of NG]}.
\end{aligned}$$

$$NH_1 \cap NH_2 = NH_2 \cap NH_3 = \{1\} \quad \text{[not NeutroSubgroups of NG]}.$$

Example 4.11. Let $(NG, *)$ be the NeutroGroup of Example 4.4 and consider the following subsets of NG.

$$NH_1 = \{1, 2, 4, 5\}, NH_2 = \{1, 2, 3, 5\}.$$

It can be shown that $(NH_i, *)$, $i = 1, 2$ are NeutroSubgroups of NG. Next consider the following:

$$\begin{aligned}
NH_1 \cup NH_2 &= \, = \{1, 2, 5\} \quad \text{[a NeutroSubgroup of NG]}.\\
NH_1 \cap NH_2 &= \{1, 2, 3, 4, 5\} \quad \text{[a trivial NeutroSubgroup of NG]}.
\end{aligned}$$

Remark 4.12. Examples 4.10 and 4.11 have shown that in the NeutroGroups of type-NG[1,2,4], we can have the following:

(i) Lagrange's theorem may hold for some NeutroSubgroups of the NeutroGroups and fail to hold for some NeutroSubgroups.

(ii) The union of two NeutroSubgroups of the NeutroGroups can be NeutroSubgroups even if one is not contained in the other.

(iii) The intersection of two NeutroSubgroups of the NeutroGroups can be NeutroSubgroups.

Definition 4.13. Let NH be a NeutroSubgroup of the NeutroGroup $(NG, *)$ and let $x \in NG$.

(i) xNH the left coset of NH in NG is defined by

$$xNH = \{xh : h \in NH\}.$$

(ii) The number of distinct left cosets of NH in NG is called the index of NH in NG denoted by $[NG : NH]$.

(iii) The set of all distinct left cosets of NH in NG denoted by NG/NH is defined by

$$NG/NH = \{xNH : x \in NG\}.$$

Example 4.14. Let $(NG, *)$ be the NeutroGroup of Example 4.1 and let $(NH_i, *)$, $i = 1, 2, 3, 4, 5, 6$ be the NeutroSubgroups of Example 4.10. The left cosets of NH_i are computed as follows.

$$\begin{aligned}
1NH_1 &= \{1,2\}, 2NH_1 = \{2,3\}, 3NH_1 = \{3,4\}, 4NH_1 = \{0,4\}.\\
1NH_2 &= \{1,3\}, 2NH_2 = \{2,4\}, 3NH_2 = \{0,3\}, 4NH_2 = \{1,4\}.\\
1NH_3 &= \{1,4\}, 2NH_3 = \{0,2\}, 3NH_3 = \{1,3\}, 4NH_3 = \{2,4\}.\\
1NH_4 &= \{1,2,3\}, 2NH_4 = \{2,3,4\}, 3NH_4 = \{0,3,4\}, 4NH_4 = \{0,1,4\}.\\
1NH_5 &= \{1,2,4\}, 2NH_5 = \{0,2,3\}, 3NH_5 = \{1,3,4\}, 4NH_5 = \{0,2,4\}.\\
1NH_6 &= \{1,3,4\}, 2NH_6 = \{0,2,4\}, 3NH_6 = \{0,1,3\}, 4NH_6 = \{1,2,4\}.\\
\therefore\ NG/NH_i &= \{1NH_i, 2NH_i, 3NH_i, 4NH_i\}, i = 1,2,3,4,5,6.
\end{aligned}$$

Example 4.15. Let $(NG, *)$ be the NeutroGroup of Example 4.4 and let $(NH_i, *)$, $i = 1, 2$ be the NeutroSubgroups of Example 4.11. The left cosets of NH_i are computed as follows.

$$\begin{aligned}
1NH_1 &= \{1,2,4,5\}, 2NH_1 = \{0,2,4,8\}, 3NH_1 = \{2,3,5,6\}, 4NH_1 = \{0,4,6,8\}, 5NH_1 = \{0,5\}.\\
1NH_2 &= \{1,2,3,5\}, 2NH_2 = \{0,2,4,6\}, 3NH_2 = \{3,5,6,9\}, 4NH_2 = \{0,2,4,8\}, 5NH_1 = \{0,5\}.\\
\therefore\ NG/NH_i &= \{1NH_i, 2NH_i, 3NH_i, 4NH_i, 5NH_i\}, i = 1,2.
\end{aligned}$$

Lemma 4.16. *Let NH be a NeutroSubgroup of the NeutroGroup $(NG, *)$ of type-NG[1,2,4] and let $x \in NG$. Then, $xNH = NH$ if and only if $x = e$ where e is the identity element in NG.*

Proof. Obvious. □

Remark 4.17. Examples 4.14 and 4.15 have shown that in the NeutroGroups of type-NG[1,2,4] distinct left cosets of NeutroSubgroups in the NeutroGroups do not necessarily partition the NeutroGroups.

Let NH be a NeutroSubgroup of a NeutroGroup $(NG, *)$ of type-NG[1,2,4] and let NG/NH be the set of distinct left cosets of NH in NG. For $xNH, yNH \in NG/NH$ with $x, y \in NG$, let $\odot$ be a binary operation defined on NG/NH by

$$xNH \odot yNH = xyNH \quad \forall\, x, y \in NG.$$

We want to investigate if the couple $(NG/NH, \odot)$ is a NeutroGroup of type-NG[1,2,4] using the following examples.

Example 4.18. Let $NG/NH_i = \{1NH_i, 2NH_i, 3NH_i, 4NH_i\}, i = 1, 2, 3, 4, 5, 6$ be as given in Example 4.14. For $i = 1$, we have

$\odot$	$1NH_1$	$2NH_1$	$3NH_1$	$4NH_1$
$1NH_1$	$1NH_1$	$2NH_1$	$3NH_1$	$4NH_1$
$2NH_1$	$2NH_1$	$3NH_1$	$4NH_1$	$0NH_1$
$3NH_1$	$3NH_1$	$4NH_1$	$0NH_1$	$1NH_1$
$4NH_1$	$4NH_1$	$0NH_1$	$1NH_1$	$2NH_1$

It is clear from the Cayley table that $(NG/NH_1, \odot)$ is a NeutroGroup of type-NG[1,2,4] with $1NH_1$ as the identity element. This is also true for $i = 2, 3, 4, 5, 6$.

Example 4.19. Let $NG/NH_i = \{1NH_i, 2NH_i, 3NH_i, 4NH_i, 5NH_i\}, i = 1, 2$ be as given in Example 4.15. For $i = 1$, we have

$\odot$	$1NH_1$	$2NH_1$	$3NH_1$	$4NH_1$	$5NH_1$
$1NH_1$	$1NH_1$	$2NH_1$	$3NH_1$	$4NH_1$	$5NH_1$
$2NH_1$	$2NH_1$	$4NH_1$	$6NH_1$	$8NH_1$	$0NH_1$
$3NH_1$	$3NH_1$	$6NH_1$	$9NH_1$	$2NH_1$	$5NH_1$
$4NH_1$	$4NH_1$	$8NH_1$	$2NH_1$	$6NH_1$	$0NH_1$
$5NH_1$	$5NH_1$	$0NH_1$	$5NH_1$	$0NH_1$	$5NH_1$

It is evident from the Cayley table that $(NG/NH_1, \odot)$ is a NeutroGroup of type-NG[1,2,4] with $1NH_1$ as the identity element.. This is also true for $i = 2$.

Remark 4.20. It is evident from Examples 4.18 and 4.19 that if NH is a NeutroSubgroup of the NeutroGroup of type-NG[1,2,4], then NG/NH the set of distinct left cosets can be made a NeutroGroup of type-NG[1,2,4] by defining appropriate binary operation $\odot$ on NG/NH. The NeutroGroup NG/NH is called the Quotient-NeutroGroup of NG factored by NH.

Definition 4.21. Let $(NG, *)$ and $(NH, \circ)$ be any two NeutroGroups of type-NG[1,2,4]. The mapping $\phi : NG \to NH$ is called a NeutroGroupHomomorphism if ϕ preserves the binary operations $*$ and $\circ$ that is if for at least a duplet $(x, y) \in G$, we have

$$\phi(x * y) = \phi(x) \circ \phi(y).$$

The Kernel of ϕ denoted by $Ker\phi$ is defined by

$$Ker\phi = \{x : \phi(x) = e_{NH}\}$$

where e_{NH} is the identity element in NH.

The Image of ϕ denoted by $Im\phi$ is defined by

$$Im\phi = \{y \in H : y = \phi(x) \text{ for some } h \in NH\}.$$

If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroGroupIsomorphism and we write $NG \cong NH$. NeutroGroupEpimorphism, NeutroGroupMonomorphism, NeutroGroupEndomorphism, and NeutroGroupAutomorphism are similarly defined.

Example 4.22. Let $(NG, \bullet)$ be the NeutroGroup of Example 4.2 and let $\phi : NG \times NG \to NG$ be a projection given by

$$\psi(x, y) = y \quad \forall x, y \in NG.$$

Then

$$\phi(1,1) = \phi(2,1) = \phi(3,1) = 1, \phi(1,2) = \phi(2,2) = \phi(3,2) = 2, \phi(1,3) = \phi(2,3) = \phi(3,3) = 3.$$

Since $\phi((2,1)(3,3)) = \phi(2,3) = 3$ and $\phi(2,1)\phi(3,3) = 1 \bullet 3 = 3$ but $\phi((2,2)(2,3)) = \phi(0,2) =?$ and $\phi(2,2)\phi(2,3) = 2 \bullet 3 = 2$, it follows that ϕ is a NeutroGroupHomomorphism. $Im\phi = \{1,2,3\} = NG$ and $Ker\phi = \{(1,1),(2,1),(3,1)\}$. The $Ker\phi$ is a NeutroSubgroup of $NG \times NG$ as can be seen in the following Cayley table

$\bullet$	$(1,1)$	$(2,1)$	$(3,1)$
$(1,1)$	$(1,1)$	$(2,1)$	$(3,1)$
$(2,1)$	$(2,1)$	$(0,1)$	$(2,1)$
$(3,1)$	$(3,1)$	$(2,1)$	$(1,1)$

.

It is evident from the table that $(Ker\phi, \bullet)$ is a NeutroGroup of type-NG[1,2,4] and since $Ker\phi \subseteq NG \times NG$, it follows that $\ker\phi$ is a NeuroSubgroup.

Example 4.23. Let $(NK, \circ)$ be the NeutroGroup of Example 4.3 and let $\psi : NK \times NK \to NK$ be a projection given by

$$\psi(x, y) = x \quad \forall x, y \in NK.$$

Then

$$\psi(1,1) = \psi(1,3) = \psi(1,5) = 1, \psi(3,1) = \psi(3,3) = \psi(3,5) = 3, \psi(5,1) = \psi(5,3) = \psi(5,5) = 5.$$

Since $\psi((1,1)(1,3)) = \psi(1,3) = 1$ and $\psi(1,1)\psi(1,3) = 1 \circ 1 = 1$ but $\psi((1,5)(5,3)) = \psi(5,7) =?$ and $\psi(1,5)\psi(5,3) = 1 \circ 5 = 5$, it follows that ψ is a NeutroGroupHomomorphism. $Im\psi = \{1,3,5\} = NK$ and $Ker\psi = \{(1,1),(1,3),(1,5)\}$. The $Ker\psi$ is a NeutroSubgroup of $NK \times NK$ as can be seen in the following Cayley table

$\circ$	$(1,1)$	$(1,3)$	$(1,5)$
$(1,1)$	$(1,1)$	$(1,3)$	$(1,5)$
$(1,3)$	$(1,3)$	$(1,1)$	$(1,7)$
$(1,5)$	$(1,5)$	$(1,7)$	$(1,1)$

.

It is evident from the table that $(Ker\psi, \circ)$ is a NeutroGroup of type-NG[1,2,4] and since $Ker\psi \subseteq NK \times NK$, it follows that $\ker\psi$ is a NeuroSubgroup.

Example 4.24. Let $NG/NH_i = \{1NH_i, 2NH_i, 3NH_i, 4NH_i\}, i = 1, 2, 3, 4, 5, 6$ be the NeutroQuotientGroup of Example 4.18. For $i = 1$, let $\psi : NG \to NG/NH_1$ be a mapping defined by

$$\psi(x) = xNH_1 \quad \forall x \in NG.$$

From Example 4.14 we have

$$\psi(1) = 1NH_1 = \{1, 2\}, \psi(2) = 2NH_1 = \{2, 3\}, \psi(3) = 3NH_1 = \{3, 4\}, \psi(4) = 4NH_1 = \{0, 4\}.$$

Next,

$$\begin{aligned}
\psi(2 * 3) &= \psi(4) = 4NH_1 = \{0, 4\}, \\
\psi(2) \odot \psi(3) &= 2NH_1 \odot 3NH_1 = 2 * 3NH_1 = 4NH_1 = \{0, 4\} \quad \text{but then,} \\
\psi(2 * 4) &= \psi(0) =? \\
\psi(2) \odot \psi(4) &= 2NH_1 \odot 4NH_1 = 2 * 4NH_1 = 0NH_1 =?.
\end{aligned}$$

This shows that ψ is a NeutroGroupHomomorphism. The $Ker\psi = 1NH_1 = \{1, 2\}$ the identity element of NG/NH_1.

Example 4.25. Let $NG/NH_i = \{1NH_i, 2NH_i, 3NH_i, 4NH_i, 5NH_i\}, i = 1, 2$ be the NeutroQuotientGroup of Example 4.19. For $i = 1$, let $\phi : NG \to NG/NH_1$ be a mapping defined by

$$\phi(x) = xNH_1 \quad \forall x \in NG.$$

From Example 4.15 we have

$$\begin{aligned}
\psi(1) &= 1NH_1 = \{1, 2, 4, 5\}, \phi(2) = 2NH_1 = \{0, 2, 4, 8\}, \phi(3) = 3NH_1 = \{2, 3, 5, 6\}, \\
&\phi(4) = 4NH_1 = \{0, 4, 6, 8\}, \phi(5) = 5NH_1 = \{0, 5\}.
\end{aligned}$$

Next,

$$\begin{aligned}
\phi(3 * 5) &= \phi(5) = 5NH_1 = \{0, 5\}, \\
\phi(3) \odot \phi(5) &= 3NH_1 \odot 5NH_1 = 3 * 5NH_1 = 5NH_1 = \{0, 5\} \quad \text{but then,} \\
\phi(3 * 2) &= \phi(6) =? \\
\phi(3) \odot \phi(2) &= 3NH_1 \odot 2NH_1 = 3 * 2NH_1 = 6NH_1 =?.
\end{aligned}$$

This shows that ψ is a NeutroGroupHomomorphism. The $Ker\psi = 1NH_1 = \{1, 2, 4, 5\}$ the identity element of NG/NH_1.

Proposition 4.26. *Let $(NG, *)$ and $(NH, \circ)$ be NeutroGroups of type-NG[1,2,4] and let e_{NG} and e_{NH} be identity elements in NG and NH respectively. Suppose that $\phi : NG \to NH$ is a NeutroGroupHomomorphism. Then:*

(i) $\phi(e_{NG}) = e_{NH}$.

(ii) $\phi\left(x^{-1}\right) = (\phi(x))^{-1}$ *for every invertible element $x \in NG$.*

(iii) $Ker\phi$ *is a NeutroSubgroup of NG.*

(iv) $Im\phi$ *is a NeutroSubgroup of NH.*

(v) ϕ *is NeutroInjective if and only if $Ker\phi = \{e_{NG}\}$.*

Proof. The same as for the classical groups and so omitted. □

Proposition 4.27. *Let NH be a NeutroSubgroup of a NeutroGroup $(NG, *)$ of type-NG[1,2,4]. The mapping $\psi : NG \to NG/NH$ defined by*

$$\psi(x) = xNH \quad \forall\, x \in NG$$

is a NeutroGroupHomomorphism and the $Ker\psi = NH$.

Proof. The same as for the classical groups and so omitted. □

Proposition 4.28. *Let $\phi : NG \to NH$ be a NeutroGroupHomomorpism and let $NK = Ker\phi$. Then the mapping $\psi : NG/NK \to Im\phi$ defined by*

$$\psi(xNK) = \phi(x) \quad \forall\, x \in NG$$

is a NeutroGroupIsomorphism.

Proof. The same as for the classical groups and so omitted. □

5 Conclusion

We have in this work studied a class of NeutroGroups $(NG, *)$ of type-NG[1,2,4]. In this class of NeutroGroups, the closure law, the axiom of associativity and existence of inverse were taking to be either partially true or partially false for some elements of NG; while the existence of identity element and axiom of commutativity were taking to be totally true for all the elements of NG. Several examples of NeutroGroups of type-NG[1,2,4] were presented along with their basic properties. It was shown that Lagrange's theorem holds for some NeutroSubgroups of a NeutroGroup and failed to hold for some NeutroSubgroups of the same NeutroGroup. It was also shown that the union of two NeutroSubgroups of a NeutroGroup can be a NeutroSubgroup even if one is not contained in the other; and that the intersection of two NeutroSubgroups may not be a NeutroSubgroup. The concepts of NeutroQuotientGroups and NeutroGroupHomomorphisms were presented and studied. It was shown that the fundamental homomorphism theorem of the classical groups is holding in the class of NeutroGroups of type-NG[1,2,4]. We hope to study AntiGroups, revisit NeutroRings, study AntiRings,NeutroVectorSpaces, AntiVectorSpaces, NeutroModules, AntiModules, NeutroHypergroups, AntiHypergroups, NeutroHyperrings, AntiHyperrings, NeutroHypervectorSpaces and AntiHypervectorSpaces in our future papers.

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