Classical Homomorphisms Between n-Refined Neutrosophic Rings

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Abstract

This paper studies classical homomorphisms between n-refined neutrosophic ring and m-refined neutrosophic ring. It proves that every m-refined neutrosophic ring $R_m(I)$ is a homomorphic image of n-refined neutrosophic ring $R_n(I)$, where $m \leq n$. Also, it presents a discussion of kernels and some corresponding isomorphisms between those rings.

Keywords:n-Refined neutrosophic ring, Ring homomorphism, Ring extension.

1.Introduction

Neutrosophy is a new kind of logic founded by Smarandache, concerns with origin, nature, and indeterminacy. Neutrosophic ideas found their way in algebra and its applications. Neutrosophical algebraic studies began with Smarandache and Kandasamy in [5]. They presented many neutrosophical structures such as neutrosophic rings, groups, and loops. Many generalizations came to light, such as n-refined neutrosophic structures, refined neutrosophic rings, n-refined neutrosophic rings, refined neutrosophic ideals, refined neutrosophic homomorphisms, and AH-substructures.See[1-8].

In [3], Abobala proved that each neutrosophic ring R(I) is a homomorphic image of the refined neutrosophic ring $R(I_1, I_2)$. This result means that a refined neutrosophic ring $R(I_1, I_2)$ is a ring extension of R(I). This extension can be represented by the homomorphism $f: R(I_1, I_2) \to R(I); f(a, bI_1, cI_2) = a + (b + c)I$. In this paper we generalize the previous result into n-refined neutrosophic rings. Also, we prove that each m-refined neutrosophic ring $R_m(I)$ is a homomorphic image of n-refined neutrosophic ring $R_n(I)$, where $m \leq n$.

All homomorphisms through this paper are taken by classical meaning in Ring Theory, not by neutrosophical meaning. For example, see[1, 2].

Motivation

This paper generalizes some results introduced in [3] about refined neutrosophic rings, into n-refined neutrosophic rings. Also, it clarifies that n-refined neutrosophic ideas have an algebraic origin, since n-refined neutrosophic ring $R_n(I)$ can be realized as a classical ring extension of the ring R.

2. Preliminaries

In this section, we show some concepts we used through the paper.

Theorem2.1: [3]

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Let (R ,+,×) be a ring and R(I), R(I_1, I_2) the related neutrosophic ring and refined neutrosophic ring respectively, we have:

(a) There is a ring homomorphism f: R$(I_1, I_2) \to R(I)$.

(b) The additive group (Ker(f),+) is isomorphic to the additive group (R,+).

Theorem 2.2: [3]

Let R be a ring, where Char(R) = 2, there is a subring of R(I_1, I_2) $say\ K$ with property $K \cong R$; R$(I_1, I_2)/K \cong R(I)$.

Definition 2.3: [6]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Addition and multiplication on R_n(I)are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i$,$\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

3. Main results

In the following section, we discuss the main results and theorems.

Lemma 3.1:

Let R be a ring with unity 1, $R_n(I), R_{n-1}(I)$ be the corresponding n-refined neutrosophic ring, and

(n-1) refined neutrosophic ring respectively. Then:

(a) $R_{n-1}(I)$is a homomorphic image of $R_n(I)$.

(b) $R_n(I)/K \cong R_{n-1}(I)$; K is a ring with property $K \cong R$.

Proof:

(a) Define the map

$f: R_n(I) \to R_{n-1}(I); f(a_0 + a_1 I_1 + \cdots + a_n I_n) = a_0 + a_1 I_1 + \cdots + a_{n-2} I_{n-2} + (a_{n-1} + a_n) I_{n-1}$.

$f$is well defined. Consider$x = \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} b_i I_i = y$, we have $a_i = b_i$ for all i, thus

$a_0 + a_1 I_1 + \cdots + a_{n-2} I_{n-2} + (a_{n-1} + a_n) I_{n-1} = b_0 + b_1 I_1 + \cdots + b_{n-2} I_{n-2} + (b_{n-1} + b_n) I_{n-1}$, this means $f(x) = f(y)$.

$f$is a classical ring homomorphism.

Let $x = \sum_{i=0}^{n} a_i I_i, y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $R_n(I)$, we have:

$x + y = \sum_{i=0}^{n} (a_i + b_i) I_i$,

$x.y = \sum_{i,j=0}^{n} (a_i.b_j) I_i I_j = \sum_{i,j=o}^{n-2} (a_i.b_j) I_i I_j + (a_{n-1} I_{n-1} + a_n I_n).(b_{n-1} I_{n-1} + b_n I_n) =$

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$\sum_{i,j=o}^{n-2}(a_i.b_j)I_iI_j + (a_{n-1}.b_{n-1} + a_{n-1}.b_n + a_nb_{n-1})I_{n-1} + a_n.b_nI_n$.

$f(x+y) = \sum_{i=0}^{n-2}(a_i+b_i)I_i + (a_{n-1} + a_n + b_{n-1} + b_n)I_{n-1} = f(x) + f(y)$.

$f(x.y) = \sum_{i,j=o}^{n-2}(a_i.b_j)I_iI_j + (a_{n-1}.b_{n-1} + a_{n-1}.b_n + a_nb_{n-1} + a_n.b_n)I_{n-1}$,

$f(x).f(y) =$

$[a_0 + a_1I_1 + \cdots + a_{n-2}I_{n-2} + (a_{n-1} + a_n)I_{n-1}].[b_0 + b_1I_1 + \cdots + b_{n-2}I_{n-2} + (b_{n-1} + b_n)I_{n-1}]$.

Hence $f(x.y) = f(x).f(y)$.

(b) $Ker(f) = \{y = \sum_{i=0}^{n} b_iI_i \in R_n(I): f(y) = 0\}$, this implies $b_i = 0\ for\ all\ 0 \le i \le n-2$ and

$a_{n-1} = -a_n$, so $Ker(f) = \{a_n(I_n - I_{n-1}); a_n \in R\}$= K.

By first isomorphism theory we find

$R_n(I)/K \cong R_{n-1}(I)$. Consider $g: K \to R; g(a_n(I_n - I_{n-1})) = a_n$. Where $a_n \in R$.

It is easy to see that g is a well defined map, f is an isomorphism:

Let $= a_n(I_n - I_{n-1}), y = b_n(I_n - I_{n-1}); a_i, b_i \in R; i \in \{n-1, n\}$ be two arbitrary elements in K,

$x + y = (a_n + b_n)(I_n - I_{n-1}), g(x+y) = a_n + b_n = g(x) + g(y)$.

$x.y = a_n.b_nI_n - a_n.b_nI_{n-1} - a_n.b_nI_{n-1} + a_n.b_nI_{n-1} = a_n.b_n(I_n - I_{n-1})$,

$g(x.y) = a_n.b_n = g(x).g(y)$.

It is clear that g is bijective. Thus we get the proof.

Theorem 3.2:

Let R be a ring with unity 1, $R_n(I), R_m(I)$ be the corresponding n-refined, m-refined neutrosophic ring with $m \le n$. Then $R_m(I)$ is a homomorphic image of $R_n(I)$.

Proof:

If $m = n$ then it is clear.

Suppose that $m < n$. Then by previous lemma, we get a series of ring homomorphisms

$R_n(I) \to_{f_n} R_{n-1}(I) \to_{f_{n-1}} R_{n-2}(I) \ldots \to_{f_{n-m+1}} R_{m+1}(I) \to_{f_{n-m}} R_m(I)$.

$f_{n-m}of_{n-m+1}o \ldots of_{n-1}of_n$is a ring homomorphism between $R_n(I), R_m(I)$ since it is a product of homomorphisms, thus our proof is complete.

Example 3.3:

Let $R = Z_6$be the ring of integers modulo 6, $R_4(I) = \{a + bI_1 + cI_2 + dI_3 + eI_4; a,b,c,d,e \in R\}$ be the corresponding 4-refined neutrosophic ring, $R_3(I) = \{a + bI_1 + cI_2 + dI_3; a,b,c,d \in R\}$ be the corresponding 3-refined neutrosophic ring. We have:

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(a) $f: R_4(I) \to R_3(I); f(a + bI_1 + cI_2 + dI_3 + eI_4) = a + bI_1 + cI_2 + (d + e)I_3$is a homomorphism.

(b) $Ker(f) = \{m(I_4 - I_3);\ m \in R\} \cong R$, and $R_4(I)/Ker(f) \cong R_3(I)$.

(c) $g: R_3(I) \to R_2(I); g(a + bI_1 + cI_2 + dI_3) = a + bI_1 + (c + d)I_2$is a homomorphism too.

(d) $gof: R_4(I) \to R_2(I); gof(a + bI_1 + cI_2 + dI_3 + eI_4) = a + bI_1 + (c + d + e)I_2$is a homomorphism between $R_4(I), R_2(I)$.

Result 3.4:

According to Theorem 3.2, if$R_n(I)$ is an n-refined neutrosophic ring. Then:

$R_n(I)/K_n \cong R_{n-1}(I), R_{n-1}(I)/K_{n-1} \cong R_{n-2}(I), \dots, R_1(I)/K_1 \cong R,$Where $K_i \cong R$.

So, we have the following series of ring extensions $R \to R_1(I) \to \cdots \to R_{n-1}(I) \to R_n(I)$. For each ring

$R_m(I); 1 \le m \le n$there is a subring $K \cong R$, with property $R_m(I)/K \cong R_{m-1}(I)$.

According to the previous result, we can understand the n-refined neutrosophic ring $R_n(I)$ as an extension of R with n steps. Each step can be represented by a ring homomorphism.

Remark 3.5:

The main application of Result 3.4 that it clarifies the algebraic nature of n-refined neutrosophic idea in the case of n-refined neutrosophic ring.

Splitting I into n subindeterminacies $I_1, \dots, I_n$ is a logical idea introduced by Smarandache in [7,8]. This work shows that it has an algebraic origin in rings, since the n-refined neutrosophic ring $R_n(I)$can be considered as a classical ring extention based on classical homomorphisms.

5. Conclusion

In this article we have studied classical homomorphisms between n-refined neutrosophic ring $R_n(I)$ and m-refined neutrosophic ring $R_m(I)$, where $m \le n$. Also,we haveproved the following results:

1) Each m-refined neutrosophic ring is a homomorphic image of n-refined neutrosophic ring, where $m \le n$.
2) Each n-refined neutrosophic ring $R_n(I)$ is a ring extension of the ring R by n steps. Each one can be represented by a ring homomorphism.

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