Introduction to NeutroGroups

Agboola[1] A.A.A.
Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
agboolaaaa@funaab.edu.ng

Abstract

The objective of this paper is to formally present the concept of NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). Several interesting results and examples of NeutroGroups, NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms are presented. It is shown that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups.

Keywords: Neutrosophy, NeutroGroup, NeutroSubgroup, NeutroCyclicGroup, NeutroQuotientGroup and NeutroGroupHomomorphism.

1 Introduction

In 2019, Florentin Smarandache[2] introduced new fields of research in neutrosophy which he called NeutroStructures and AntiStructures respectively. The concepts of NeutroAlgebras and AntiAlgebras were recently introduced by Smarandache in.[3] Smarandache in[4] revisited the notions of NeutroAlgebras and AntiAlgebras where he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and showed that NeutroAlgebras are generalization of Partial Algebras. In,[1] Agboola et al examined NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. The mention of NeutroGroup by Smarandache in[2] motivated us to write the present paper. The concept of NeutroGroup is formally presented in this paper by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). We study NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. We present several interesting results and examples. It is shown that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups.

For more details about NeutroAlgebras, AntiAlgebras, NeutroAlgebraic Structures and AntiAlgebraic Structures, the readers should see.[1–4]

2 Formal Presentation of NeutroGroup and Properties

In this section, we formally present the concept of a NeutroGroup by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element) and we present its basic properties.

Definition 2.1. Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a NeutroGroup if the following conditions are satisfied:

(i) $*$ is NeutroAssociative that is there exits at least one triplet $(a, b, c) \in G$ such that

$$a * (b * c) = (a * b) * c \tag{1}$$

and there exists at least one triplet $(x, y, z) \in G$ such that

$$x * (y * z) \neq (x * y) * z. \tag{2}$$

[1]Correspondence:aaaola2003@yahoo.com

Doi :10.5281/zenodo.3840761

Received: February 26, 2020 Revised: March 17, 2020 Accepted: May19, 2020

(ii) There exists a NeutroNeutral element in G that is there exists at least an element $a \in G$ that has a single neutral element that is we have $e \in G$ such that

$$a * e = e * a = a \tag{3}$$

and for $b \in G$ there does not exist $e \in G$ such that

$$b * e = e * b = b \tag{4}$$

or there exist $e_1, e_2 \in G$ such that

$$b * e_1 = e_1 * b = b \quad \text{or} \tag{5}$$

$$b * e_2 = e_2 * b = b \quad \text{with} \quad e_1 \neq e_2 \tag{6}$$

(iii) There exists a NeutroInverse element that is there exists an element $a \in G$ that has an inverse $b \in G$ with respect to a unit element $e \in G$ that is

$$a * b = b * a = e \tag{7}$$

or there exists at least one element $b \in G$ that has two or more inverses $c, d \in G$ with respect to some unit element $u \in G$ that is

$$b * c = c * b = u \tag{8}$$

$$b * d = d * b = u. \tag{9}$$

In addition, if $*$ is NeutroCommutative that is there exists at least a duplet $(a, b) \in G$ such that

$$a * b = b * a \tag{10}$$

and there exists at least a duplet $(c, d) \in G$ such that

$$c * d \neq d * c, \tag{11}$$

then $(G, *)$ is called a NeutroCommutativeGroup or a NeutroAbelianGroup.

If only condition (i) is satisfied, then $(G, *)$ is called a NeutroSemiGroup and if only conditions (i) and (ii) are satisfied, then $(G, *)$ is called a NeutroMonoid.

Definition 2.2. Let $(G, *)$ be a NeutroGroup. G is said to be finite of order n if the cardinality of G is n that is $o(G) = n$. Otherwise, G is called an infinite NeutroGroup and we write $o(G) = \infty$.

Example 2.3. Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $G = \{a, b, c, d\}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on G as shown in the Cayley table below:

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

.

It is clear from the table that:

$$a * (b * c) = (a * b) * c = d,$$

$$b * (d * c) = a, \text{ but } (b * d) * c = b.$$

This shows that $*$ is NeutroAssociative and hence $(G, *)$ is a NeutroSemiGroup.

Next, let N_x and I_x represent the neutral element and the inverse element respectively with respect to any element $x \in G$. Then

$$N_a = d,$$

$$I_a = c,$$

$$N_b, N_c, N_d \quad \text{do not exist,}$$

$$I_b, I_c, I_d \quad \text{do not exist.}$$

This in addition to $(G, *)$ being a NeutroSemiGroup implies that $(G, *)$ is a NeutroGroup.

It is also clear from the table that $*$ is NeutroCommutative. Hence, $(G, *)$ is a NeutroAbelianGroup.

Doi :10.5281/zenodo.3840761

Example 2.4. Let $G = \mathbb{Z}_{10}$ and let $*$ be a binary operation on G defined by $x * y = x + 2y$ for all $x, y \in G$ where $+$ is addition modulo 10. Then $(G, *)$ is a NeutroAbeliaGroup. To see this, for $x, y, z \in G$, we have

$$x * (y * z) = x + 2y + 4z, \tag{12}$$
$$(x * y) * z = x + 2y + 2z. \tag{13}$$

Equating (16) and (17) we obtain $z = 0, 5$. Hence only the triplets $(x, y, 0), (x, y, 5)$ can verify associativity of $*$ and not any other triplet $(x, y, z) \in G$. Hence, $*$ is NeutroAssociative and therefore, $(G, *)$ is a NeutroSemigroup.

Next, let $e \in G$ such that $x * e = x + 2e = x$ and $e * x = e + 2x$. Then, $x + 2e = e + 2x$ from which we obtain $e = x$. But then, only $5 * 5 = 5$ in G. This shows that G has a NeutroNeutral element. It can also be shown that G has a NeutroInverse element. Hence,$(G, *)$ is a NeutroGroup.

Lastly,

$$x * y = x + 2y, \tag{14}$$
$$y * x = y + 2x. \tag{15}$$

Equating (18) and (19), we obtain $x = y$. Hence only the duplet $(x, x) \in G$ can verify commutativity of $*$ and not any other duplet $(x, y) \in G$. Hence, $*$ is NeutroCommutative and thus $(G, *)$ is a NeutroAbelianGroup.

Remark 2.5. General NeutroGroup is a particular case of general NeutroAlgebra which is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). Therefore, a NeutroGroup is a group that has either one NeutroOperation (partially well-defined, partially indeterminate, and partially outer-defined), or atleast one NeutroAxiom (NeutroAssociativity, NeutroElement, or NeutroInverse).

It is possible to define NeutroGroup in another way by considering only one NeutroAxiom or by considering two NeutroAxioms.

Theorem 2.6. *Let $(G_i, *), i = 1, 2, \cdots, n$ be a family of NeutroGroups. Then*

(i) $G = \bigcap_{i=1}^{n} G_i$ *is a NeutroGroup.*

(ii) $G = \prod_{i=1}^{n} G_i$ *is a NeutroGroup.*

Proof. Obvious. □

Definition 2.7. Let $(G, *)$ be a NeutroGroup. A nonempty subset H of G is called a NeutroSubgroup of G if $(H, *)$ is also a NeutroGroup.

The only trivial NeutroSubgroup of G is G.

Example 2.8. Let $(G, *)$ be the NeutroGroup of **Example 2.3** and let $H = \{a, c, d\}$. The compositions of elements of H are given in the Cayley table below.

$*$	a	c	d
a	b	d	a
c	d	b	d
d	a	c	a

It is clear from the table that:

$$\begin{aligned}
a * (c * d) &= (a * c) * d = a, \\
d * (c * a) &= a, \text{ but } (d * c) * a = d \neq a. \\
N_a &= d, \\
I_a &= c, \\
N_c, N_d & \quad \text{do not exist,} \\
I_c, I_d & \quad \text{do not exist.} \\
a * d &= d * a = a, \text{ but } c * d = d, d * c = c \neq d.
\end{aligned}$$

All these show that $(H, *)$ is a NeutroAbelianGroup. Since $H \subset G$, it follows that H is a NeutroSubgroup of G.

It should be observed that the order of H is not a divisor of the order of G. Hence, Lagrange's theorem does not hold.

Theorem 2.9. *Let $(G,*)$ be a NeutroGroup and let $(H_i,*), i=1,2,\cdots,n$ be a family of NeutroSubgroups of G. Then*

(i) $H=\bigcap_{i=1}^{n} H_i$ *is a NeutroSubgroup of G.*

(ii) $H=\prod_{i=1}^{n} H_i$ *is a NeutroSubgroup of G.*

Proof. Obvious. □

Definition 2.10. Let H be a NeutroSubgroup of the NeutroGroup $(G,*)$ and let $x\in G$.

(i) xH the left coset of H in G is defined by

$$xH=\{xh : h\in H\}. \tag{16}$$

(i) Hx the right coset of H in G is defined by

$$Hx=\{hx : h\in H\}. \tag{17}$$

Example 2.11. Let $(G,*)$ be the NeutroGroup of **Example 2.3** and let H be the NeutroSubgroup of **Example 2.8**. H_l the set of distinct left cosets of H in G is given by

$$H_l=\{\{a,b,d\},\{a,c\},\{b,d\}\}$$

and H_r the set of distinct right cosets of H in G is given by

$$H_r=\{\{a,b,d\},\{a,b,c\},\{b,c,d\},\{a,d\}\}.$$

It should be observed that H_l and H_r do not partition G. This is different from what is obtainable in the classical groups. However, the order of H_l is 3 which is not a divisor of the order of G and therefore, $[G:H]$ the index of H in G is 3, that is $[G:H]=3$. Also, the order of H_r is 4 which is a divisor of G. Hence, $[G:H]=4$.

Example 2.12. Let $\mathbb{U}=\{a,b,c,d\}$ be a universe of discourse and let $G=\{a,b,c\}$ be a NeutroGroup given in the Cayley table below:

$*$	a	b	c
a	a	c	b
b	c	a	c
c	a	c	d

.

Let $H=\{a,b\}$ be a NeutroSubgroup of G given in the Cayley table below:

$*$	a	b
a	a	c
b	c	a

.

Then, the sets of distinct left and right cosets of H in G are respectively obtained as:

$$\begin{aligned} H_l &= \{\{a,c\}\}, \\ H_r &= \{\{a,\},\{c\},\{b,c\}\}. \end{aligned}$$

In this example, the order of H_l the set of distinct left cosets of H in G is 1 which is not a divisor of the order of G and therefore, $[G:H]=1$. However, the order of H_r the set of distinct right cosets of H in G is 3 which is a divisor of the order of G and therefore, $[G:H]=3$. This is also different from what is obtainable in the classical groups.

Consequent on **Examples 2.8, 2.11 and 2.12**, we state the following theorem:

Theorem 2.13. *Let H be a NeutroSubgroup of the finite NeutroGroup $(G,*)$. Then generally:*

(i) $o(H)$ *is not a divisor of* $o(G)$.

(ii) There is no 1-1 correspondence between any two left(right) cosets of H in G.

(iii) There is no 1-1 correspondence between any left(right) coset of H in G and H.

(iv) If $N_x = e$ that is $xe = ex = x$ for any $x \in G$, then $eH \neq H$, $He \neq H$ and $\{e\}$ is not a NeutroSubgroup of G.

(v) $o(G) \neq [G : H]o(H)$.

(vi) The set of distinct left(right) cosets of H in G is not a partition of G.

Definition 2.14. Let $(G, *)$ be a NeuroGroup. Since $*$ is associative for at least one triplet $(x, x, x) \in G$, the powers of x are defined as follows:

$$\begin{aligned} x^1 &= x \\ x^2 &= xx \\ x^3 &= xxx \\ \vdots &\quad \vdots \\ x^n &= xxx \cdots x \quad n \text{ factors } \forall n \in \mathbb{N}. \end{aligned}$$

Theorem 2.15. *Let $(G, *)$ be a NeutroGroup and let $x \in G$. Then for any $m, n \in \mathbb{N}$, we have:*

(i) $x^m x^n = x^{m+n}$.

(ii) $(x^m)^n = x^{mn}$.

Definition 2.16. Let $(G, *)$ be a NeutroGroup. G is said to be cyclic if G can be generated by an element $x \in G$ that is

$$G =< x >= \{x^n : n \in \mathbb{N}\}. \tag{18}$$

Example 2.17. Let $(G, *)$ be the NeutroGroup given in **Example 2.3** and consider the following:

$$\begin{aligned} a^1 &= a, a^2 = b, a^3 = c, a^4 = d. \\ b^1 &= b, b^2 = d, b^3 = c, b^4 = a. \\ c^1 &= c, c^2 = b, c^3 = a, c^4 = d. \\ d^1 &= d, d^2 = a, d^3 = a, d^4 = a. \\ \therefore \quad G &= < a >=< b >=< c >, G \neq < d > . \end{aligned}$$

These show that G is cyclic with the generators a, b, c. The element $d \in G$ does not generate G.

Definition 2.18. Let H be a NeutroSubgroup of the NeutroGroup $(G, *)$. The sets $(G/H)_l$ and $(G/H)_r$ are defined by:

$$(G/H)_l = \{xH : x \in G\} \tag{19}$$

$$(G/H)_r = \{Hx : x \in G\}. \tag{20}$$

Let $xH, yH \in (G/H)_l$ and let $\odot_l$ be a binary operation defined on $(G/H)_l$ by

$$xH \odot_l yH = x * yH \quad \forall\, x, y \in G. \tag{21}$$

Also, let $xH, yH \in (G/H)_r$ and let $\odot_r$ be a binary operation defined on $(G/H)_r$ by

$$Hx \odot_r Hy = Hx * y \quad \forall\, x, y \in G. \tag{22}$$

It can be shown that the couples $((G/H)_l, \odot_l)$ and $((G/H)_r, \odot_r)$ are NeutroGroups.

Example 2.19. Let G and H be as given in **Example 2.12** and consider

$$(G/H)_l = \{aH, bH, cH\} = \{aH\} = \{\{a, c\}\}.$$

Then $((G/H)_l, \odot_l)$ is a NeutroGroup.

Definition 2.20. Let $(G, *)$ and $(H, \circ)$ be any two NeutroGroups. The mapping $\phi : G \to H$ is called a NeutroGroupHomomorphism if ϕ preserves the binary operations $*$ and $\circ$ that is if for all $x, y \in G$, we have

$$\phi(x * y) = \phi(x) \circ \phi(y). \tag{23}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_H\} \tag{24}$$

where $e_H \in H$ is such that $N_h = e_H$ for at least one $h \in H$.

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in H : y = \phi(x) \text{ for some } h \in H\}. \tag{25}$$

If in addition ϕ is a bijection, then ϕ is called a NeutroGroupIsomorphism and we write $G \cong H$. NeutroGroupEpimorphism, NeutroGroupMonomorphism, NeutroGroupEndomorphism, and NeutroGroupAutomorphism are similarly defined.

Example 2.21. Let $(G, *)$ be a NeutroGroup of **Example 2.12** and let $\psi : G \times G \to G$ be a projection given by

$$\psi((x, y)) = x \quad \forall x, y \in G.$$

Then ψ is a NeutroGroupHomomorphism. The $Ker\psi = \{(a, a), (a, b), a, c)\}$ which is a NeutroSubgroup of $G \times G$ as shown in the Cayley table below.

$*$	(a,a)	(a,b)	(a,c)
(a,a)	(a,a)	(a,c)	(a,b)
(a,b)	(a,c)	(a,a)	(a,c)
(a,c)	(a,a)	(a,c)	(a,d)

and $Im\psi = \{a, b, c\} = G$.

Consequent on **Example 2.21**, we state the following theorem:

Theorem 2.22. *Let $(G, *)$ and $(H, \circ)$ be NeutroGroups and let $N_x = e_G$ such that $e_G * x = x * e_G = x$ for at least one $x \in G$ and let $N_y = e_H$ such that $e_H * y = y * e_H = y$ for at least one $y \in H$. Suppose that $\phi : G \to H$ is a NeutroGroupHomomorphism. Then:*

(i) $\phi(e_G) = e_H$.

(ii) $Ker\phi$ *is a NeutroSubgroup of G.*

(iii) $Im\phi$ *is a NeutroSubgroup of H.*

(iv) ϕ *is injective if and only if* $Ker\phi = \{e_G\}$.

Example 2.23. Considering **Example 2.19**, let $\phi : G \to G/H$ be a mapping defined by $\phi(x) = xH$ for all $x \in G$. Then, $\phi(a) = \phi(b) = \phi(c) = aH = \{a, c\}$ from which we have that ϕ is a NeutroGroupHomomorphism.

$$Ker\phi = \{x \in G : \phi(x) = e_{G/H}\} = \{x \in G : xH = e_{G/H} = e_{\{\{a,c\}\}}\} \neq H.$$

Consequent on **Example 2.23**, we state the following theorem:

Theorem 2.24. *Let H be a NeutroSubgroup of a NeutroGroup $(G, *)$. The mapping $\psi : G \to G/H$ defined by*

$$\psi(x) = xH \quad \forall\, x \in G$$

is a NeutroGroupHomomorphism and the $Ker\psi \neq H$.

Theorem 2.25. *Let $\phi : G \to H$ be a NeutroGroupHomomorpism and let $K = Ker\phi$. Then the mapping $\psi : G/K \to Im\phi$ defined by*

$$\psi(xK) = \phi(x) \quad \forall\, x \in G$$

is a NeutroGroupEpimorphism and not a NeutroGroupIsomorphism.

Proof. That ψ is a well defined surjective mapping is clear. Let $xK, yK \in G/K$ be arbitrary. Then

$$\begin{aligned}
\psi(xKyK) &= \psi(xyK) \\
&= \phi(xy) \\
&= \phi(x)\phi(y) \\
&= \psi(xK)\psi(yK). \\
Ker\psi &= \{xK \in G/K : \psi(xK) = e_{\phi(x)}\} \\
&= \{xK \in G/K : \phi(x) = e_{\phi(x)}\} \\
&\neq \{e_{G/K}\}.
\end{aligned}$$

This shows that ψ is a surjective NeutroGroupHomomorphism and therefore it is a NeutroGroupEpimorphism. Since ψ is not injective, it follows that $G/K \ncong Im\phi$ which is different from what is obtainable in the classical groups. □

Theorem 2.26. *NeutroGroupIsomorphism of Neutrogroups is an equivalence relation.*

Proof. The same as the classical groups. □

3 Conclusion

We have formally presented the concept of NeutroGroup in this paper by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). We presented and studied several interesting results and examples on NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. We have shown that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups. Further studies of NeutroGroups will be presented in our future papers. Other NeutroAlgebraicStructures such as NeutroRings, NeutroModules, NeutroVectorSpaces etc are opened to studies for Neutrosophic Researchers.

4 Appreciation

The author is very grateful to all anonymous reviewers for valuable comments and suggestions which we have found very useful in the improvement of the paper.

References

[1] Agboola A.A.A., Ibrahim M.A. and Adeleke E.O.,"Elementary Examination of NeutroAlgebras and AntiAlgebras viz-a-viz the Classical Number Systems", International Journal of Neutrosophic Science,Volume 4 , Issue 1, pp. 16-19 , 2020.

[2] Smarandache, F., "Introduction to NeutroAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories," Pons Publishing House Brussels, Belgium, Ch. 6, pp. 240-265, 2019.

[3] Smarandache, F., "Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232.

[4] Smarandache, F., "NeutroAlgebra is a Generalization of Partial Algebra", International Journal of Neutrosophic Science, vol. 2 (1), pp. 08-17, 2020.