Some results on fixed point in generalized metric spaces via an auxiliary function

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Abstract

In this manuscript, we elegantly delineate the concept of JS_φ-contractions within the realm of JS-metric spaces, as articulated by Jleli and Samet. Utilizing these contractions, we have formulated a groundbreaking fixed point theorem that paves the way for a diverse array of fixed point results. Additionally, we demonstrate a fixed point result specifically for P-contractions in JS-metric spaces. To further enrich our discourse, we provide several examples that vividly illustrate the essence of our principal theorem.

Keywords: Fixed point; Generalized metric space; JS-metric; JS_φ-contraction

1 Introduction

Since 1922, fixed point theory has intrigued a multitude of researchers, largely due to the famous Banach's fixed point theorem.[1] This field is supported by a vast array of literature and remains a highly active and dynamic area of inquiry today. Fixed point theorems are distinguished for their focus on the existence and properties of fixed points. These theorems are of considerable importance as they provide essential methodologies for demonstrating the existence and uniqueness of solutions across various mathematical models. Such models address a broad spectrum of phenomena found in numerous disciplines, including, but not limited to, chemical equations, economic models, and fluid dynamics. Theorems of this kind are applicable in areas such as differential equations, integral and partial differential equations, variational inequalities, numerical analysis, and real analysis, among others.

Numerous researchers have investigated the concept of fixed points. In,[2] the study focused on multivalued G-contractions within the framework of Hausdorff b-Gauge spaces. The work in[3] introduced several fixed point theorems pertaining to Suzuki mappings in modular spaces. Additionally,[4] provided proofs of fixed point results specifically for weakly Chatterjea-type cyclic contractions. Furthermore, in,[5] the authors presented theorems regarding tripled coincidence points for weak Φ-contractions in partially ordered metric spaces.

The fixed point methodology has been widely utilized in numerous areas of mathematics, with particular significance in fractional calculus, as highlighted by various citations and related literature, such as.[6] By employing the Caputo-Hadamard fractional-order operator, researchers have established the existence and uniqueness of solutions for generalized Sturm-Liouville and Langevin equations, as noted in.[7] Additionally, a novel analytical approach was proposed for solving partial differential equations through the tensor product theory of Banach spaces. This method has yielded atomic precise solutions for several fractional partial differential equations within Banach spaces, as documented in.[8]

2 Preliminary

Many expansions of the notion of metric spaces have resulted in the creation of a large number of distance spaces that go beyond the conventional definition of metric spaces. Examples of these generalizations include the introduction of the idea of contraction mappings in b-metric spaces by.[9,10]In addition, a notable expansion of metric spaces, referred to as the generalized metric space or JS-metric, was proposed by Jleli and Samet[11] in 2015. This framework integrates various distance spaces and establishes fixed point theorems within this context.

In the following, we recall some basic notions related to JS-metric spaces that we discuss in this section.

Definition 2.1. [11] Let $\mathcal{H}$ be a nonempty set, and let $\Delta : \mathcal{H} \times \mathcal{H} \to [0, \infty]$ be a function. We define Δ as a JS-metric on $\mathcal{H}$ if it satisfies the following conditions.

1. For all pairs (ϑ, y) in the Cartesian product $\mathcal{H} \times \mathcal{H}$, the following conditions hold: If $\Delta(\vartheta, y) = 0$, then it follows that $\vartheta = y$.
2. The distance function satisfies $\Delta(\vartheta, y) = \Delta(y, \vartheta)$ for every (ϑ, y) in $\mathcal{H} \times \mathcal{H}$.
3. There exists a constant $C > 0$ such that for any (ϑ, y) in $\mathcal{H} \times \mathcal{H}$, if there is a sequence $\{\vartheta_n\}$ for which $C \lim_{n\to\infty} \Delta(\vartheta_n, \vartheta) = 0$, then it holds that $\Delta(\vartheta, y) \leq C \limsup_{n\to\infty} \Delta(\vartheta_n, y)$.

In this context, we denote the triple $(\mathcal{H}, \Delta, C)$ as a JS-metric space defined by a constant $C > 0$.

From this point forward, $\mathbb{R}^+$ will refer to the collection of all non-negative real numbers, while $(\mathcal{H}, \Delta, C)$ signifies a JS-metric space characterized by the constant C.

Definition 2.2. [11] In a JS-metric space $(\mathcal{H}, \Delta, C)$, consider a sequence $\{\vartheta_n\}$ within the set $\mathcal{H}$ and an element ϑ also belonging to $\mathcal{H}$. We define the sequence $\{\vartheta_n\}$ to be Δ-convergent to the element ϑ if the limit of the distance $\Delta(\vartheta_n, \vartheta)$ approaches zero as n approaches infinity. i.e. if $\lim_{n\to\infty} \Delta(\vartheta_n, \vartheta) = 0$.

Proposition 2.3. [11] *Let $(\mathcal{H}, \Delta, C)$ represent a JS-metric space, and consider a sequence $\{\vartheta_n\}$ within $\mathcal{H}$ along with a pair (ϑ, y) in the Cartesian product $\mathcal{H} \times \mathcal{H}$. If the sequence $\{\vartheta_n\}$ Δ-converges to the point ϑ and also Δ-converges to the point y, it follows that ϑ must be equal to y.*

Definition 2.4. [11] Let $(\mathcal{H}, \Delta, C)$ represent a JS-metric space. Consider a sequence $\{\vartheta_n\}$ within $\mathcal{H}$ and an element $\vartheta \in \mathcal{H}$. We define the sequence $\{\vartheta_n\}$ to be a Δ-Cauchy sequence if it holds that $\lim_{n,m\to\infty} \Delta(\vartheta_n, \vartheta_{n+m}) = 0$.

Definition 2.5. [11] Let $(\mathcal{H}, \Delta, C)$ represent a JS-metric space. It is considered Δ-complete if every Cauchy sequence within ϑ converges to an element that belongs to $\mathcal{H}$.

Jleli and Samet demonstrated several fixed point results in JS-metric spaces, a contribution that has been further explored by numerous authors since then, as referenced in.[12,13]

3 Main Results

To enhance our forthcoming results, we introduce the following class of functions that will be employed in the construction of our contraction.

Definition 3.1. Let Φ denote the collection of all functions $\varphi : \mathrm{R}^{+^2} \to \mathrm{R}^+$ that satisfy the condition

$$\varphi(a, b) \leq a + b.$$

Definition 3.2. Let $(\mathcal{H}, \Delta, C)$ represent a JS-metric space characterized by a constant $C > 0$, and let f denote a self-map on $\mathcal{H}$. We define f as a JS$_\varphi$-contraction if the following condition holds:

$$\Delta(f\vartheta, fy \leq k[\Delta(\vartheta, y) + \varphi(\Delta(\vartheta, f\vartheta), \Delta(y, fy)] \tag{1}$$

for some value of k such that $0 \leq k < \min\{\frac{1}{3}, \frac{1}{C}\}$.

Lemma 3.3. *Let $(\mathcal{H}, \Delta, C)$ be a JS-metric space with constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ be JS$_\varphi$-contraction. Then for any $\vartheta_0 \in \mathcal{H}$ with $\Delta(\vartheta_0, f\vartheta_0) < \infty$, if for each $n \in \mathbb{N}$, $\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \neq 0$, then*

$$\lim_{n\to\infty} \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) = 0.$$

Proof. Let ϑ_0 be arbitrary point of $\mathcal{H}$.Then we have

$$\begin{aligned}\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) &\leq k[\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) + \varphi(\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0), \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)]\\ &\leq k[2(\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) + \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)].\end{aligned}$$

So,

$$\Delta(f^n(\vartheta_0), f^{n+1}(\vartheta_0)) \leq \left(\frac{2k}{1-k}\right) \Delta(f^{n-1}(\vartheta_0), f^n(\vartheta_0)).$$

By continuing this process, we get

$$\Delta(f^n(\vartheta_0), f^{n+1}(\vartheta_0)) \leq \left(\frac{2k}{1-k}\right)^n \Delta(\vartheta_0, f(\vartheta_0)).$$

By taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) = 0.$$

□

Proposition 3.4. *Assume that f is a P_φ-contraction. In this case, any fixed point $\zeta \in \mathcal{H}$ of the function f fulfills the condition $\Delta(\zeta, \zeta) < \infty$, leading to the conclusion that $\Delta(\zeta, \zeta) = 0$.*

Proof. Let $\zeta \in \mathcal{H}$ denote a fixed point of the function f, where it holds that $\Delta(\zeta, \zeta) < \infty$. Given that f is a P_φ-contraction, we can conclude that.

$$\begin{aligned}\Delta(\zeta, \zeta) &= \Delta(f(\zeta), f(\zeta))\\ &\leq k(\Delta(\zeta, \zeta) + \varphi(\Delta(\zeta, f(\zeta)), \Delta(\zeta, f(\zeta))\\ &\leq k(\Delta(\zeta, \zeta) + \Delta(\zeta, f(\zeta)) + \Delta(\zeta, f(\zeta))\\ &= 3k\Delta(\zeta, \zeta).\end{aligned}$$

So, $(1 - 3k)\Delta(\zeta, \zeta) \leq 0$, which implies that $\Delta(\zeta, \zeta) = 0$. □

To enhance the efficiency of our work, we will adopt the following notation: For each element $\vartheta \in \mathcal{H}$, define $\delta(\Delta, f, \vartheta) = \sup_{i,j\in\mathbb{N}} \Delta(f^i\vartheta, f^j\vartheta)$.

Theorem 3.5. *Let $(\mathcal{H}, \Delta, C)$ represent a complete JS-metric space characterized by a constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ denote a mapping. Assume that the subsequent conditions are satisfied:*

1. *f is a JS_{φ}-contraction;*
2. *there exists $\vartheta_0 \in \mathcal{H}$ such that $\delta(\Delta, f, \vartheta_0) < \infty$;*

The sequence $\{f^n\vartheta_0\}$ converges to a point $\zeta \in \mathcal{H}$, which is a fixed point of the function f. Furthermore, if there exists another fixed point $\zeta' \in \mathcal{H}$ such that the distance $\Delta(\zeta, \zeta') < \infty$, it follows that ζ must be equal to ζ'.

Proof. if there is some $n_0 \in \mathbb{N}$ such that $\Delta(f^{n_0}(\vartheta_0), f^{n_0+1}(\vartheta_0)) = 0$, then $f^{n_0}(\vartheta_0) = f^{n_0+1}(\vartheta_0)$, and so, $f^{n_0}(\vartheta_0)$ is a fixed point for f. So, we assume that for each $n \in \mathbb{N}$, $\Delta(f^n(\vartheta_0), f^{n+1}(\vartheta_0)) \neq 0$

Now, for $i, j \in \mathbb{N}$, we have

$$\begin{aligned}\Delta(f^{n+i}(\vartheta_0), f^{n+j}(\vartheta_0)) &\leq k[\Delta(f^{n-1+i}(\vartheta_0), f^{n-1+j}(\vartheta_0)) + \varphi(\Delta(f^{n-1+i}(\vartheta_0), f^{n+i}\vartheta_0), \Delta(f^{n-1+j}\vartheta_0, f^{n+j}\vartheta_0))] \\ &\leq k[\Delta(f^{n-1+i}\vartheta_0, f^{n-1+j}\vartheta_0) + \Delta(f^{n-1+i}\vartheta_0, f^{n+i}\vartheta_0) + \Delta(f^{n-1+j}\vartheta_0, f^{n+j}\vartheta_0)].\end{aligned}$$

So,

$$\delta(D, f, f^n(\vartheta_0)) \leq 3k\delta(\Delta, f, f^{n-1}\vartheta_0).$$

By continuing this procedure, we get

$$\delta(D, f, f^n(\vartheta_0)) \leq (3k)^n\delta(\Delta, f, \vartheta_0).$$

Now, for $n, m \in \mathbb{N}$, we have

$$\Delta(f^n(\vartheta_0), f^{n+m}\vartheta_0) \leq \delta(\Delta, f, f^n\vartheta_0) \leq (3k)^n\delta(\Delta, f, \vartheta_0).$$

Since $\delta(\Delta, f, \vartheta_0) < \infty$ and $k \in (0, \frac{1}{3})$, we obtain

$$\lim_{n,m\to\infty} \Delta(f^n(\vartheta_0), f^{n+m}(\vartheta_0)) = 0,$$

which implies that $(f^n(\vartheta_0))$ is a Δ-Cauchy sequence .
Given that $(\mathcal{H}, \Delta, C)$ is complete, there exists an element $\zeta \in \mathcal{H}$ such that the sequence $(f^n(\vartheta_0))$ converges to ζ with respect to the metric Δ. According to condition 1, we have

$$\begin{aligned}\Delta(f^{n+1}(\vartheta_0), f(\zeta)) &\leq k(\Delta(f^n(\vartheta_0), \zeta) + \varphi(\Delta(f^n(\vartheta_0), f^{n+1}(\vartheta_0)), \Delta(\zeta, f(\zeta))) \\ &\leq k[\Delta(f^n\vartheta_0, \zeta) + \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) + \Delta(\zeta, f\zeta)]\end{aligned}$$

By taking limsup to both sides as $n \to \infty$, we get

$$\limsup_{n\to\infty} \Delta(f^{n+1}(\vartheta_0), f\zeta) \leq k\Delta(\zeta, f\zeta)$$

By D3, we get

$$\Delta(\zeta, f\zeta) \leq C \limsup_{n\to\infty} \Delta(f^n\vartheta_0, f\zeta)$$

So,

$$\Delta(\zeta, f\zeta) \leq Ck\Delta(\zeta, f\zeta).$$

Hence, $(1 - Ck)\Delta(\zeta, f\zeta) \leq 0$, and so, $\zeta = f\zeta$.

To prove the uniqueness, assume that ζ' is another fixed point for f such that $\Delta(\zeta, \zeta') < \infty$.
By Condition 1, we get

$$\begin{aligned}
\Delta(\zeta, \zeta') &= \Delta(f\zeta, f\zeta') \\
&\leq k[\Delta(\zeta, \zeta') + \varphi(\Delta(\zeta, f\zeta), \Delta(\zeta', f\zeta'))] \\
&\leq k[\Delta(\zeta, \zeta') + \Delta(\zeta, f\zeta) + \Delta(\zeta', f\zeta')] \\
&\leq k\Delta(\zeta, \zeta').
\end{aligned}$$

Thus, $(1 - k)\Delta(\zeta, \zeta') \leq 0$, and so, $\zeta = \zeta'$. □

Utilizing the definition of φ in conjunction with Theorem 3.5, we can derive the subsequent results.

Corollary 3.6. *Let $(\mathcal{H}, \Delta, C)$ denote a complete JS-metric space characterized by a constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ be a mapping. We assume that for every pair of elements $\vartheta, y \in \mathcal{H}$, the following inequality is satisfied:*

$$\Delta(f\vartheta, fy) \leq k[\Delta(\vartheta, y) + \Delta(\vartheta, f\vartheta) + \Delta(y, fy)],$$

where $0 \leq k < \min\{\frac{1}{3}, \frac{1}{C}\}$. Furthermore, we consider the existence of an element $\vartheta_0 \in \mathcal{H}$ such that

$$\sup\{\Delta(f^i\vartheta_0, f^j\vartheta_0) : i, j \in \mathbb{N}\} < \infty,$$

then $\{f^n\vartheta_0\}$ converges to a fixed point of the mapping f. Additionally, it is established that f has a unique fixed point.

Corollary 3.7. *Let $(\mathcal{H}, \Delta, C)$ denote a complete JS-metric space characterized by a constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ be a defined mapping. We assume that for every pair of elements $\vartheta, y \in \mathcal{H}$, the following inequality is satisfied:*

$$\Delta(f\vartheta, fy) \leq k[\Delta(\vartheta, y) + \max\{\Delta(\vartheta, f\vartheta), \Delta(y, fy)\}],$$

where $0 \leq k < \min\{\frac{1}{3}, \frac{1}{C}\}$. Furthermore, we consider the existence of an element $\vartheta_0 \in \mathcal{H}$ such that

$$\sup\{\Delta(f^i\vartheta_0, f^j\vartheta_0) : i, j \in \mathbb{N}\} < \infty,$$

then $\{f^n\vartheta_0\}$ converges to a fixed point of the mapping f. Additionally, it is established that f has a unique fixed point.

Corollary 3.8. *Let $(\mathcal{H}, \Delta, C)$ denote a complete JS-metric space characterized by a constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ be a defined mapping. We assume that for every pair of elements $\vartheta, y \in \mathcal{H}$, the following inequality is satisfied:*

$$\Delta(f\vartheta, fy) \leq k[\Delta(\vartheta, y) + 2\min\{\Delta(\vartheta, f\vartheta), \Delta(y, fy)\}],$$

where $0 \leq k < \min\{\frac{1}{3}, \frac{1}{C}\}$. Furthermore, we consider the existence of an element $\vartheta_0 \in \mathcal{H}$ such that

$$\sup\{\Delta(f^i\vartheta_0, f^j\vartheta_0) : i, j \in \mathbb{N}\} < \infty,$$

then $\{f^n\vartheta_0\}$ converges to a fixed point of the mapping f. Additionally, it is established that f has a unique fixed point.

Corollary 3.9. *Let $(\mathcal{H}, \Delta, C)$ represent a complete JS-metric space with a constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ denote a mapping. Assume that for all $\vartheta, y \in \mathcal{H}$, the following inequality holds:*

$$\Delta(f\vartheta, fy) \leq k[\Delta(\vartheta, y) + \log(1 + \Delta(\vartheta, f\vartheta) + \Delta(y, fy))],$$

for some constant $0 \leq k < \min\{\frac{1}{3}, \frac{1}{C}\}$. If there exists an element $\vartheta_0 \in \mathcal{H}$ such that

$$\sup\{\Delta(f^i\vartheta_0, f^j\vartheta_0) : i, j \in \mathbb{N}\} < \infty,$$

then the sequence $\{f^n\vartheta_0\}$ converges to a fixed point of the mapping f. Furthermore, f possesses a unique fixed point.

Example 3.10. If we take $\mathcal{H} = \mathbb{R}$ with a metric Δ defined as $\Delta(\vartheta, y) = |\vartheta - y|$. Then $(\mathbb{R}, \Delta, 1)$ is a complete JS-metric. Let $f : \mathbb{R} \to \mathbb{R}$, be defined as $f(\vartheta) = \frac{\vartheta}{4}$, and define the function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(a, b) = a$. Then,

$$\begin{aligned}
\Delta(f\vartheta, fy) &= |\tfrac{\vartheta}{4} - \tfrac{y}{4}| \\
&= \tfrac{1}{4}|\vartheta - y| \\
&\leq \tfrac{1}{4}[|\vartheta - y| + |\tfrac{3\vartheta}{4}|] \\
&= \tfrac{1}{4}[|\vartheta - y| + |\vartheta - \tfrac{\vartheta}{4}|] \\
&= \tfrac{1}{4}[\Delta(\vartheta, y) + \Delta(\vartheta, f\vartheta)] \\
&= \tfrac{1}{4}[\Delta(\vartheta, y) + \varphi(\Delta(\vartheta, f\vartheta), \Delta(y, fy))].
\end{aligned}$$

So, $\Delta(f\vartheta, fy) \leq \frac{1}{4}[\Delta(\vartheta, y) + \varphi(\Delta(\vartheta, f\vartheta), \Delta(y, fy))]$.
Hence f satisfies all conditions of Theorem 3.5, and so, f meets a unique fixed point.

Example 3.11. Let $\mathcal{H} = \mathbb{R}$, and let $\Delta : \mathcal{H} \times \mathcal{H} \to \mathrm{R}^+$ be defined by

$$\Delta(\vartheta, y) = \left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y|.$$

Then $(\mathbb{R}, \Delta, 1)$ is a JS-metric space.

Proof. We first show that Δ satisfies all conditions of Definition 2.1.

(1) if $\Delta(\vartheta, y) = 0 \Rightarrow \vartheta = y$
$\Delta(\vartheta, y) = 0 \Rightarrow \left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y| = 0 \Rightarrow |\vartheta - y| = 0 \Rightarrow \vartheta = y$

(2) $\Delta(\vartheta, y) = \Delta(y, \vartheta)$
$\Delta(\vartheta, y) = \left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y| = \left(\frac{1}{2}\right)^{|y - \vartheta|} |y - \vartheta| = \Delta(y, \vartheta)$

(3) For every $\vartheta, y \in \mathcal{H} \times \mathcal{H}, \{\vartheta_n\} \subset \mathcal{H}$ such that $\lim_{n\to\infty} \Delta(\vartheta_n, \vartheta) = 0$, we have

$$\lim_{n\to\infty} \Delta(\vartheta_n, \vartheta) = 0 \Rightarrow \lim_{n\to\infty} \left(\frac{1}{2}\right)^{|\vartheta_n - \vartheta|} |\vartheta_n - \vartheta| = 0 \Rightarrow \lim_{n\to\infty} |\vartheta_n - \vartheta| = 0.$$

Now by definition of Δ and by properties of Limsup for every $(\vartheta, y) \in \mathcal{H} \times \mathcal{H}$ we show that $\Delta(\vartheta, y) \leq C \limsup_{n\to\infty} \Delta(\vartheta_n, y)$.

Since $\{\vartheta_n\}$ converges to ϑ then $\lim_{n\to\infty} |\vartheta_n - y| = |\vartheta - y|$ this implies that

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^{|\vartheta_n - y|} |\vartheta_n - y| = \left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y|.$$

Thus, we can write

$$\left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y| \leq \limsup_{n\to\infty} \left(\frac{1}{2}\right)^{|\vartheta_n - y|} |\vartheta_n - y|.$$

So,
$\Delta(\vartheta, y) \leq \limsup_{n\to\infty} \Delta(\vartheta_n, y)$.

□

Example 3.12. If we take $\mathcal{H} = [0, 2]$ with a metric Δ defined as
$\Delta(\vartheta, y) = \left(\frac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y|$. Then, $(\mathcal{H}, \Delta, 1)$ is a complete JS-metric. Let $f : [0, 4] \to [0, 4]$, be defined as $f(\vartheta) = \frac{\vartheta}{4} + 1$, and define the function $\varphi :^+ \times^+ \to^+$ by $\varphi(a, b) = a$. Then f is P_φ-contraction.

Proof. In Example 3.11, we showed that $(\mathcal{H}, \Delta, 1)$ is a JS-metric space. Now, we show that f is a P_φ-contraction.

$$\begin{aligned}
\Delta(f\vartheta, fy) &= \left(\tfrac{1}{2}\right)^{|\frac{1}{4}\vartheta - \frac{1}{4}y|} |\tfrac{1}{4}\vartheta - \tfrac{1}{4}y| \\
&= \left(\tfrac{1}{2}\right)^{\frac{1}{4}|\vartheta - y|} \tfrac{1}{4}|\vartheta - y| \\
&= \tfrac{1}{4}[\left(\tfrac{1}{2}\right)^{\frac{1}{4}|\vartheta - y|} |\vartheta - y|] \\
&\leq \tfrac{1}{4}[\left(\tfrac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y|] \\
&\leq \tfrac{1}{4}[\left(\tfrac{1}{2}\right)^{|\vartheta - y|} |\vartheta - y| + \left(\tfrac{1}{2}\right)^{|\frac{3\vartheta}{4} - 1|} |\tfrac{3\vartheta}{4} - 1|] \\
&= \tfrac{1}{4}[\Delta(\vartheta, y) + \Delta(\vartheta, f\vartheta)] \\
&= \tfrac{1}{4}[\Delta(\vartheta, y) + \varphi(\Delta(\vartheta, f\vartheta), \Delta(y, fy))].
\end{aligned}$$

Thus,

$$\Delta(f\vartheta, fy) \leq \frac{1}{4}[\Delta(\vartheta, y) + \varphi(\Delta(\vartheta, f\vartheta), \Delta(y, fy))].$$

□

So f is P_φ-contraction.

We will subsequently demonstrate a fixed point theorem for P-contractions within the framework of JS-metric spaces.

Definition 3.13. Let $(\mathcal{H}, \Delta, C)$ represent a JS-metric space where the constant C is greater than zero. We define a function f to be a P-contraction if there exists a constant $0 \leq k < \min\{1, \frac{1}{C}\}$ such that the following inequality holds:

$$\Delta(f\vartheta, fy) \leq k \left[\Delta(\vartheta, fy) + |\Delta(\vartheta, f\vartheta) - \Delta(y, fy)|\right].$$

Theorem 3.14. *Let $(\mathcal{H}, \Delta, C)$ be a complete JS-metric space with constant $C > 0$, and let $f : \mathcal{H} \to \mathcal{H}$ be a mapping. Suppose that the following conditions hold:*

1. *f is a P-contraction;*
2. *there exists $\vartheta_0 \in \mathcal{H}$ such that $\delta(\Delta, f, \vartheta_0) < \infty$.*

Then $\{f^n(\vartheta_0)\}$ converges to $\zeta \in \mathcal{H}$, a fixed point of f. More over , if $\zeta' \in \mathcal{H}$ is another fixed point of f such that $\Delta(\zeta, \zeta') < \infty$ then $\zeta = \zeta'$

Proof. If there is $n_0 \in \mathbb{N}$ such that $\Delta(f^{n_0}\vartheta, f^{n_0+1}\vartheta) = 0$, then

$$f^{n_0}\vartheta_0 = f^{n_0+1}\vartheta_0 = ff^{n_0}\vartheta_0.$$

So, $f^{n_0}\vartheta_0$ is a fixed point for f. So, we may assume that $\forall n \in \mathbb{N},\ \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \neq 0$.
Since f is P-contraction, then

$$\begin{aligned}\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) &= \Delta(f^n\vartheta_0, ff^n\vartheta_0)\\ &\leq k[\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) + |\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) - \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)|]\end{aligned}$$

If $\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) \leq \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)$, then

$$\begin{aligned}\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) &\leq k[\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) + \Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) - \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)]\\ &\leq k[\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)]\end{aligned}$$

So, $(1-k)\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \leq 0$, which implies that $\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) = 0$, a contradiction.

Hence, $\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \leq \Delta(f^{n-1}\vartheta_0, f^n\vartheta_0)$.
So,

$$\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \leq k[2\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) - \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0)].$$

Hence,

$$\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \leq \frac{2k}{1+k}\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0).$$

By continuing this process, we get

$$\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) \leq \left(\frac{2k}{1+k}\right)^n \Delta(\vartheta_0, f\vartheta_0).$$

Thus, by taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) = 0.$$

Now, for $n, m \in \mathbb{N}$, we have

$$\begin{aligned}\Delta(f^n\vartheta_0, f^{n+m}\vartheta_0) &\leq k\left[\Delta(f^{n-1}\vartheta_0, f^{n-1+m}\vartheta_0) + \left|\Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) - \Delta(f^{n-1+m}\vartheta_0, f^{n+m}\vartheta_0)\right|\right]\\ &= k\left[\Delta(f^{n-1}\vartheta_0, f^{n-1+m}\vartheta_0) + \Delta(f^{n-1}\vartheta_0, f^n\vartheta_0) - \Delta(f^{n-1+m}\vartheta_0, f^{n+m}\vartheta_0)\right]\\ &\leq k\left[\Delta(f^{n-1}\vartheta_0, f^{n-1+m}\vartheta_0) + \Delta(f^{n-1}\vartheta_0, f^n\vartheta_0)\right].\\ &\leq k\left[\Delta(f^{n-1}\vartheta_0, f^{n-1+m}\vartheta_0) + \left(\tfrac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0)\right].\end{aligned}$$

By the same way, we get

$$\Delta(f^{n-1}\vartheta_0, f^{n-1+m}\vartheta_0) \leq k\left[\Delta(f^{n-2}\vartheta_0, f^{n-2+m}\vartheta_0) + \left(\frac{2k}{1+k}\right)^{n-2}\Delta(\vartheta_0, f\vartheta_0)\right].$$

So,

$$\Delta(f^{n}\vartheta_0, f^{n+m}\vartheta_0) \leq k^2\Delta(f^{n-2}\vartheta_0, f^{n-2+m}\vartheta_0)+k^2\left(\frac{2k}{1+k}\right)^{n-2}\Delta(\vartheta_0, f\vartheta_0)+k\left(\frac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0).$$

Continuing this process leads to

$$\begin{aligned}\Delta(f^{n}\vartheta_0, f^{n+m}\vartheta_0) &\leq k^n\Delta(\vartheta_0, f^m\vartheta_0) + \left[k^n + k^{n-1}\left(\tfrac{2k}{1+k}\right) + \cdots + k^2\left(\tfrac{2k}{1+k}\right)^{n-2} + k\left(\tfrac{2k}{1+k}\right)^{n-1}\right]\Delta(\vartheta_0, f\vartheta_0)\\ &= k^n\Delta(\vartheta_0, f^m\vartheta_0) + \left[\left(\tfrac{1+k}{2}\right)^{n-1} + \left(\tfrac{1+k}{2}\right)^{n-2} + \cdots + \left(\tfrac{1+k}{2}\right) + 1\right]k\left(\tfrac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0)\\ &\leq k^n\Delta(\vartheta_0, f^m\vartheta_0) + \left[1 + \left(\tfrac{1+k}{2}\right) + \cdots\right]k\left(\tfrac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0)\\ &= k^n\Delta(\vartheta_0, f^m\vartheta_0) + \left(\tfrac{2}{1-k}\right)k\left(\tfrac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0)\\ &\leq k^n\delta(\Delta, f, \vartheta_0) + \left(\tfrac{2}{1-k}\right)k\left(\tfrac{2k}{1+k}\right)^{n-1}\Delta(\vartheta_0, f\vartheta_0).\end{aligned}$$

Since $\delta(D, f, \vartheta_0) < \infty$ and $k < 1$ we obtain

$$\lim_{n,m\to\infty}\Delta(f^n\vartheta_0, f^{n+m}\vartheta_0) = 0.$$

Therefore, $\{f^n\vartheta_0\}$ is D-Cauchy sequence.
Given that $(\mathcal{H}, D, C)$ is D-Complete, there exists an element $\zeta \in \mathcal{H}$ such that the sequence $f^n(\vartheta_0)$ converges to ζ in the D sense.
Now,

$$\Delta(f^{n+1}\vartheta_0, f\zeta) \leq k[\Delta(f^n\vartheta_0, \zeta) + |\Delta(f^n\vartheta_0, f^{n+1}\vartheta_0) - \Delta(\zeta, f\zeta)|].$$

So,

$$\limsup_{n\to\infty}\Delta(f^{n+1}\vartheta_0, f(\zeta)) \leq k(\Delta(\zeta, f\zeta)).$$

Also, by D_3, we get

$$\begin{aligned}\Delta(\zeta, f\zeta) &\leq C\limsup_{n\to\infty}\Delta(f^n\vartheta_0, f\zeta)\\ &\leq Ck\Delta(\zeta, f\zeta).\end{aligned}$$

Hence, $(1 - Ck)\Delta(\zeta, f\zeta) \leq 0$, and so, $\Delta(\zeta, f\zeta) = 0$,which implies that ζ is a fixed point for f.

Let us consider that ζ' represents another fixed point of the function f, and that the distance between ζ and ζ' is finite, denoted as $\Delta(\zeta, \zeta') < \infty$. Given that f is a P-contraction, we can conclude that.

$$\begin{aligned}\Delta(\zeta, \zeta') &= \Delta(f\zeta, f\zeta')\\ &\leq k(\Delta(\zeta, \zeta') + |\Delta(\zeta, f\zeta) - \Delta(\zeta', f\zeta')|).\end{aligned}$$

Thus, $(1 - k)\Delta(\zeta, \zeta') \leq 0$, and so, $\Delta(\zeta, \zeta') = 0$. Hence, $\zeta = \zeta'$. □

4 Conclusion

The notion of fixed points plays a crucial role in both theoretical and practical mathematics, finding extensive applications in multiple disciplines. Building on Banach's pioneering contributions to metric spaces, numerous scholars have endeavored to extend the Banach contraction principle in various directions. In this research, we formulated several fixed point theorems within the context of JS-metric spaces, as introduced by Jleli and Samet, and provided several examples to elucidate our results. Subsequent investigations may aim to generalize our contraction mappings or examine outcomes in more expansive distance spaces.

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