Applying Block Method for the Numerical Solutions of the Second Order n-Refined Neutrosophic ODE for n=2, 3

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Abstract

In this paper, we study the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discuss the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for n=2, and n=3. In addition, we list the numerical tables of our results.

Keywords: Weak Fuzzy Complex (WFC) Numbers; Weak Fuzzy Complex Functions,; Differential Equations (DE)

1. Introduction

Numerical analysis in its various branches aims to find approximate numerical solutions and errors resulting from the approximation process for many mathematical problems that are difficult to solve using traditional methods [1-3]. Block method is considered one of the most important methods used in finding numerical solutions too many differential problems, which has many applications in various sciences such as physics, economics, and even computer science [4-6].

In [7-9], we find the concept of refined neutrosophic structures a generalization of classical. They have been studied widely in [10-12]. Many authors [13-17] have studied Neutrosophic numerical analysis with its applications, where we can find some examples for numerical algorithms and numerical tables for neutrosophic differential problems in [18-19].

In this work, we are motivated to study the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discuss the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for n=2, and n=3. In addition, we list the numerical tables of our results.

2. Main discussion

Definition 2.1

Consider the following second-order refined neutrosophic equation:

$$y_0'' + y_1''I_1 + y_2''I_2 = f(x_0 + x_1I_1 + x_2I_2, y_0 + y_1I_1 + y_2I_2, y_0' + y_1'I_1 + y_2'I_2),$$

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$$\begin{cases} (y_0 + y_1 I_1 + y_2 I_2)[a_0 + a_1 I_1 + a_2 I_2] = y_0^* + y_1^* I_1 + y_2^* I_2, \\ (y_0' + y_1' I_1 + y_2' I_2)[a_0 + a_1 I_1 + a_2 I_2] = (y_0^*)' + (y_1^*)' I_1 + (y_2^*)' I_2, \end{cases}$$

The Method:

The approximated solution of $y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2, y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2, \dots, y_{n+4}^{(0)} + y_{n+4}^{(1)} I_1 + y_{n+4}^{(2)} I_2$ are computed in the block at the points:

$x_{n+1}^{(0)} + x_{n+1}^{(1)} I_1 + x_{n+1}^{(2)} I_2, x_{n+2}^{(0)} + x_{n+2}^{(1)} I_1 + x_{n+2}^{(2)} I_2, \dots, x_{n+4}^{(0)} + x_{n+4}^{(1)} I_1 + x_{n+4}^{(2)} I_2$ in the k-block.

The refined neutrosophic interval $[a_0 + a_1 I_1 + a_2 I_2, b_0 + b_1 I_1 + b_2 I_2]$ is divided into a series of blocks can be gotten from Euler's formula.

$\left(x_n^{(i)}, f_n^{(i)}\right)$ are derived by Lagrange's interpolation polynomial, as follows:

$$(y_{n+1}')^0 + (y_{n+1}')^1 I_1 + (y_{n+1}')^2 I_2 = \left(y'^{(0)}_n + y'^{(1)}_n I_1 + y'^{(2)}_n I_2\right) + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} [A_1];$$

Where:

$A_1 = 251\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 646\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) - 246\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 106\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) - 19\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right).$

$y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+2}^{(2)} I_2 = y_n^{(0)} + y_n^{(1)} I_1 + y_n^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(y'^{(0)}_n + y'^{(1)}_n I_1 + y'^{(2)}_n I_2\right) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_2$; where:

$A_2 = 367\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 540\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) - 282\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 116\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) - 21\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

$y'^{(0)}_{n+2} + y'^{(1)}_{n+2} I_1 + y'^{(2)}_{n+2} I_2 = y'^{(0)}_{n+1} + y'^{(1)}_{n+1} I_1 + y'^{(2)}_{n+1} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_3,$

Where:

$A_3 = -19\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 346\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) + 456\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) - 74\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) + 11\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

$y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 = y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(y'^{(0)}_{n+1} + y'^{(1)}_{n+1} I_1 + y'^{(2)}_{n+1} I_2\right) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_4$; where:

$A_4 = -21\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 472\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) + 330\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) - 72\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) + 11\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

$y'^{(0)}_{n+3} + y'^{(1)}_{n+3} I_1 + y'^{(2)}_{n+3} I_2 = y'^{(0)}_{n+2} + y'^{(1)}_{n+2} I_1 + y'^{(2)}_{n+2} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_5,$

Where:

$A_5 = 11\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) - 74\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) + 456\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 346\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) - 19\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

$y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 = y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(y'^{(0)}_{n+2} + y'^{(1)}_{n+2} I_1 + y'^{(2)}_{n+2} I_2\right) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_6$; where:

$A_6 = 11\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) - 76\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) + 582\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 220\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) - 17\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

$y'^{(0)}_{n+4} + y'^{(1)}_{n+4} I_1 + y'^{(2)}_{n+4} I_2 = y'^{(0)}_{n+3} + y'^{(1)}_{n+3} I_1 + y'^{(2)}_{n+3} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_7,$

$A_7 = -19\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 106\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) - 264\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 646\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) + 251\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$

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$$y_{n+4}^{(0)} + y_{n+4}^{(1)} I_1 + y_{n+4}^{(2)} I_2 = y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(y'^{(0)}_{n+3} + y'^{(1)}_{n+3} I_1 + y'^{(2)}_{n+3} I_2\right) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_8;$$

$$A_8 = -17\left(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2\right) + 96\left(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2\right) - 246\left(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2\right) + 752\left(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2\right) + 135\left(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2\right),$$

Implementation:

For making the selection of the next step size, we use:

$$y'^{(0)}_{n+i} + y'^{(1)}_{n+i} I_1 + y'^{(2)}_{n+i} I_2 = y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)} I_1 + y'^{(2)}_{n+(i-1)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(f^{(0)}_{n+(i-1)} + f^{(1)}_{n+(i-1)} I_1 + f^{(2)}_{n+(i-1)} I_2\right),$$

$$y'^{(0)}_{n+i} + y'^{(1)}_{n+i} I_1 + y'^{(2)}_{n+i} I_2 = y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)} I_1 + y'^{(2)}_{n+(i-1)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)\left(y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)} I_1 + y'^{(2)}_{n+(i-1)} I_2\right) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{2}\left[f^{(0)}_{n+(i-1)} + f^{(0)}_{n+(i-1)} I_1 + f^{(0)}_{n+(i-1)} I_2\right] \; ; i = 1,2,3,4, \ldots..$$

The corrector iteration converges by:

$$\left| y^{c(0)}_{n+4,r+1} + y^{c(1)}_{n+4,r+1} I_1 + y^{c(2)}_{n+4,r+1} I_2 - y^{c(0)}_{n+4,r} + y^{c(1)}_{n+4,r} I_1 + y^{c(2)}_{n+4,r} I_2 \right| < 0.1 + 0.1\, I_1 + 0.1 I_2.$$

(r) is the number of iterations.

Numerical tests: (Refined neutrosophic two-body orbit)

$$\begin{cases} y_1''^{(0)} + y_1''^{(1)} I_1 + y_1''^{(2)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r I_2)^3}\left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2\right) \\ y_2''^{(0)} + y_2''^{(1)} I_1 + y_2''^{(2)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r I_2)^3}\left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2\right) \end{cases}$$

With: $\begin{cases} \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2\right)(0) = 1 \\ \left(y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2\right)(0) = 0 \end{cases}$ and:

$$\begin{cases} \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2\right)(0) = 0 \\ \left(y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2\right)(0) = 1 \\ r_0 + r_1 I_1 + r I_2 = \sqrt{\left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2\right)^2 + \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2\right)^2} \end{cases}$$

Refined neutrosophic first order systems:

$$\begin{cases} y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2 = y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2 \\ \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2\right)(0) = 1 \end{cases}$$

$$\begin{cases} y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2 = y_4^{(0)} + y_4^{(1)} I_1 + y_4^{(2)} I_2 \\ \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2\right)(0) = 0 \end{cases}$$

$$\begin{cases} y_3'^{(0)} + y_3'^{(1)} I_1 + y_3'^{(2)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r I_2)^3}\left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2\right) \\ \left(y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2\right)(0) = 0 \end{cases}$$

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$$\begin{cases} y_4'^{(0)} + y_4'^{(1)} I_1 + y_4'^{(2)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r I_2)^3} \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 \right) \\ \left(y_4^{(0)} + y_4^{(1)} I_1 + y_4^{(2)} I_2 \right)(0) = 1 \end{cases}$$

Problem (2):

Refined neutrosophic van der pol oscillator:

$(y_0 + y_1 I_1 + y_2 I_2)'' - 2m\,[1 - (y_0 + y_1 I_1 + y_2 I_2)^2](y_0 + y_1 I_1 + y_2 I_2)' + (y_0 + y_1 I_1 + y_2 I_2) = 0$, with:

$$\begin{cases} (y_0 + y_1 I_1 + y_2 I_2)(0) = 0 \\ (y_0 + y_1 I_1 + y_2 I_2)'(0) = \dfrac{1}{2} \end{cases}$$

The first order system:

$$\begin{cases} \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 \right)' = y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2, \\ \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 \right)(0) = 0 \end{cases}$$

$$\begin{cases} y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 = 2m \left[1 - \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 \right)^2 \right] \left(y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 \right)' - \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} \right); \\ m = \dfrac{25}{1000} \left(y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 \right)(0) = \dfrac{1}{2} \end{cases}$$

Table 1: Numerical results for the first problem (A)

Tol	MTD	Ts	MaxE	AVE	FCN
10^{-4}	4 PrED 4 PosB	$43+12I_1+6I_2$ $27+4I_1+6I_2$	$6.03214e\text{-}04+4.0983I_1+I_2$ $2.9181e\text{-}05+2.09987I_1+I_2$	$7.1337e\text{-}05+0.765I_1+I_2$ $9.0450e\text{-}06+0.4432I_1+I_2$	$1017+11I_1+12I_2$ $529+22I_1+6I_2$
10^{-6}	4 PrED 4 PosB	$125+I_1+I_2$ $57+3I_1+I_2$	$1.2363e\text{-}06+123I_1+I_2$ $6.4539e\text{-}06+I_1+I_2$	$2.1477e\text{-}07+13I_1+I_2$ $1.4363e\text{-}06+12I_1+I_2$	$2749+56I_1+I_2$ $909+45I_1+I_2$
10^{-8}	4 PrED 4 PosB	$207+67I_1+I_2$ $121+89I_1+I_2$	$5.9493e\text{-}09+I_1+I_2$ $3.0989e\text{-}08+I_1+I_2$	$8.8773e\text{-}10+I_1+I_2$ $6.6848e\text{-}09+I_1+I_2$	$5317+I_1+I_2$ $1937+I_1+I_2$
10^{-10}	4 PrED 4 PosB	$637+21I_1+I_2$ $261+14I_1+I_2$	$2.1775e\text{-}11+17I_1+I_2$ $1.3901e\text{-}10+12I_1+I_2$	$4.6349e\text{-}12+I_1+I_2$ $2.9203e\text{-}11+I_1+I_2$	$14861+8I_1+I_2$ $4177+16I_1+I_2$
10^{-12}	4 PrED 4 PosB	$911+I_1+I_2$ $312+I_1+I_2$	$2.172375e\text{-}11+17I_1+I_2$ $1.3901e\text{-}10+12I_1+I_2$	$4.6349e\text{-}12+I_1+I_2$ $2.9203e\text{-}11+I_1+I_2$	$15861+8I_1+I_2$ $5177+16I_1+I_2$
10^{-14}	4 PrED 4 PosB	$1013+I_1+I_2$ $402+I_1+I_2$	$2.172375e\text{-}11+17I_1+I_2$ $1.3901e\text{-}10+12I_1+I_2$	$4.6349e\text{-}12+I_1+I_2$ $2.9203e\text{-}11+I_1+I_2$	$16861+8I_1+I_2$ $4777+16I_1+I_2$
10^{-16}	4 PrED 4 PosB	$1201+I_1+I_2$ $572+I_1+I_2$	$2.172375e\text{-}11+17I_1+I_2$ $1.3901e\text{-}10+12I_1+I_2$	$4.6349e\text{-}12+I_1+I_2$ $2.9203e\text{-}11+I_1+I_2$	$18861+8I_1+I_2$ $7177+16I_1+I_2$

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Table 2: Problem (2) (A)

Tol	MTD	Ts	Absolutes error	
			$y(x)$	$y'(x)$
10^{-4}	RK 45 4 PosB	285+I_1+I_2 43+I_1+I_2	7.016927e-0410+12I_1+I_2 9.610276e-0410+12I_1+I_2	4.157618e-04+16I_1+I_2 4.888968e-04+16I_1+I_2
10^{-6}	RK 45 4 PosB	713+15I_1+21I_2 62+15I_1+33I_2	6.567087e-0611+I_1+I_2 4.267339e-011+I_1+$I_2$6	5.359170e-07+I_1+I_2 5.667588e-06+I_1+I_2
10^{-8}	RK 45 4 PosB	1789+17I_1+5I_2 122+57I_1+43I_2	5.764219e-0810+12I_1+I_2 4.399870e-0810+12I_1+I_2	2.270758e-08+16I_1+I_2 6.230616e-08+16I_1+I_2
10^{-10}	RK 45 4 PosB	4489+13I_1+5I_2 267+21I_1+5I_2	5.331400e-1011+I_1+I_2 4.851652e-111+I_1+$I_2$0	2.876250e-10+I_1+I_2 5.869338e-10+I_1+I_2

Definition 2.2 Consider the following second-order 3-refined neutrosophic equation:

$$y_0^{''} + y_1^{''}I_1 + y_2^{''}I_2 + y_3^{''}I_3 = f(x_0 + x_1I_1 + x_2I_2 + x_3I_3, y_0 + y_1I_1 + y_2I_2 + y_3I_3, y_0' + y_1'I_1 + y_2'I_2 + y_3'I_3),$$

$$\begin{cases} (y_0 + y_1I_1 + y_2I_2 + y_3I_3)[a_0 + a_1I_1 + a_2I_2 + a_3I_3] = y_0^* + y_1^*I_1 + y_2^*I_2 + y_3^*I_3, \\ (y_0' + y_1'I_1 + y_2'I_2 + y_3'I_3)[a_0 + a_1I_1 + a_2I_2 + a_3I_3] = (y_0^*)' + (y_1^*)'I_1 + (y_2^*)'I_2 + (y_3^*)'I_3, \end{cases}$$

The Method:

The approximated solution of $y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+1}^{(2)}I_2 + y_{n+1}^{(3)}I_3, y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3, \dots, y_{n+4}^{(0)} + y_{n+4}^{(1)}I_1 + y_{n+4}^{(2)}I_2 + y_{n+4}^{(3)}I_3$ are computed in the block at the points:

$x_{n+1}^{(0)} + x_{n+1}^{(1)}I_1 + x_{n+1}^{(2)}I_2 + x_{n+1}^{(3)}I_3, x_{n+2}^{(0)} + x_{n+2}^{(1)}I_1 + x_{n+2}^{(2)}I_2 + x_{n+2}^{(3)}I_3, \dots, x_{n+4}^{(0)} + x_{n+4}^{(1)}I_1 + x_{n+4}^{(2)}I_2 + x_{n+4}^{(3)}I_3$ in the k-block.

The refined neutrosophic interval $[a_0 + a_1I_1 + a_2I_2 + a_3I_3, b_0 + b_1I_1 + b_2I_2 + b_3I_3]$ is divided into a series of blocks can be gotten from Euler's formula.

$(x_n^{(i)}, f_n^{(i)})$ are derived by Lagrange's interpolation polynomial, as follows:

$$(y'_{n+1})^0 + (y'_{n+1})^1I_1 + (y'_{n+1})^2I_2 + (y'_{n+1})^3I_3 = \left(y'^{(0)}_n + y'^{(1)}_nI_1 + y'^{(2)}_nI_2 + y'^{(3)}_nI_2\right) + \frac{h_0 + h_1I_1 + h_2I_2 + h_3I_3}{720}[A_1];$$

Where:

$A_1 = 251\left(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3\right) + 646\left(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3\right) - 246\left(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3\right) + 106\left(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3\right) - 19\left(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3\right).$

$y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3 = y_n^{(0)} + y_n^{(1)}I_1 + y_n^{(2)}I_2 + y_n^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)\left(y'^{(0)}_n + y'^{(1)}_nI_1 + y'^{(2)}_nI_2 + y'^{(3)}_nI_3\right) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_2$; where:

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$A_2 = 367(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 540(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 282\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 116\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 21\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$y'^{(0)}_{n+2} + y'^{(1)}_{n+2}I_1 + y'^{(2)}_{n+2}I_2 + y'^{(3)}_{n+2}I_3 = y'^{(0)}_{n+1} + y'^{(1)}_{n+1}I_1 + y'^{(2)}_{n+1}I_2 + y'^{(3)}_{n+1}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_3,$

Where:

$A_3 = -19(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 346(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 456\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) - 74\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 11\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3 = y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+1}^{(2)}I_2 + y_{n+1}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'^{(0)}_{n+1} + y'^{(1)}_{n+1}I_1 + y'^{(2)}_{n+1}I_2 + y'^{(3)}_{n+1}I_3) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_4$; where:

$A_4 = -21(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 472(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 330\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) - 72\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 11\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$y'^{(0)}_{n+3} + y'^{(1)}_{n+3}I_1 + y'^{(2)}_{n+3}I_2 + y'^{(3)}_{n+3}I_2 = y'^{(0)}_{n+2} + y'^{(1)}_{n+2}I_1 + y'^{(2)}_{n+2}I_2 + y'^{(3)}_{n+2}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_5,$

Where:

$A_5 = 11(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) - 74(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 456\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 346\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 19\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$y_{n+3}^{(0)} + y_{n+3}^{(1)}I_1 + y_{n+3}^{(2)}I_2 + y_{n+3}^{(3)}I_3 = y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'^{(0)}_{n+2} + y'^{(1)}_{n+2}I_1 + y'^{(2)}_{n+2}I_2) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_6$; where:

$A_6 = 11(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) - 76(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 582\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 220\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 17\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$y'^{(0)}_{n+4} + y'^{(1)}_{n+4}I_1 + y'^{(2)}_{n+4}I_2 + y'^{(3)}_{n+4}I_3 = y'^{(0)}_{n+3} + y'^{(1)}_{n+3}I_1 + y'^{(2)}_{n+3}I_2 + y'^{(3)}_{n+3}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_7,$

$A_7 = -19(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 106(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 264\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 646\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 251(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

$$\begin{aligned} y_{n+4}^{(0)} + y_{n+4}^{(1)}I_1 + y_{n+4}^{(2)}I_2 + y_{n+4}^{(3)}I_3 &= y_{n+3}^{(0)} + y_{n+3}^{(1)}I_1 + y_{n+3}^{(2)}I_2 + y_{n+3}^{(3)}I_3 \\ &+ (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'^{(0)}_{n+3} + y'^{(1)}_{n+3}I_1 + y'^{(2)}_{n+3}I_2 + y'^{(3)}_{n+3}I_3) \\ &+ \frac{(h_0 + h_1I_1 + h_2I_2 + h_3I_3)^2}{1440}A_8; \end{aligned}$$

$A_8 = -17(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 96(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 246\,(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 752\,(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 135\,(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$

Implementation:

For making the selection of the next step size, we use:

$y'^{(0)}_{n+i} + y'^{(1)}_{n+i}I_1 + y'^{(2)}_{n+i}I_2 + y'^{(3)}_{n+i}I_3 = y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)}I_1 + y'^{(2)}_{n+(i-1)}I_2 + y'^{(3)}_{n+(i-1)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)\left(f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(1)}I_1 + f_{n+(i-1)}^{(2)}I_2 + f_{n+(i-1)}^{(3)}I_3\right),$

$y'^{(0)}_{n+i} + y'^{(1)}_{n+i}I_1 + y'^{(2)}_{n+i}I_2 + y'^{(3)}_{n+i}I_3 = y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)}I_1 + y'^{(2)}_{n+(i-1)}I_2 + y'^{(3)}_{n+(i-1)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)\left(y'^{(0)}_{n+(i-1)} + y'^{(1)}_{n+(i-1)}I_1 + y'^{(2)}_{n+(i-1)}I_2 + y'^{(3)}_{n+(i-1)}I_3\right) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{2}\left[f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(0)}I_1 + f_{n+(i-1)}^{(0)}I_2 + f_{n+(i-1)}^{(0)}I_3\right]$; $i = 1,2,3,4, \ldots..$

The corrector iteration converges by:

$\left|y^{c(0)}_{n+4,r+1} + y^{c(1)}_{n+4,r+1}I_1 + y^{c(2)}_{n+4,r+1}I_2 + y^{c(3)}_{n+4,r+1}I_3 - y^{c(0)}_{n+4,r} - y^{c(1)}_{n+4,r}I_1 - y^{c(2)}_{n+4,r}I_2 - y^{c(3)}_{n+4,r}I_2\right| <$ $0.1 + 0.1\,I_1 + 0.1I_2 + 0.1I_3$.

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(r) is the number of iterations.

Numerical tests: (Refined neutrosophic two-body orbit)

$$\begin{cases} y_1^{''(0)} + y_1^{''(1)} I_1 + y_1^{''(2)} I_2 + y_1^{''(3)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r_2 I_2 + r_3 I_3)^3} \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 + y_1^{(3)} I_3\right) \\ y_2^{''(0)} + y_2^{''(1)} I_1 + y_2^{''(2)} I_2 + y_2^{''(3)} I_2 = \dfrac{-1}{(r_0 + r_1 I_1 + r_2 I_2 + r_3 I_3)^3} \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3\right) \end{cases}$$

With: $\begin{cases} \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 + y_1^{(3)} I_3\right)(0) = 1 \\ \left(y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2 + y_1'^{(3)} I_3\right)(0) = 0 \end{cases}$ and:

$$\begin{cases} \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3\right)(0) = 0 \\ \left(y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2 + y_2'^{(3)} I_3\right)(0) = 1 \\ r_0 + r_1 I_1 + r_2 I_2 + r_3 I_3 = \sqrt{\left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 + y_1^{(3)} I_3\right)^2 + \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3\right)^2} \end{cases}$$

Refined neutrosophic first order systems:

$$\begin{cases} y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2 + y_1'^{(3)} I_3 = y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2 + y_3^{(3)} I_3 \\ \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 + y_1^{(3)} I_3\right)(0) = 1 \end{cases}$$

$$\begin{cases} y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2 + y_2'^{(3)} I_3 = y_4^{(0)} + y_4^{(1)} I_1 + y_4^{(2)} I_2 + y_4^{(3)} I_3 \\ \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3\right)(0) = 0 \end{cases}$$

$$\begin{cases} y_3'^{(0)} + y_3'^{(1)} I_1 + y_3'^{(2)} I_2 + y_3'^{(3)} I_3 = \dfrac{-1}{(r_0 + r_1 I_1 + r_2 I_2 + r_3 I_3)^3} \left(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 + y_1^{(3)} I_3\right) \\ \left(y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2 + y_3^{(3)} I_3\right)(0) = 0 \end{cases}$$

$$\begin{cases} y_4'^{(0)} + y_4'^{(1)} I_1 + y_4'^{(2)} I_2 + y_4'^{(3)} I_3 = \dfrac{-1}{(r_0 + r_1 I_1 + r_2 I_2 + r_3 I_3)^3} \left(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3\right) \\ \left(y_4^{(0)} + y_4^{(1)} I_1 + y_4^{(2)} I_2 + y_4^{(3)} I_3\right)(0) = 1 \end{cases}$$

Problem (2):

Refined neutrosophic van der pol oscillator:

$(y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3)'' - 2m\,[1 - (y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3)^2](y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3)' + (y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3) = 0$, with:

$$\begin{cases} (y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3)(0) = 0 \\ (y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3)'(0) = \dfrac{1}{2} \end{cases}$$

The first order system:

$$\begin{cases} \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 + y_2^{(1)} I_3\right)' = y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 + y_2^{(3)} I_3, \\ \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 + y_2^{(1)} I_3\right)(0) = 0 \end{cases}$$

$$\begin{cases} y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 + y_2^{(2)} I_3 = 2m\left[1 - \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)} I_2 + y_2^{(1)} I_3\right)^2\right] \\ \left(y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 + y_2^{(2)} I_3\right)' - \left(y_0^{(1)} + y_1^{(1)} I_1 + y_2^{(1)}\right); m = \dfrac{25}{1000} \\ \left(y_0^{(2)} + y_1^{(2)} I_1 + y_2^{(2)} I_2 + y_2^{(2)} I_3\right)(0) = \dfrac{1}{2} \end{cases}$$

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Table 3: Numerical results for the first problem (B)

Tol	MTD	Ts	Max E	AVE	FCN
10^{-4}	4 PrED 4 PosB	$43+12I_1+6I_2+53I_3$ $27+4I_1+6I_2+53I_3$	6.03214e-04+4.0983I_1+I_2 + 53I_3 2.9181e-05+2.09987I_1+I_2	7.1337e-05+0.765I_1+I_2 9.0450e-+53I_3 06+0.4432I_1+I_2 + 53I_3	$1017+11I_1+12I_2+53I_3$ $529+22I_1+6I_2+53I_3$
10^{-6}	4 PrED 4 PosB	$125+I_1+I_2+53I_3$ $57+3I_1+I_2+53I_3$	1.2363e-06+123I_1+I_2 + 53I_3 6.4539e-06+I_1+I_2 + 53I_3	2.1477e-07+13I_1+I_2 1.4363e-+53I_3 06+12I_1+I_2 + 53I_3	$2749+56I_1+I_2+53I_3$ $909+45I_1+I_2+53I_3$
10^{-8}	4 PrED 4 PosB	$207+67I_1+I_2+53I_3$ $121+89I_1+I_2+53I_3$	5.9493e-09+I_1+I_2 3.0989e-08+I_1+I_2	8.8773e-10+I_1+I_2 6.6848e-09+I_1+I_2	$5317+I_1+I_2$ $1937+I_1+I_2$
10^{-10}	4 PrED 4 PosB	$637+21I_1+I_2+53I_3$ $261+14I_1+I_2+53I_3$	2.1775e-11+17I_1+I_2 1.3901e-+53I_3 10+12I_1+I_2 + 53I_3	4.6349e-12+I_1+I_2 + 53I_3 2.9203e-11+I_1+I_2 + 53I_3	$14861+8I_1+I_2+53I_3$ $4177+16I_1+I_2+53I_3$
10^{-12}	4 PrED 4 PosB	$911+I_1+I_2+53I_3$ $312+I_1+I_2+53I_3$	2.172375e-11+17I_1+I_2 1.3901e-+53I_3 10+12I_1+I_2 + 53I_3	4.6349e-12+I_1+I_2 + 53I_3 2.9203e-11+I_1+I_2 + 53I_3	$15861+8I_1+I_2+53I_3$ $5177+16I_1+I_2+53I_3$
10^{-14}	4 PrED 4 PosB	$1013+I_1+I_2+53I_3$ $402+I_1+I_2+53I_3$	2.172375e-11+17I_1+I_2 + 53I_3 1.3901e-10+12I_1+I_2	4.6349e-12+I_1+I_2 + 53I_3 2.9203e-11+I_1+I_2	$16861+8I_1+I_2+53I_3$ $4777+16I_1+I_2$
10^{-16}	4 PrED 4 PosB	$1201+I_1+I_2+53I_3$ $572+I_1+I_2+53I_3$	2.172375e-11+17I_1+I_2 + 53I_3 1.3901e-10+12I_1+I_2 + 53I_3	4.6349e-12+I_1+I_2 + 53I_3 2.9203e-11+I_1+I_2 + 53I_3	$18861+8I_1+I_2+53I_3$ $7177+16I_1+I_2+53I_3$

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Table 4: Problem (2) (B)

Tol	**MTD**	**Ts**	**Absolutes error**	
			$y(x)$	$y'(x)$
10^{-4}	RK 45 4 PosB	$285+I_1+I_2+22I_3$ $43+I_1+I_2+22I_3$	7.016927e-0410+12I_1+I_2 + 22I_3 9.610276e-0410+12I_1+I_2 + 22I_3	4.157618e-04+16I_1+I_2 + 22I_3 4.888968e-04+16I_1+I_2 + 22I_3
10^{-6}	RK 45 4 PosB	$713+15I_1+21I_2+22I_3$ $62+15I_1+33I_2+22I_3$	6.567087e-0611+I_1+I_2 + 22I_3 4.267339e-011+I_1+I_2 + 22I_3	5.359170e-07+I_1+I_2 5.667588e-06+I_1+I_2 + 22I_3
10^{-8}	RK 45 4 PosB	$1789+17I_1+5I_2+22I_3$ $122+57I_1+43I_2+22I_3$	5.764219e-0810+12I_1+I_2 4.399870e-0810+12I_1+I_2	2.270758e-08+16I_1+I_2 6.230616e-08+16I_1+I_2
10^{-10}	RK 45 4 PosB	$4489+13I_1+5I_2+22I_3$ $267+21I_1+5I_2+22I_3$	5.331400e-1011+I_1+I_2 + 22I_3 4.851652e-111+I_1+I_2 + 22I_3	2.876250e-10+I_1+I_2 + 22I_3 5.869338e-10+I_1+I_2 + 22I_3

3. Conclusion

In this paper, we have studied the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discussed the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for n=2, and n=3. In addition, we listed the numerical tables of our results.

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References

[1] J. B. Rosser, Runge-Kutta for all seasons, in *SIAM Rev.* 9, 417-452 (1967).
[2] L. F. Shampine and H. F. Watts, Block implicit one-step method, in *Math. Comp*, 23, 731-740(1969).
[3] P. B. Worland, Parallel methods for the numerical solutions of ordinary differential equations, in *IEEE Transactions on Computers*, 25, 1045-1048 (1976).
[4] Z. A. Majid, M. B. Suleiman, F. Ismail and M. Othman, 2-point implicit block one-step method half Gauss-Seidel for solving first order ordinary differential equations, in *Matematika, Jabatan UTM*, 19, 91-100 (2003).
[5] P. C. Chakravarti and P. B. Worland, A class of self-starting methods for the numerical solution of y f (x, y), in *BIT*, 11, 368-383 (1971).

DOI: https://doi.org/10.54216/IJNS.250420

[6] Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Al-Qadri, M. O., & Al-Husban, A. (2024). On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2≤ 3. *Neutrosophic Sets and Systems*, 68, 8-25.

[7] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, *Neutrosophic Sets and Systems*, 67,169-178.

[8] S. H. Karim, A. S. Jain, and R. Y. Choudhary, "Properties of neutrosophic ideals in semigroups and their applications," *Journal of Mathematical Structures*, vol. 48, no. 4, pp. 217-230, 2022.

[9] S. M. R. S. V. Sarma, R. Kumar, and P. V. S. R. K. Kumar, "Complex cubic intuitionistic fuzzy sets and their applications to semigroups and ring theory," *Mathematical Logic Quarterly*, vol. 64, no. 2, pp. 150-160, 2024.

[10] F. Smarandache, "Neutrosophic singular boundary value problems and their applications," *Neutrosophic Sets and Systems*, vol. 35, pp. 108-115, 2023.

[11] J. Zhang and Q. Yang, "The k-threshold conversion number for graph products and its generalizations in neutrosophic logic," *Mathematical Logic Quarterly*, vol. 61, no. 6, pp. 612-627, 2023.

[12] Raed Hatamleh, Abdallah Al-Husban, K. Sundareswari, G.Balaj, M.Palanikumar (2025). Complex Tangent Trigonometric Approach Applied to (γ, τ)-Rung Fuzzy Set using Weighted Averaging, Geometric Operators and its Extension. *Communications on Applied Nonlinear Analysis*, 32 (5), PP: 133-144.

[13] M. G. Ali, S. M. Amin, and H. R. Ghaffari, "Generalized weighted operators and their applications in fuzzy decision-making systems based on trigonometric functions," *Journal of Applied Mathematics and Computational Mechanics*, vol. 20, no. 4, pp. 113-127, 2024.

[14] A. M. Peralta and D. Cabezas, "Linear orthogonality preservers between function spaces associated with commutative JB*-triples," *J. Math. Phys.*, vol. 63, no. 5, p. 053501, 2022, doi: 10.1063/5.0083580.

[15] Zolotarev, V. A., & Koudelka, L. (2021). The behavior of fuzzy systems under uncertainty in environmental modeling. *Journal of Computational and Applied Mathematics*, 381, 112874.

[16] E. G. Belyaev and A. I. Lisin, "Commutative and Noncommutative Operator Models in Quantum Information Theory," *J. Math. Phys.*, vol. 62, no. 7, p. 073502, 2021, doi: 10.1063/5.0052819.

[17] M. L. Liao, Z. S. Zhou, and Z. X. Liu, "A new approach for solving fuzzy Diophantine equations with applications to cryptography," *Fuzzy Optimization and Decision Making*, vol. 22, no. 3, pp. 293-307, 2023.

[18] X. Liu, H. Li, and H. Zhang, "A hybrid cryptographic protocol based on fuzzy logic and split-complex numbers," *Journal of Mathematical Cryptography*, vol. 27, pp. 45-60, 2024.

[19] T. A. Doe, P. R. Singh, and L. V. Gupta, "Study of neutrosophic ideals in ternary semigroups and their algebraic properties," *Journal of Algebraic Structures and Applications*, vol. 42, no. 3, pp. 155-170, 2024..

DOI: https://doi.org/10.54216/IJNS.250420