# Geometric Properties of Neutrosophic $q$ -Poisson distribution Series through $\mathfrak{P}_{\mathfrak{m}\aleph}$ Operator

**Layla Esmet Jalil[1,*], Mohammad El-Ityan[2], Rafid Habib Buti[3]**
[1]Department of Mathematics, College of Science University of Kirkuk -Kirkuk, Iraq
[2]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan
[3]Department of Mathematics and Computer Applications, College of Science, Al Muthanna University, Iraq
Emails: laylaismet@uokirkuk.edu.iq; Mohammad65655vv22@gmail.com; Sci.rafid@mu.edu.iq

**Abstract**

This paper investigates the $\mathfrak{P}_{\mathfrak{m}\aleph}$ operator, constructed from the Neutrosophic $q$-Poisson distribution series. The study examines this operator within the realm of geometric function theory, focusing on key characteristics such as coefficient bounds, growth and distortion behavior, and the determination of convexity and star likeness radii. Additionally, the paper explores the weighted and arithmetic means of functions associated with this operator and analyzes its closure properties under the Hadamard product.

**Keywords:** Neutrosophic$q$-Poisson distribution; Coefficient bounds; Growth; Hadamard product; Pmℵ operator

## 1. Introduction

Letting $\mathfrak{Z}$ represent the class of function $\mathfrak{J}$ that are analytic within the open unit disk $\mathcal{D} = \{z: z \in \mathbb{C}, |z| < 1\}$ and can be expressed in the form.

$$\mathfrak{J}(z) = z + \sum_{\mathfrak{m}=2}^{\infty} a_{\mathfrak{m}} z^{\mathfrak{m}} \quad (1.1)$$

Suppose $\mathfrak{M}$ represent the subclass of functions within $\mathfrak{Z}$ characterized by the following form:

$$\mathfrak{J}(z) = z - \sum_{\mathfrak{m}=2}^{\infty} a_{\mathfrak{m}} z^{\mathfrak{m}} \quad (1.2)$$

If $\mathfrak{J}(z), \mathcal{G}(z) \in \mathfrak{M}$ then their Hadamard product is

$$\mathfrak{J}(z) = z - \sum_{\mathfrak{m}=2}^{\infty} a_{\mathfrak{m}} z^{\mathfrak{m}}, \mathcal{G}(z) = z - \sum_{\mathfrak{m}=2}^{\infty} b_{\mathfrak{m}} z^{\mathfrak{m}} \quad (1.3)$$

A function $\mathfrak{J}(z)$ is considered starlike in the domain $\mathcal{D}$ if $\mathfrak{J}: \mathcal{D} \to \mathbb{C}$ is univalent and maps $\mathcal{D}$ onto a starlike region with respect to the origin. Furthermore, $\mathfrak{J}(z) \in \mathfrak{Z}$ is defined to be starlike of order $\propto$ if it satisfies the following condition.

$$\Re e\left(\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}\right) > \propto \quad (1.4)$$

for some $\propto$, $0 \le \propto < 1$, and for all $z \in \mathcal{D}$, refer to [1, 2].



DOI: https://doi.org/10.54216/IJNS.250419

A univalent function $\mathfrak{I}(z) \in \mathcal{S}$ is considered convex of order $\propto$ if and only if $z\mathfrak{I}'(z)$ is starlike of order $\propto$. To rephrase, if

$$\Re e\left(1+\frac{z\mathfrak{I}''(z)}{\mathfrak{I}'(z)}\right)>\propto \tag{1.5}$$

for some $\propto$, $0 \leq \propto < 1$, and for all $z \in \mathcal{D}$. Furthermore, a univalent function $\mathfrak{I}(z) \in \mathcal{S}$ is said to be close-to-convex of order $\propto$ if

$$\Re e\left(z\mathfrak{I}'(z)\right)>\propto \tag{1.6}$$

for some $\propto$, $0 \leq \propto < 1$, and for all $z \in \mathcal{D}$. Symbolize by $S^*_\propto$ and $K_\propto$ the classes of univalent starlike and univalent convex functions of order $\propto$, respectively see[2].

Neutrosophic theory, introduced by Florentin Smarandache, enhances classical and fuzzy logic by incorporating three elements: truth (T), falsehood (F), and indeterminacy (I). It is designed to handle uncertainty, ambiguity, as well as contradictions, making it suitable for the examination of complex and unclear phenomena.

This concept is widely employed in fields like artificial intelligence, decision-making, and data analysis, offering solutions for problems associated with inadequate or inconsistent information. Neutrosophic concepts have shown effective in disciplines such as engineering, computer science, as well as social sciences over time [3-5].

[6,7] A discrete random variable $\Theta$ is described as following a $q$-Poisson distribution if it takes the values 0, 1, 2, 3, . . . with corresponding probabilities.

$$e_q^{-\mathcal{M}}, \frac{\mathcal{M}e_q^{-\mathcal{M}}}{[1]_q!}, \frac{\mathcal{M}^2 e_q^{-\mathcal{M}}}{[2]_q!}, \frac{\mathcal{M}^3 e_q^{-\mathcal{M}}}{[3]_q!}, \cdots \tag{1.7}$$

where $\mathcal{M}$ is the parameter, $e_q^{\Theta}$ is the $q$-exponential function defined as

$$e_q^{\Theta} = 1 + \frac{\Theta}{[1]_q!} + \frac{\Theta^2}{[2]_q!} + \frac{\Theta^3}{[3]_q!} + \cdots = \sum_{m=0}^{\infty}\frac{\Theta^m}{[m]_q!} \tag{1.8}$$

$[m]_q! = [m]_q[m-1]_q, \cdots, [2]_q[1]_q$ is the $q$-factorial function see[8].

For a $q$-Poisson distribution, the probability mass function is expressed as

$$\mathcal{P}_q(\Theta = m) = \frac{e_q^{-\mathcal{M}}\mathcal{M}^m}{[m]_q!}, m = 0,1,2,\ldots. \tag{1.9}$$

The corresponding $q$-Poisson distribution series is defined by

$$\mathcal{P}_q(z) = z + \sum_{m=2}^{\infty}\frac{e_q^{-\mathcal{M}}\mathcal{M}^{m-1}}{[m-1]_q!}z^m, z \in \mathcal{D} \tag{1.10}$$

The radius of convergence for this series is infinite, as determined by the ratio test [7]. Extending this concept, neutrosophic theory—introduced by Smarandache in 1995—provides a framework for handling imprecise parameters. For instance, the neutrosophic $q$-Poisson distribution modifies the classical distribution by considering an interval-valued parameter $m\aleph$. Its probability mass function is

$$\aleph\mathcal{P}_q(\Theta = k) = \frac{e_q^{-m\aleph}(m\aleph)^k}{[k]_q!} \tag{1.11}$$

where $\aleph = d + I$ represents a neutrosophic statistical number, and both the expectation and variance are $m\aleph$. The neutrosophic $q$-Poisson distribution series is given by

$$\mathbb{M}_q(m\aleph, z) = z + \sum_{m=2}^{\infty}\frac{e_q^{-m\aleph}(m\aleph)^{m-1}}{[m-1]_q!}z^m, z \in \mathcal{D} \tag{1.12}$$

According to [3, 9], the linear operator $\mathfrak{P}_{m\aleph}: \mathcal{S} \to \mathcal{S}$, defined via convolution (Hadamard product), is given by

$$\mathfrak{P}_{m\aleph}(\mathfrak{I}(z)) = \mathbb{M}_q(m\aleph, z) * \mathfrak{I}(z) = z + \sum_{m=2}^{\infty}\frac{e_q^{-m\aleph}(m\aleph)^{m-1}}{[m-1]_q!}a_m z^m, z \in \mathcal{D} \tag{1.13}$$

This new class is introduced as a generalization of the classes defined in [2, 10, 11].



DOI: https://doi.org/10.54216/IJNS.250419

**Definition:** A function $\mathfrak{I}(\mathfrak{z})$ in $\mathfrak{M}$ is in the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$ if and only if it satisfies the condition:

$$\left|\frac{\varrho\mathfrak{z}^2[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]'' + \gamma\left(\mathfrak{z}[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' - \mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))\right)}{\delta\lambda[\mathfrak{z}\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' + (1-\gamma)\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))}\right| < \eta$$

where $\mathfrak{z} \in \mathcal{D}$, $0 \le \gamma < 1$, $0 \le \lambda < 1$, $0 < \eta < 1$, $\delta \ge 0$, and $0 \le \varrho \le 1$.

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This article gives linear model, which is the direct simplex method using neutrosophic logic, the logic that is the new vision of modelling and is designed to effectively address the uncertainties inherent in the real world founded by the Romanian mathematician Florentine Smarandache [1, 2]. In addition to that, Ahmed A. Salama presented the theory of neutrosophic classical categories as a generalization of the theory of classical categories [12,20], also, he developed, introduced, and formulated new concepts in the various disciplinary of mathematics, statistics, computer science by neutrosophic theory [17,18,19,22,28].

## 2. Coefficient Estimate

This theorem delineates the requisite and adequate condition for a function to be classified inside the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$.

**Theorem 2.1**. Define the function $\mathfrak{I}$ as specified in (1.2). Then $\mathfrak{I}(\mathfrak{z})$ is an element of $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$ if and only if

$$\sum_{\mathfrak{m}=2}^{\infty} (\mathfrak{m}\aleph)^{\mathfrak{m}-1} e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\, a_{\mathfrak{m}} \le \eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}! \qquad (2.1)$$

where $\mathfrak{z} \in \mathcal{D}$, $0 \le \gamma < 1$, $0 \le \lambda < 1$, $0 < \eta < 1$, $\delta \ge 0$, and $0 \le \varrho \le 1$.

The result (2.1) is sharp for the function

$$\mathfrak{I}(\mathfrak{z}) = \mathfrak{z} - \frac{\eta(\lambda\delta+1-\gamma)[\mathfrak{m}-1]_{q}!}{\sum_{\mathfrak{m}=2}^{\infty} (\mathfrak{m}\aleph)^{\mathfrak{m}-1} e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}\mathfrak{z}^{\mathfrak{m}}. \quad (2.2)$$

**Proof**: Assume that inequality (2.1) is valid and that $|\mathfrak{z}| = 1$. Subsequently, we acquire

$$\left|\varrho\mathfrak{z}^2[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]'' + \gamma\left(\mathfrak{z}[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' - \mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))\right)\right| - \eta\left|\delta\lambda[\mathfrak{z}\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' + (1-\gamma)\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))\right|$$

(2.3)

$$= \left|-\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}\mathfrak{z}^{\mathfrak{m}}\right| - \eta\left|(\delta\lambda+(1-\gamma))\mathfrak{z} - \sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}\delta\lambda+1-\gamma)\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}\mathfrak{z}^{\mathfrak{m}}\right| \qquad (2.4)$$

$$\le \sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}} - \eta(\mathfrak{m}\delta\lambda+1-\gamma) \le 0$$

(2.5)

Hence, by the maximum modulus principle, $\mathfrak{I}(\mathfrak{z}) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$,

Now, assume that $\mathfrak{I}(\mathfrak{z}) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho, \gamma, \lambda, \eta)$ so that.

$$\left|\frac{\varrho z^2[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]'' + \gamma\left(\mathfrak{z}[\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' - \mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))\right)}{\delta\lambda[\mathfrak{z}\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))]' + (1-\gamma)\,\mathfrak{P}_{\mathfrak{m}\aleph}(\mathfrak{I}(\mathfrak{z}))}\right| < \eta$$



DOI: https://doi.org/10.54216/IJNS.250419

Hence

$$\left|\varrho z^2[\mathfrak{P}_{\mathfrak{m}\aleph}(\Im(z))]'' + \gamma\left(z[\mathfrak{P}_{\mathfrak{m}\aleph}(\Im(z))]' - \mathfrak{P}_{\mathfrak{m}\aleph}(\Im(z))\right)\right| - \eta\left|\delta\lambda[z\,\mathfrak{P}_{\mathfrak{m}\aleph}(\Im(z))]' + (1-\gamma)\,\mathfrak{P}_{\mathfrak{m}\aleph}(\Im(z))\right|$$

Therefore, we get:

$$\left|-\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_q^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_q!}a_{\mathfrak{m}}z^{\mathfrak{m}}\right| - \eta\left|(\delta\lambda+(1-\gamma))z - \sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}\delta\lambda+1-\gamma)\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_q^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_q!}a_{\mathfrak{m}}z^{\mathfrak{m}}\right|$$

Thus

$$\sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_q^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_q!}a_{\mathfrak{m}} \le \eta(\delta\lambda+1-\gamma) \tag{2.6}$$

Thus, the proof is complete.

**Corollary 2.2**. If $\Im(z) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$ then we have:

$$a_{\mathfrak{m}} \le \frac{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_q!}{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_q^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]} \tag{2.7}$$

## 3. Growth and Distortion Theorems

In this section, we delineate the growth and distortion constraints for the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.

**Theorem 3.1.** If $\Im(z)$ is an analytic function given by (1.2) and $\Im(z) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$, then for $0 < |z| = r < 1$,

$$|\Im(z)| \ge r - \frac{\eta(\delta\lambda+1-\gamma)}{(\mathfrak{m}\aleph)\,e_q^{-\mathfrak{m}\aleph}[2\varrho+\gamma+\eta(2\delta\lambda+1-\gamma)]}r^2$$

and

$$|\Im(z)| \le r + \frac{\eta(\delta\lambda+1-\gamma)}{(\mathfrak{m}\aleph)\,e_q^{-\mathfrak{m}\aleph}[2\varrho+\gamma+\eta(2\delta\lambda+1-\gamma)]}r^2 \tag{3.1}$$

These bounds are precise, as they are achieved by the function.

$$\Im(z) = z - \frac{\eta(\delta\lambda+1-\gamma)}{(\mathfrak{m}\aleph)\,e_q^{-\mathfrak{m}\aleph}[2\varrho+\gamma+\eta(2\delta\lambda+1-\gamma)]}z^2 \tag{3.2}$$

Proof. According to Theorem 2.1, we get

$$\sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_q^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_q!}a_{\mathfrak{m}} \le \eta(\delta\lambda+1-\gamma)$$

and

$$(\mathfrak{m}\aleph)e_q^{-\mathfrak{m}\aleph}[2\varrho+\gamma+\eta(2\delta\lambda+1-\gamma)]\sum_{\mathfrak{m}=2}^{\infty}a_{\mathfrak{m}} \le \eta(\delta\lambda+1-\gamma), \tag{3.3}$$

Thus,

$$\sum_{\mathfrak{m}=2}^{\infty}a_{\mathfrak{m}} \le \frac{\eta(\delta\lambda+1-\gamma)}{(\mathfrak{m}\aleph)e_q^{-\mathfrak{m}\aleph}[2\varrho+\gamma+\eta(2\delta\lambda+1-\gamma)]} \tag{3.4}$$

For $\Im(z) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$ , we obtain:



DOI: https://doi.org/10.54216/IJNS.250419

$$|\mathfrak{I}(z)| = \left| z - \sum_{m=2}^{\infty} a_m z^m \right| \leq |z| + |z|^2 \sum_{m=2}^{\infty} a_m$$
$$\leq r + \frac{\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r^2 \qquad (3.5)$$

Similarly,

$$|\mathfrak{I}(z)| = \left| z - \sum_{m=2}^{\infty} a_m z^m \right| \geq |z| - |z|^2 \sum_{m=2}^{\infty} a_m$$
$$\geq r - \frac{\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r^2 \qquad (3.6)$$

This completes the proof.

**Theorem 3.2.** If $\mathfrak{I}$ is an analytic function defined by (1.2) within the class $\mathcal{F}_{m\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$, then for $0 < |z| = r < 1$,

$$|\mathfrak{I}'(z)| \geq 1 - \frac{2\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r,$$

and

$$|\mathfrak{I}'(z)| \leq 1 + \frac{2\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r. \qquad (3.7)$$

The result is exact for the function $\mathfrak{I}(z)$defined in equation (3.1).

**Proof:** For $\mathfrak{I}(z) \in \mathcal{F}_{m\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$, we have

$$|\mathfrak{I}'(z)| = \left| 1 - \sum_{m=2}^{\infty} m a_m z^{m-1} \right| \leq 1 + |z| \sum_{m=2}^{\infty} m a_m \qquad (3.8)$$

Substituting the bounds of $\sum_{m=2}^{\infty} m a_m$, we obtain

$$|\mathfrak{I}'(z)| \leq 1 + \frac{2\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r. \qquad (3.9)$$

Alternatively, we also obtain.

$$|\mathfrak{I}'(z)| = \left| 1 - \sum_{m=2}^{\infty} m a_m z^{m-1} \right| \geq 1 - |z| \sum_{m=2}^{\infty} m a_m \qquad (3.10)$$

which leads to

$$|\mathfrak{I}'(z)| \geq 1 - \frac{2\eta(\delta\lambda + 1 - \gamma)}{(m\aleph)\, e_q^{-m\aleph}[2\varrho + \gamma + \eta(2\delta\lambda + 1 - \gamma)]} r. \qquad (3.11)$$

This completes the proof.

## 4. Convex Set

**Theorem 4.1.** The class $\mathcal{F}_{m\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$ is a convex set.

**Proof.** Letting $\mathfrak{I}$ and $\mathcal{G}$ be functions belonging to the class $\mathcal{F}_{m\aleph}(\varrho, \gamma, \lambda, \delta, \eta)$. We need to demonstrate that for every$0 \leq \varrho \leq 1$, the function formed by the convex combination of $\mathfrak{I}$ and $\mathcal{G}$ also belongs to the same class.

$$(1 - \varrho)\mathfrak{I}(z) + \varrho\mathcal{G}(z) \in \mathcal{F}_{m\aleph}(\varrho, \gamma, \lambda, \eta) \qquad (4.1)$$



DOI: https://doi.org/10.54216/IJNS.250419

We have

$$(1-\varrho)\Im(\mathfrak{z})+\varrho\mathcal{G}(\mathfrak{z})=\mathfrak{z}-\sum_{\mathfrak{m}=2}^{\infty}[(1-\varrho)a_{\mathfrak{m}}+\varrho b_{\mathfrak{m}}]\,\mathfrak{z}^{\mathfrak{m}} \tag{4.2}$$

Using Theorem 2.1, we find

$$\sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}[(1-\varrho)a_{\mathfrak{m}}+\varrho b_{\mathfrak{m}}]$$
$$=(1-\varrho)\sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}$$
$$+\varrho\sum_{\mathfrak{m}=2}^{\infty}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}}{[\mathfrak{m}-1]_{q}!}b_{\mathfrak{m}}. \tag{4.3}$$

Since both $\Im(\mathfrak{z})$ and $\mathcal{G}(\mathfrak{z})$ satisfy the class conditions, it follows that:
$\leq(1-\varrho)\eta(\delta\lambda+1-\gamma)+\varrho\eta(\delta\lambda+1-\gamma)=\eta(\delta\lambda+1-\gamma)$ (4.4)

Thus,

$$(1-\varrho)\Im(\mathfrak{z})+\varrho g(\mathfrak{z})\in\mathcal{F}_{\mathfrak{m}\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$$

completing the proof.

**5. Radii of Convexity and Star likeness**

In the subsequent theorems, we ascertain the radii of convexity and starlikeness for the functions belonging to the class $\mathcal{F}_{\mathfrak{m}\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.

**Theorem 5.1** Let $\Im(\mathfrak{z})\in\mathcal{F}_{\mathfrak{m}\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$. Then the function $\Im(\mathfrak{z})$ is univalent convex of order $\psi(0\leq\psi<1)$ in the disk $|\mathfrak{z}|<\mathcal{R}_1$, where

$$\mathcal{R}_1=\inf_{\mathfrak{m}\geq2}\left[\frac{(1-\psi)(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\mathfrak{m}(\mathfrak{m}-\psi))(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}\right]^{\frac{1}{\mathfrak{m}-1}} \tag{5.1}$$

The outcome is sharp for the function $\Im(\mathfrak{z})$ given by (2.2).

**Proof:** It is enough to show that

$$\left|\frac{\mathfrak{z}\Im''(\mathfrak{z})}{\Im'(\mathfrak{z})}\right|\leq1-\psi,\ 0\leq\psi<1 \tag{5.2}$$

for $|z|<\mathcal{R}_1$. Expanding $\Im'(\mathfrak{z})$ in a power series, we have:

$$\Im'(\mathfrak{z})=1-\sum_{\mathfrak{m}=2}^{\infty}\mathfrak{m}a_{\mathfrak{m}}\mathfrak{z}^{\mathfrak{m}-1} \tag{5.3}$$

Thus,

$$\left|\frac{\mathfrak{z}\Im''(\mathfrak{z})}{\Im'(\mathfrak{z})}\right|\leq\frac{\sum_{\mathfrak{m}=2}^{\infty}\mathfrak{m}(\mathfrak{m}-1)a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}}{1-\sum_{\mathfrak{m}=2}^{\infty}\mathfrak{m}a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}} \tag{5.4}$$

For the inequality to hold, it suffices that:

$$\frac{\sum_{\mathfrak{m}=2}^{\infty}\mathfrak{m}(\mathfrak{m}-1)a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}}{1-\sum_{\mathfrak{m}=2}^{\infty}\mathfrak{m}a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}}\leq1-\psi \tag{5.5}$$

Using Theorem 2.1, this is true if:

$$\frac{\mathfrak{m}(\mathfrak{m}-\psi))}{1-\psi}|\mathfrak{z}|^{\mathfrak{m}-1}\leq\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!} \tag{5.6}$$

Then



DOI: https://doi.org/10.54216/IJNS.250419

$$|\mathfrak{z}|^{\mathfrak{m}-1} = \inf_{\mathfrak{m}\geq 2}\left[\frac{(1-\psi)(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)(\mathfrak{m}(\mathfrak{m}-\psi))[\mathfrak{m}-1]_{q}!}\right] \quad (5.7)$$

Setting $|\mathfrak{z}| = \mathcal{R}_1$, we get the desired result.

**Theorem 5.2** Let $\mathfrak{I}(\mathfrak{z}) \in \mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$. Then the function $\mathfrak{I}(\mathfrak{z})$ is univalent starlike of order $\psi(0 \leq \psi < 1)$ in the disk $|\mathfrak{z}| < \mathcal{R}_2$, where

$$\mathcal{R}_2 = \inf_{\mathfrak{m}\geq 2}\left[\frac{(1-\psi)(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\mathfrak{m}-\psi)(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}\right]^{\frac{1}{\mathfrak{m}-1}} \quad (5.8)$$

The result is definitive for the function $\mathfrak{I}(\mathfrak{z})$ as specified in (2.2).

**Proof:** It suffices to demonstrate that:

$$\left|\frac{\mathfrak{z}\mathfrak{I}'(\mathfrak{z})}{\mathfrak{I}(\mathfrak{z})}-1\right| \leq 1-\psi,\ 0\leq\psi<1 \quad (5.9)$$

for $|\mathfrak{z}| < R_2$. Expanding $\mathfrak{I}(\mathfrak{z})$ in a power series, we have:

$$\left|\frac{\mathfrak{z}\mathfrak{I}'(\mathfrak{z})}{\mathfrak{I}(\mathfrak{z})}-1\right| \leq \frac{\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}-1)a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}}{1-\sum_{\mathfrak{m}=2}^{\infty}a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}} \quad (5.10)$$

Thus,

$$\frac{\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}-1)a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}}{1-\sum_{\mathfrak{m}=2}^{\infty}a_{\mathfrak{m}}|\mathfrak{z}|^{\mathfrak{m}-1}} \leq 1-\psi. \quad (5.11)$$

For the inequality to hold, it suffices that

$$\frac{\mathfrak{m}-\psi}{1-\psi}|\mathfrak{z}|^{\mathfrak{m}-1} \leq \frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!} \quad (5.12)$$

Using Theorem 2.1, this is true if

$$|z|^{\mathfrak{m}-1} \leq \frac{(1-\psi)(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\mathfrak{m}-\psi)(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!} \quad (5.13)$$

Setting $|z| = \mathcal{R}_2$, we get the desired result.

## 6. Weighted Mean and Arithmetic Mean

**Definition 6.1:** The weighted mean $\mathfrak{D}_{\zeta}(\mathfrak{z})$ of $\mathfrak{I}(\mathfrak{z})$ and $\mathcal{G}(\mathfrak{z})$ is defined by

$$\mathfrak{D}_{\zeta}(\mathfrak{z}) = \frac{1}{2}[(1-\zeta)\mathfrak{I}(\mathfrak{z})+(1+\zeta)\mathcal{G}(\mathfrak{z})], \qquad 0<\zeta<1. \quad (6.1)$$

**Theorem 6.2:** Let $\mathfrak{I}(\mathfrak{z})$ and $\mathcal{G}(\mathfrak{z})$ be in the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$. Then the weighted mean of $\mathfrak{I}(\mathfrak{z})$ and $\mathcal{G}(\mathfrak{z})$ is also in the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.

**Proof:** According to the definition of the weighted mean, we have

$$\mathfrak{D}_{\zeta}(\mathfrak{z}) = \frac{1}{2}[(1-\zeta)\mathfrak{I}(\mathfrak{z})+(1+\zeta)\mathcal{G}(\mathfrak{z})] = \mathfrak{z}-\sum_{\mathfrak{m}=2}^{\infty}\frac{1}{2}[(1-\zeta)a_{\mathfrak{m}}+(1+\zeta)b_{\mathfrak{m}}]\mathfrak{z}^{\mathfrak{m}} \quad (6.2)$$

Since $\mathfrak{I}(\mathfrak{z})$ and $g(\mathfrak{z})$ are in the class $\mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$, by Theorem 2.1, we have

$$\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]a_{\mathfrak{m}} \leq \eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!$$

(6.3)

and

$$\sum_{\mathfrak{m}=2}^{\infty}(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]b_{\mathfrak{m}} \leq \eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!$$

(6.4)

Thus,



DOI: https://doi.org/10.54216/IJNS.250419

$$\sum_{\mathrm{m}=2}^{\infty}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]\frac{(\mathrm{m}\aleph)^{\mathrm{m}-1}e_{q}^{-\mathrm{m}\aleph}}{[\mathrm{m}-1]_{q}!}\left[\frac{1}{2}(1-\zeta)a_{\mathrm{m}}+\frac{1}{2}(1+\zeta)b_{\mathrm{m}}\right]\le\eta(\delta\lambda+1-\gamma)$$

(6.5)

$$=\sum_{\mathrm{m}=2}^{\infty}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]\frac{(\mathrm{m}\aleph)^{\mathrm{m}-1}e_{q}^{-\mathrm{m}\aleph}}{[\mathrm{m}-1]_{q}!}\left(\frac{1}{2}(1-\zeta)a_{\mathrm{m}}\right)\quad(6.6)$$

$$+\sum_{\mathrm{m}=2}^{\infty}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]\frac{(\mathrm{m}\aleph)^{\mathrm{m}-1}e_{q}^{-\mathrm{m}\aleph}}{[\mathrm{m}-1]_{q}!}\left(\frac{1}{2}(1+\zeta)b_{\mathrm{m}}\right)$$

$$=\frac{1}{2}\eta(1-\zeta)(\delta\lambda+1-\gamma)+\frac{1}{2}\eta(1+\zeta)(\delta\lambda+1-\gamma)=\eta(\delta\lambda+1-\gamma)\quad(6.7)$$

This shows that $\mathfrak{D}_{\zeta}(z)\in\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.

**Theorem 6.3**: Let $\mathfrak{I}_1(z),\mathfrak{I}_2(z),\ldots,\mathfrak{I}_k(z)$, defined by

$$\mathfrak{I}_{\tau}(z)=z-\sum_{\mathrm{m}=2}^{\infty}a_{\mathrm{m},\tau}z^{\mathrm{m}},\ \left(a_{\mathrm{m},\tau}\ge0,\tau=1,2,\ldots,\kappa,\mathrm{m}\ge2\right)\quad(6.7)$$

be members of the class $\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$. Then the arithmetic mean of $\mathfrak{I}_{\tau}(z)$, where $\tau=1,2,\ldots,\kappa$, defined by

$$\hbar(z)=\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}\mathfrak{I}_{\tau}(z)\quad(6.8)$$

is also in the class $\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.
**Proof:** By the definitions of $\mathfrak{I}_{\tau}(z)$ and $\hbar(z)$, we have:

$$\hbar(z)=\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}\left(z-\sum_{\mathrm{m}=2}^{\infty}a_{\mathrm{m},\tau}z^{\mathrm{m}}\right)=z-\sum_{\mathrm{m}=2}^{\infty}\left(\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}a_{\mathrm{m},\tau}\right)z^{\mathrm{m}}\quad(6.9)$$

Since $\mathfrak{I}_{\tau}(z)\in\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$ for every $\tau=1,2,\ldots,\kappa$, by Theorem 2.1, we have:

$$\sum_{\mathrm{m}=2}^{\infty}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]\frac{(\mathrm{m}\aleph)^{\mathrm{m}-1}e_{q}^{-\mathrm{m}\aleph}}{[\mathrm{m}-1]_{q}!}\left(\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}a_{\mathrm{m},\tau}\right)\quad(6.10)$$

$$\le\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}\left(\sum_{\mathrm{m}=2}^{\infty}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]\frac{(\mathrm{m}\aleph)^{\mathrm{m}-1}e_{q}^{-\mathrm{m}\aleph}}{[\mathrm{m}-1]_{q}!}a_{\mathrm{m},\tau}\right)\quad(6.11)$$

$$\le\frac{1}{\kappa}\sum_{\tau=1}^{\kappa}\eta(\delta\lambda+1-\gamma)=\eta(\delta\lambda+1-\gamma)$$

The proof is complete.

## 7. Hadamard Product

In the following theorem, we obtain the convolution result for functions belonging to the class $\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$.
**Theorem 7.1:** Let $\mathfrak{I},\mathcal{G}\in\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$. Then $\mathfrak{I}*\mathcal{G}\in\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$ for

$$\mathfrak{I}(z)=z-\sum_{\mathrm{m}=2}^{\infty}a_{\mathrm{m}}z^{\mathrm{m}},\ \mathcal{G}(z)=z-\sum_{\mathrm{m}=2}^{\infty}b_{\mathrm{m}}z^{\mathrm{m}}\quad(7.1)$$

and

$$(\mathfrak{I}*g)(z)=z-\sum_{\mathrm{m}=2}^{\infty}a_{\mathrm{m}}b_{\mathrm{m}}z^{\mathrm{m}}\quad(7.2)$$

where

$$\xi\le\frac{\left(\eta^{2}(\mathrm{m}-1)(\delta\lambda+1-\gamma)[\mathrm{m}-1]_{q}!\,(\varrho\mathrm{m}+\gamma)\right)}{(m_{\aleph})^{\mathrm{m}-1}e_{q}^{-m\aleph}[(\mathrm{m}-1)(\varrho\mathrm{m}+\gamma)+\eta(\mathrm{m}\delta\lambda+1-\gamma)]^{2}-\left(\eta^{2}(\delta\lambda+1-\gamma)(\mathrm{m}\delta\lambda+1-\gamma)[\mathrm{m}-1]_{q}!\right)}.$$

**Proof:** Since $\mathfrak{I}(z)\in\mathcal{F}_{\mathrm{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$, we have



DOI: https://doi.org/10.54216/IJNS.250419

$$\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}\leq 1 \quad (7.3)$$

and similarly for $\mathcal{G}\in\mathcal{F}_{\mathfrak{m}\aleph}(\varrho,\gamma,\lambda,\delta,\eta)$,

$$\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}b_{\mathfrak{m}}\leq 1 \quad (7.4)$$

We need to find the smallest $\xi$ such that

$$\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\xi(\mathfrak{m}\delta\lambda+1-\gamma)]}{\xi(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}b_{\mathfrak{m}}\leq 1 \quad (7.5)$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}\sqrt{a_{\mathfrak{m}}b_{\mathfrak{m}}}\leq 1 \quad (7.6)$$

Thus, it is sufficient to show that

$$\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\xi(\mathfrak{m}\delta\lambda+1-\gamma)]}{\xi(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}a_{\mathfrak{m}}b_{\mathfrak{m}}$$

$$\leq\sum_{\mathfrak{m}=2}^{\infty}\frac{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}\sqrt{a_{\mathfrak{m}}b_{\mathfrak{m}}} \quad (7.7)$$

Rewriting this, we get

$$\sqrt{a_{\mathfrak{m}}b_{\mathfrak{m}}}\leq\frac{[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\xi}{[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\xi(\mathfrak{m}\delta\lambda+1-\gamma)]\eta} \quad (7.8)$$

From (7.6), we know

$$\sqrt{a_{\mathfrak{m}}b_{\mathfrak{m}}}\leq\frac{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]} \quad (7.9)$$

Therefore, it suffices to show that

$$\frac{\eta(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!}{(\mathfrak{m}\aleph)^{\mathfrak{m}-1}e_{q}^{-\mathfrak{m}\aleph}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]}$$

$$\leq\frac{[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]\xi}{[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\xi(\mathfrak{m}\delta\lambda+1-\gamma)]\eta}. \quad (7.10)$$

Simplifying this, we obtain

$$\xi\leq\frac{\left(\eta^{2}(\mathfrak{m}-1)(\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!\,(\varrho\mathfrak{m}+\gamma)\right)}{(m_{\aleph})^{\mathfrak{m}-1}e_{q}^{-m_{\aleph}}[(\mathfrak{m}-1)(\varrho\mathfrak{m}+\gamma)+\eta(\mathfrak{m}\delta\lambda+1-\gamma)]^{2}-\left(\eta^{2}(\lambda+1-\gamma)(\mathfrak{m}\delta\lambda+1-\gamma)[\mathfrak{m}-1]_{q}!\right)}$$

## 8. Conclusion

This paper analyzed the $\mathfrak{P}_{\mathfrak{m}\aleph}$ operator, which is derived from the Neutrosophic $q$-Poisson distribution series, within the context of geometric function theory. The analysis uncovered key properties such as coefficient bounds, growth and distortion limits, and the radii of convexity and star likeness, providing deeper insights into the function classes associated with this operator. Furthermore, the research examined the weighted and arithmetic means of functions under the operator and explored its closure properties in the context of the Hadamard product, demonstrating its adaptability and potential for further applications. These results contribute to the development of mathematical frameworks in geometric function theory and its practical uses.

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