Relations between W_d-fuzzy implication algebras and other logical algebras

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Abstract

In this paper, we continue the study W_d-fuzzy implication algebras which are subalgebras of fuzzy implication algebras. Properties and axiomatic systems for W_d-fuzzy implication algebras are presented, then a few new results on W_d-fuzzy implication algebras have been added. We showed that there are relations between W_d-fuzzy implication algebras and some of other fuzzy logical algebras such as FI-algebras, RFI-algebras, CFI-algebras, HFI-algebras. In particular, the relations between W_d-fuzzy implication algebras and L-algebras are investigated, and we prove that every W_d-fuzzy implication algebras is a proper subclass of L-algebras. Finally, we introduce the notions of GW_d-FI algebras, whose some properties of it are investigated. The relations between distributive GW_d-FI-algebras, Hilbert algebras, BE-algebras and W-eo algebras have been obtained.

Keywords: Fuzzy implication algebra; W_d-Fuzzy implication algebras; L-algebra; GW_d-fuzzy implication algebras

1 Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [1]. Wu introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990 [2]. Recently, some authors studied fuzzy implications from different perspectives [3]. Naturally, it is meaningful to investigating the common properties of some important fuzzy implications used in fuzzy logic. Various interesting properties of FI-algebras [4,5], regular FI-algebras [1,6], W_d- FI-algebras [7], and other kinds of FI-algebras [8] were reported and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed [2,3,9]. Relationships among FI-algebra and BCK-algebra [10,11], MV-algebras [12], Rough algebras [13], BL-algebras were partly investigated, and FI-algebras were axiomatized [14,15].

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump (2008) [16]. Many examples shown that L-algebras are very useful. Yang and Rump (2012)[17], characterized pseudo-MV-algebras and Bosbachs non-commutative bricks as L-algebras. Wu and Yang proved that orthomodular lattices form a special class of L-algebras in different ways (2020) [18]. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. (2019) [19]. Basic algebras and L-algebras, State L-algebras and derivations of L-algebras, L-algebras in logic, algebra, geometry, and topology, Prime L-algebras and right-angled Artin groups were systematically discussed [20-24].

The paper is organized as follows: preliminary notions and results are introduced in Section 2. Main properties of W_d -FI algebras are included in Section 3. In Section 4, relationships between W_d -FI algebras and several classes of important logical algebras are discussed. In Sect.5, we introduce a new algebraic structures known as generalized W_d -fuzzy implication algebras, and we construct some examples to show the existence of the structures.

2 Preliminary notions of W_d -FI algebras

In this part, we firstly review some relevant concepts and definitions.

Definition 2.1 ([2]) Let X be a set with the algebraic constant $0 \in X$, and $\to$ be a binary operation on X, where $0 \to 0 = 1$. A (2,0)-type algebra $(X, \to, 0)$ is called a fuzzy implication algebra, shortly, FI-algebra, if the following five conditions hold for all $x, y, z \in X$:

$(I_1) x \to (y \to z) = y \to (x \to z)$;

$(I_2)(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

$(I_3) x \to x = 1$;

$(I_4) x \to y = y \to x = 1 \Rightarrow x = y$;

$(I_5) 0 \to x = 1$.

Example1. Let $X = \{0, a, b, 1\}$ be a finite set of distinct elements, and $0 < a < b < 1$. We define

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

It is easy to check that $(X, \to, 0)$ is an FI- algebra.

Definition 2.2 ([2]) Let X be a set with the algebraic constant $0 \in X$, and $\to$ be a binary operation on X, where $1 = 0 \to 0$. A (2,0)-type algebra $(X, \to, 0)$ is called an Heyting fuzzy implication algebra, shortly, HFI-algebra, if the following five conditions hold for all $x, y, z \in X$:

$(HFI_1) x \to (y \to x) = 1$;

$(HFI_2)(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

$(HFI_3) 1 \to x = 1 \Rightarrow x = 1$;

$(HFI_4) x \to y = y \to x = 1 \Rightarrow x = y$;

$(HFI_5) 0 \to x = 1$.

On an FI-algebra $(X, \to, 0)$, one can define a binary relation $\leqslant$ and operators $', T, S$ as follows.

$$x \leqslant y \Longleftrightarrow x \to y = 1, x, y \in X; \quad (1)$$

$$x^{'} = x \to 0, x \in X; \quad (2)$$

$$T(x,y) = (x \to y^{'})^{'}, x, y \in X. \quad (3)$$

$$S(x,y) = x^{'} \to y, x, y \in X. \quad (4)$$

Usually, we also say that X is an FI-algebra for convenience.

Obviously, the relation "$\leqslant$" is a partial ordering on X, i.e., the relation is reflexive, antisymmetric and transitive(see [2]). In fuzzy logic, the property (1) is called the ordering property.

The operator"$'$" defined in the above definition is a negation on X, i.e., the operator is order-inverting and satisfies $0^{'} = 1$, and $1^{'} = 0$. "$\leqslant$" and "$'$"are called the partial ordering and the negation induced by the FI-algebra X, respectively.

Definition 2.3 ([2]) Let X be an FI-algebra.

(i) X is a regular FI-algebra, or an RFI-algebra, if $x^{''} = x$, for all $x \in X$.

(ii) X is commutative, or a CFI-algebra, if the binary operation "T" defined by (3) is commutative, or the following condition (I_6) holds for all $x, y \in x$:

$(I_6)(x \to y) \to y = (y \to x) \to x.$

For the related background of W_d - Fuzzy implication algebra, we refer to Deng and Li(1996). A W_d - Fuzzy implication algebra is an algebra of type(2,0). The notion was first formulated in 1996 by Deng and some properties were obtained[7]. This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu [2].

Definition 2.4 ([7]) A (2,0)-type algebra $(X, \to, 0)$ is a W_d-Fuzzy implication algebra, shortly, W_d-FI algebra, if the following conditions hold for all $x, y, z \in X$:

$(W_1)x \to (y \to z) = y \to (x \to z);$

$(W_2)(x \to y) \to z = (z \to y) \to x;$

$(W_3)x \to x = 1;$

$(W_4)x \to y = y \to x = 1 \Rightarrow x = y;$

$(W_5)0 \to x = 1$, where $1 = 0 \to 0$.

Example 2. Consider $X = [0,1]$, for every $x, y \in X$, defined $x \to y = 1$, then $(X, \to, 0)$ is a W_d-FI algebra.

Example 3. Let $X = \{0, a, 1\}$ be a finite set of distinct elements. We define

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Then $(X, \to, 0)$ is a FI- algebra, but not W_d-FI algebras. In fact, $(1 \to a) \to 0 = a \to 0 = 0$, but $(0 \to a) \to 1 = 1 \to 1 = 1$, so (W_2) does not hold.

In Sect.4, we will prove that any W_d -FI algebra is an FI-algebra, but the inverse is not true. Every W_d-FI algebra is an RFI- algebra, we show that an RFI- algebra is W_d-FI algebra is not correct in general.

3 Main properties of W_d -FI algebra

Given a W_d -FI algebra $(X, \rightarrow, 1)$, a binary relation " $\leqslant$ " is defined by

$$x \leqslant y \Leftrightarrow x \rightarrow y = 1,$$

Then " $\leqslant$ " is a partial order which is called the natural order. Relative to the natural order on X, 1 is the greatest element.

In fact, by Definition 2.4 , $(W_3) x \rightarrow x = 1$ we get $x \leqslant x$, i.e., reflexive hold, using (W_4), we have antisymmetric are satisfied,i.e., $x \leqslant y, y \leqslant x \Rightarrow x = y$. Next, by $x \rightarrow z = x \rightarrow (1 \rightarrow z) = x \rightarrow ((y \rightarrow z) \rightarrow z) = x \rightarrow ((z \rightarrow z) \rightarrow y) = x \rightarrow (1 \rightarrow y) = x \rightarrow y = 1$. Hence, transitive hold. So, $\leqslant$ is a partial order on X.

Proposition 3.1 If $(X, \rightarrow, 0)$ is a W_d -FI algebra and $x, y, z \in X$, then:

$(W_6) x \rightarrow 1 = 1, 1 \rightarrow x = x,$

$(W_7) x \rightarrow y = 1, y \rightarrow z = 1 \Rightarrow x \rightarrow z = 1,$

$(W_8)(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1,$

$(W_9)(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1,$

$(W_{10})(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = x.$

Proof. (W_6) Indeed $x \rightarrow 1 = x \rightarrow (0 \rightarrow x) = 0 \rightarrow (x \rightarrow x) = 0 \rightarrow 1 = 1$. Thus, we have verified that $x \rightarrow 1 = 1$.

Besides, $(1 \rightarrow x) \rightarrow x = (x \rightarrow x) \rightarrow 1 = 1 \rightarrow 1 = 1, x \rightarrow (1 \rightarrow x) = 1 \rightarrow (x \rightarrow x) = 1 \rightarrow 1 = 1$. That is $1 \rightarrow x = x$.

(W_7) If $x \rightarrow y = 1, y \rightarrow z = 1$ holds, then $x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = ((x \rightarrow z) \rightarrow y) \rightarrow x = ((y \rightarrow z) \rightarrow x) \rightarrow x = (1 \rightarrow x) \rightarrow x = x \rightarrow x = 1$, Thus, $x \rightarrow z = 1$.

(W_8) Indeed $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow y)) = (x \rightarrow y) \rightarrow (((z \rightarrow z)) \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1$.

Hence, for all $x, y, z \in X$, $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$.

(W_9) We have $(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow z) \rightarrow y) = (x \rightarrow y) \rightarrow (((z \rightarrow z) \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow ((1 \rightarrow x) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = 1$.

(W_{10}) Let $x, y, z \in X$. Applying (W_1) and (W_2), we have

$(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (y \rightarrow (x \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow ((y \rightarrow (x \rightarrow z)) \rightarrow (x \rightarrow z)) = (x \rightarrow y) \rightarrow (((x \rightarrow z) \rightarrow (x \rightarrow z)) \rightarrow y) = (x \rightarrow y) \rightarrow (1 \rightarrow y) = (x \rightarrow y) \rightarrow y = (y \rightarrow y) \rightarrow x = 1 \rightarrow x = x$.

Proposition 3.2 Let X be an W_d -FI algebra, for all x, y, z, we have

$(W_{11}) y \leqslant x \rightarrow y; x \rightarrow y \leqslant (z \rightarrow x) \rightarrow (z \rightarrow y).$

$(W_{12}) x \leqslant y \Rightarrow z \rightarrow x \leqslant z \rightarrow y, y \rightarrow z \leqslant x \rightarrow z.$

$(W_{13}) x^{''} = x; (x^{''} \rightarrow x)^{'} = 0.$

$(W_{14}) x^{'''} = x^{'};$

$(W_{15}) x^{'} \to y^{'} = y \to x;$

$(W_{16}) x^{'} \to y = y^{'} \to x;$

$(W_{17}) (x \to y)^{'} = x;$

$(W_{18}) (y \to x) \to y^{'} \leqslant x^{'};$

$(W_{19}) (x^{'} \to y^{'}) \to x = x^{'} \to ((x^{'} \to y^{'})^{'} = ((x \to y) \to x^{'})^{'} = y;$

$(W_{20}) (Commutativity) T(x, y) = T(y, x), S(x, y) = S(y, x);$

$(W_{21}) (Associativity) T(T(x, y), z) = T(x, T(y, z)), S(S(x, y), z) = S(x, S(y, z));$

$(W_{22}) (Monotonicity) x \leqslant y \Rightarrow T(x, z) \leqslant T(y, z), S(x, Z) \leqslant S(y, z);$

$(W_{23}) (Identity) T(x, 1) = x, S(x, 0) = x;$

$(W_{24}) (Duality) S(x, y) = T(x^{'}, y^{'})^{'}, T(x, y) = (S(x^{'}), y^{'})^{'};$

$(W_{25}) S(x, x^{'}) = 1, T(x, x^{'}) = 0;$

$(W_{26}) x \to (y \to z) = T(x, y) \to z;$

$(W_{27}) T((z \to x), (z \to y)) = z \to T(x, y);$

$(W_{28}) x^{'} = x \to 0 = 0 \Rightarrow x = 1.$

Proof. (W_{11}) Applying (W_1), we have $y \to (x \to y) = x \to (y \to y) = x \to 1 = 1$. So $y \leqslant x \to y$;

By $(x \to y) \to ((z \to x) \to (z \to y)) = (z \to x) \to ((x \to y) \to (z \to y)) = (z \to x) \to (((z \to y) \to y) \to x) = (z \to x) \to (((y \to y) \to z) \to x) = (z \to x) \to ((1 \to z) \to x) = (z \to x) \to (z \to x) = 1$, we have that for all $x, y, z \in X$, $(x \to y) \leqslant (z \to x) \to (z \to y)$.

(W_{12}) Assume that $x \leqslant y$. Then $(z \to x) \to (z \to y) = z \to ((z \to x) \to y) = z \to ((y \to x) \to z) = (y \to x) \to (z \to z) = (y \to x) \to 1 = (1 \to x) \to y = x \to y = 1$, which implies that $z \to x \leqslant z \to y$.

Similarly, we have $(y \to z) \to (x \to z) = ((x \to z) \to z) \to y = ((z \to z) \to x) \to y = (1 \to x) \to y = x \to y = 1$.

$(W_{13}) x^{''} = (x \to 0) \to 0 = (0 \to 0) \to x = 1 \to x = x.$

$(x^{''} \to x)^{'} = (x \to x)^{'} = 1^{'} = 1 \to 0 = 0.$

$(W_{14}) x^{'''} = ((x \to 0) \to 0) \to 0 = (0 \to 0) \to (x \to 0) = 1 \to (x \to 0) = x^{'}.$

$(W_{15}) x^{'} \to y^{'} = (x \to 0) \to (y \to 0) = y \to ((x \to 0) \to 0) = y \to ((0 \to 0) \to x) = y \to (1 \to x) = y \to x.$

$(W_{16}) x^{'} \to y = (x \to 0) \to y = (y \to 0) \to x = y^{'} \to x.$

$(W_{17}) (x \to y)^{'} = (x \to y) \to 0 = (0 \to y) \to x = 1 \to x = x.$

$(W_{18}) ((y \to x) \to y^{'})) \to x^{'} = (x^{'} \to y^{'}) \to (y \to x) = (y \to x) \to (y \to x) = 1.$ So, $(y \to x) \to y^{'} \leqslant x^{'}$.

$(W_{19}) (x^{'} \to y^{'}) \to x = (y \to x) \to x = (x \to x) \to y = 1 \to y = y, x \to (x^{'} \to y^{'})^{'} = x^{'} \to (y \to x)^{'} = (y \to x) \to x = (x \to x) \to y = 1 \to y = y$. Therefore, $(x^{'} \to y^{'}) \to x = x^{'} \to (x^{'} \to y^{'})^{'} = y$.

Similarly, we have $((x \to y) \to x^{'})^{'} = ((x^{'} \to y) \to x)^{'} = (((x \to 0) \to y) \to x)^{'} = (((y \to 0) \to x) \to x)^{'} = ((x \to x) \to (y \to 0))^{'} = (y \to 0) \to 0 = y$.

$(W_{20}) T(x, y) = (x \to y^{'})^{'} = (y \to x^{'})^{'} = T(y, x)$. Similarly, we have $S(x, y) = S(y, x)$.

$(W_{21}) T(T(x, y), z) = T((x \to y^{'})^{'}, z) = ((x \to y^{'})^{'} \to z^{'})^{'} = (((x \to y^{'}) \to 0) \to z^{'})^{'} = (((x \to y^{'}) \to 0) \to (z \to 0))^{'} = (z \to (x \to y^{'}))^{'} = (x \to (z \to y^{'}))^{'} = (x \to (z \to (y \to 0)))^{'} = (x \to (y \to (z \to 0)))^{'} = T(x, T(y, z))$.

Similarly, we have $S(S(x, y), z) = S(x, S(y, z))$.

(W_{22}) Due to $x \leqslant y \Leftrightarrow x \to y = 1$, then for all $x, y, z \in X$, it is $(x \to z^{'})^{'} \to (y \to z^{'})^{'} = ((x \to z^{'}) \to 0) \to ((y \to z^{'}) \to 0) = (y \to z^{'}) \to (x \to z^{'}) = x \to y = 1$. Hence, $T(x, z) \leqslant T(y, z)$. $S(x, Z) \leqslant S(y, z)$. Similarly, we have $S(x, z) \leqslant S(y, z)$.

$(W_{23}) T(x, 1) = (x \to 1^{'})^{'} = (x \to 1^{'}) \to 0 = (0 \to 1^{'}) \to x = 1 \to x = x$.

$S(x, 0) = x^{'} \to 0 = (x \to 0) \to 0 = (0 \to 0) \to x = 1 \to x = x$.

$(W_{24}) (T(x^{'}, y^{'}))^{'} = (x^{'} \to y^{''})^{''} = x^{'} \to y = S(x, y)$.

$(S(x^{'}, y^{'}))^{'} = (x^{''} \to y^{'})^{'} = (x \to y^{'})^{'} = T(x, y)$.

$(W_{25}) S(x, x^{'}) = x^{'} \to x^{'} = 1; T(x, x^{'}) = (x \to x^{''})^{'} = (x \to x)^{'} = 1^{'} = 1 \to 0 = 0$.

$(W_{26}) T(x, y) \to z = (x \to z^{'})^{'} \to z = ((x \to y^{'}) \to 0) \to z = ((x \to (y \to 0)) \to 0) \to z = (z \to 0) \to (x \to (y \to 0)) = x \to ((z \to 0) \to (y \to 0) = x \to (y \to z)$.

$(W_{27}) T((z \to x), (z \to y)) = ((z \to x), (z \to y)^{'})^{'} = ((z \to x) \to ((z \to y) \to 0)) \to 0 = z \to x$, and $z \to T(x, y) = z \to (x \to y^{'})^{'} = z \to ((x \to (y \to 0)) \to 0 = z \to x$.

(W_{28}) By assume that $x^{'} = x \to 0 = 0 \Rightarrow 1 = 0 \to 0 = (x \to 0) \to 0 = (0 \to 0) \to x = 1 \to x = x$, we get $x = 1$.

Proposition 3.3 Let X be a W_d -FI algebra and $x, y, z \in X$, then $(x \to y) \to z = (z \to y) \to x$ implies $x \to (y \to z) = y \to (x \to z)$.

Proof. By (W_6), we have $x \to (y \to z) = (1 \to x) \to (y \to z) = ((y \to z) \to x) \to 1 = ((x \to z) \to y) \to 1 = (1 \to y) \to (x \to z) = y \to (x \to z)$. Hence, $x \to (y \to z) = y \to (x \to z)$.

Theorem 3.4 A (2,0)-type algebra $(X, \to, 0)$ is a W_d-fuzzy implication algebra if and only if it satisfies that

$(W1^{'})$ $(x \to y) \to z = (z \to y) \to x$;

$(W2^{'})$ $1 \to x = x$;

$(W3^{'})$ $x \to x = 1$;

$(W4^{'})$ If $x \to y = y \to x = 1$, then $x = y$;

$(W5^{'})$ $0 \to x = 1$, where $1 = 0 \to 0$.

Proof. Immediate from Proposition 3.3 and Definition 2.3.

By $(W1^{'})$ of Theorem 3.4, for distinct $x, y \in X$, we obtain

$(W_{29}) ((x \to y) \to y) \to y = x \to y$.

Theorem 3.5 Let$(X, \rightarrow, 0)$ be a W_d-fuzzy implication algebras with negation, we define the partial addition

$$x + y := y^{'} \rightarrow x. \tag{5}$$

Then $(X, +, 0)$ is an Abelian monoid.

Proof. It is easy to check that 0 is partial addition unit.,i.e $0 + x = x + 0 = x$.

Next, we will show that $x + (y + z) = (x + y) + z, x + y = y + x$.

In fact, by (5), we have $(x + y) + z = z^{'} \rightarrow (x + y) = (z \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow x) = (z \rightarrow 0) \rightarrow (x \rightarrow 0) \rightarrow y) = (x \rightarrow 0) \rightarrow (z \rightarrow 0) \rightarrow y) = (x \rightarrow 0) \rightarrow (y + z) = ((y + z) \rightarrow 0) \rightarrow x = x + (y + z)$.

Similarly, it can be proved that $x + y = y + x$, for any $x, y \in X$. Therefore, $(X, +, 0)$ is an Abelian monoid.

4 Relation between W_d -FI algebras and other logical algebras

4.1 Relation with FI-algebras

In this subsection, we establish the connections between: **Proposition 4.1** Every W_d -FI algebra is an FI - algebra, but the inverse is not true.

Proof. From the Definition 2.3 and (3) of Proposition 3.1, it is easy to see that any W_d -FI algebra must be an FI - algebra. By example 3, W_d -FI algebra may be an proper subalgebra of FI-algebra, but FI algebra must not be W_d -FI algebra.

Example 4. Let $X = \{0, a, b, 1\}$ be a finite set of distinct elements. We define

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then, $(X, \rightarrow, 1)$ is an FI-algebra which is not W_d-FI algebra, because $(a \rightarrow b) \rightarrow b = b \rightarrow b = 1 \neq (b \rightarrow b) \rightarrow a = 1 \rightarrow a = a$.

4.2 Relation with RFI-algebras

Proposition 4.2 Every W_d-FI algebra is an RFI- algebra.

Proof. By Proposition 4.1 and Definition 2.3, we have $\forall \in X$,$x^{''} = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x$. Therefore $(X, \rightarrow, 0)$ is an RFI- algebra.

In the following example, we show that converse of Proposition 4.3 is not correct in general.

Example 5. Let $X = [0, 1]$, we define $x \rightarrow y = min(1, 1 - x + y)$. It is easy to verify that $(X, \rightarrow, 0)$ is an RFI-algebra. But the converse of above Proposition is not true.

In fact, we put $x = 0.1, y = 0.2, z = 0.3$, because $(0.1 \rightarrow 0.2) \rightarrow 0.3 = min(1, 1 - 0.1 + 0.2) \rightarrow 0.3 = 1 \rightarrow 0.3 = 0.3$, on the other hand, $(0.3 \rightarrow 0.2) \rightarrow 0.1 = min(1, 1 - 0.3 + 0.2) \rightarrow 0.1 = 0.9 \rightarrow 0.1 = min(1, 1 - 0.9 + 0.1) = 0.2$, hence, it is not satisfies condition (W_2).

We have see that W_d-FI algebra classes are a subclasses of RFI-algebras.

4.3 Relation with CFI-algebras

Proposition 4.3 W_d -FI algebra must not be CFI- algebra.

Proof. It is easy to prove that if $x \neq y$, then the condition (I_6) does not hold, i.e., suppose

$$(x \to y) \to y = (y \to x) \to x,$$

then clearly

$$(y \to y) \to x = (x \to x) \to y.$$

Thus, we have $x = y$, which contradicts to assertion.

4.4 Relation with HFI-algebras

Proposition 4.4 ([7]) The relation between W_d -FI algebra and HFI- algebra is as following :

(1) If $(X, \to, 0)$ is a W_d-FI algebra such that, for all $x, y, z \in X$,

$$x \to (y \to z) = (x \to y) \to (x \to z) \tag{6}$$

holds, then $(X, \to, 0)$ is a HFI-algebra.

(2)If $(X, \to, 0)$ is an HFI-algebra such that, for all $x, y, z \in X$,

$$(x \to y) \to z = (z \to y) \to x$$

holds, then $(X, \to, 0)$ is a W_d -FI algebra.

Proof. (1) Assume that $(X, \to, 0)$ is a W_d -FI algebra, it is easy to verify that conditions $(HFI_4), (HFI_5)$ of HFI-algebra hold. By $(W_1) x \to (y \to z) = y \to (x \to z)$, and put $z = x$, we have $x \to (y \to x) = y \to 1 = 1$. So (HFI_1) holds for all $x, y \in X$.

Next, by $x \to (y \to z) = (x \to y) \to (x \to z)$, we have $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$, which implies (HFI_2) holds for all $x, y, z \in X$. By $1 \to x = x$ and $1 \to x = 1 \Rightarrow x = 1$, so (HFI_3) holds for all $x \in X$. Hence, $(X, \to, 0)$ is a HFI-algebra.

(2) It is easy to prove that if an HFI-algebra $(X, \to, 0)$ satisfies the condition $(x \to y) \to z = (z \to y) \to x$, then $(X, \to, 0)$ is an W_d -FI algebra.

Corollary 4.5 Every self distributive W_d -FI algebra is an HFI-algebra.

Proof. By Proposition 4.4 (1), the proof is clear.

4.5 Relation with BCK-algebras

In 1966, Imai and Iseki [28] introduced two classes of abstract algebras, BCK-algebras. It is well known that the class of MV-algebras is a proper subclass of the class of BCK-algebras. Therefore, both BCK-algebras and MV-algebras are important for the study of fuzzy logic.

In this subsection, we investigate the relation between W_d-FI-algebras and BCK-algebras.

Definition 4.1 [28] An algebraic structure $(A, \to, 1)$ of type (2, 0) is called a BCK-algebra, if for any $x, y, z \in A$, the following conditions hold:

$(BCK_1)(x \to y) \to ((y \to z) \to (x \to z)) = 1,$

$(BCK_2) 1 \to x = x$,

$(BCK_3) x \to 1 = 1$,

(L_3) if $x \to y = y \to x = 1$, then $x = y$.

Theorem 4.6 Every W_d-FI-algebras is a BCK-algebras.

Proof. By Definition 2.4 and $(W_6), (W_8)$ in Proposition 3.1, the proof is clear.

By the following example we show that every BCK-algebra is not W_d-FI-algebra, in general.

Example 6. Let $X = \{a, b, c, 1\}$ be a finite set of distinct elements. We define

$\to$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	a	1	1
1	a	b	c	1

Then $(X, \to, 1)$ is a BCK-algebra, and it is not a W_d-FI-algebra, since W_2 is not satisfied

$$(b \to c) \to a = a \to a = 1 \neq a = a \to b = (a \to c) \to b.$$

4.6 Relation with L-algebras

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [16]. It turns out that the concept of L-algebra is fundamental in the sense that various algebraic structures, even with several operations like Heyting algebras, (one-side) hoops, (pseudo) MV-algebras or l-group cones, are definable as L-algebra[17-24].

First, we recall some definitions and properties about L-algebras.

Definition 4.2([16]) An L-algebras is an algebra $(L, \to, 1)$ of type (2,0) satisfying for all $x, y, z \in L$,

$(L_1) x \to x = x \to 1 = 1, 1 \to x = x$;

$(L_2) (x \to y) \to (x \to z) = (y \to x) \to (y \to z)$;

$(L_3) x \to y = y \to x = 1 \Rightarrow x = y$.

From the above definition of Rump ([16]), we know that Hilbert algebras are equivalent to implicative BCK-algebras, and they are special L-algebras. L-subalgebra $\{y \to x | y, x \in L\}$ of an L-algebra is an MV-algebra. Next, we give characterizations of L-algebras and discuss relations between W_d-FI algebra and L-algebras.

Theorem 4.7 ([16])Let $(X, \to, 1)$ be an L-algebra. The following hold for all $x, y, z \in X$:

(1) $x \leqslant y \Rightarrow z \to x \leqslant z \to y$;

(2) $\leqslant$ is a partial order on X;

(3) $x = y$ if and only if $x \to z = y \to z$, for all $x, y, z \in X$;

(4) $x, y \leqslant z$ such that $z \to x = z \to y$, then $x = y$;

(5) $((x \to y) \to y) \to y = x \to y$;

(6) $((x \to y) \to y) \to x) \to ((x \to y) \to y) \to z) = x \to z$.

(7) If X satisfies condition $(x \to (y \to z)) \to (y \to (x \to z)) = 1$, then

$$x \to (y \to z) = y \to (x \to z).$$

Theorem 4.8 Any W_d-FI algebra is an L-algebra, but the inverse is not true.

Proof. Let $(X, \to, 0)$ is a W_d-FI algebra, any $x, y, z \in X$, by equations (W_1), (W_2) and (W_6), respectively, it is easy to see that

$(x \to y) \to (x \to z) = x \to ((x \to y) \to z) = x \to ((z \to y) \to x) = (z \to y) \to (x \to x) = (z \to y) \to 1 = 1$,

$(y \to x) \to (y \to z) = y \to ((y \to x) \to z) = y \to ((z \to x) \to y) = (z \to x) \to (y \to y) = (z \to x) \to 1 = 1$,

Hence, $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$. Thus (L_2) holds in X.

From definition 2.3, we see that (L_1 and (L_3) hold in X. Therefore, $(X, \to, 0)$ be a L-algebra.

Example 7. Let $X = \{0, a, b, c, d, 1\}$ be a set and operation $\to$ be defined as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(X, \to, 1)$ is an L-algebra which is not a W_d-FI algebra, since $(b \to c) \to d = 1 \neq (d \to c) \to b = d \to b = b$. Therefore, the class of W_d-FI algebras is a proper subclass of L-algebras.

Theorem 4.9 Let $(X, \to, 1)$ is an L-algebra satisfying the condition

$$(x \to y) \to z = (z \to y) \to x$$

for $x, y, z \in X$, then $(X, \to, 1)$ is a W_d-FI algebra.

Proof. By Definition 2.4 and Theorem 4.3 we have $x \to (y \to z) = (1 \to x) \to (y \to z) = ((y \to z) \to x) \to 1 = ((x \to z) \to y) \to 1 = (1 \to y) \to (x \to z) = y \to (x \to z)$. So, $x \to (y \to z) = y \to (x \to z)$. Thus, L-algebra $(X, \to, 1)$ is a W_d-FI algebra.

Definition 4.3 ([21]) Let $(X, \to, 1)$ be an L-algebra. The following hold for all $x, y \in X$:

(1) If L-algebra $(X, \to, 1)$ satisfies

$$x \to (y \to x) = 1,$$

then it is called a KL-algebra.

(2) If L-algebra $(X, \to, 1)$ satisfies

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1,$$

then it is called a CL-algebra.

Proposition 4.10 ([21]) Any CL-algebra is a KL-algebra.

Theorem 4.11 If $(X, \to, 0)$ is a W_d-FI algebra, then $(X, \to, 0)$ is a CL-algebra and it is also a KL-algebra.

Proof. By **Theorem 4.7**, any W_d-FI algebra $(X, \to, 0)$ is an L-algebra, using $(W_1) and (W_3)$, we get $(x \to (y \to z)) \to (y \to (x \to z)) = (y \to (x \to z)) \to (y \to (x \to z)) = 1$. Hence we get $(X, \to, 0)$ is a CL-algebra. By **Proposition 4.8**, we get $(X, \to, 0)$ is a KL-algebra.

5 Generalized W_d -FI algebras

In this Sect., the concept of GW_d-FI-algebra is introduced, and some properties of it are investigated. We discuss on a distributive GW_d-FI-algebra.The relationship between distributive GW_d-FI-algebra and Hilbert algebra is considered, and we prove that every distributive GW_d-FI-algebra is a Hilbert algebra, but the converse may not be true, with an example being given to illustrate it. Moreover, we prove that every GW_d-FI-algebra is a BE-algebras.

5.1 Notes of Generalized W_d -FI algebras

The condition (W_2) of Definition 2.4 is a strong condition, in this Sect., we abandon the condition W_2, redefine generalized W_d -fuzzy implication algebras which is a generalization of W_d-FI-algebra. In order to show the existence of the structures, we presented some examples. We then explore some properties of these structures. We start with the following definition.

Definition 5.1 A (2,0)-type algebra $(X, \to, 0)$ is said to be a generalized W_d-Fuzzy implication algebra, shortly, GW_d-FI algebra, if the following conditions hold for all $x, y, z \in X$:

$(GW_1) x \to x = 1$;

$(GW_2) 1 \to x = x$;

$(GW_3) x \to (y \to z) = y \to (x \to z)$;

(GW_4) if $x \to y = y \to x = 1$, then $x = y$;

$(GW_5) 0 \to x = 1$, where $1 = 0 \to 0$.

Let us give two examples.

Example 8. Let $X = \{0, a, b, c, 1\}$ be a set and operation $\to$ be defined as follows:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	1	1	b	c	1
b	0	a	1	1	1
c	1	1	b	1	1
1	0	a	b	c	1

Then, $(X, \to, 0)$ is a GW_d-FI-algebra.

Example 9. Let $X = \{0, a, b, c, d, 1\}$ be a set and operation $\to$ be defined as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	d	1
b	0	a	1	c	d	1
c	0	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Then, $(X, \to, 0)$ is a GW_d-FI-algebra.

Let us state and prove some properties.

Proposition 5.1 Let$(X, \to, 0)$ be a generalized GW_d-FI algebra and $x, y \in X$. Then

(5-1)$x \to 1 = 1$;

(5-2)$x \to (y \to x) = 1$.

Proof. For any $x \in X$, we have $x \to 1 = x \to (0 \to x) = 0 \to (x \to x) = 0 \to 1 = 1$. Hence, $x \to 1 = 1$.

$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1$. So, $x \to (y \to x) = 1$.

Definition 5.2 ([25,26])A BE-algebra is a non-empty set X with a constant 1 and a binary operation $\to$ satisfying the following axioms:

$(BE_1) x \to x = 1$,

$(BE_2) x \to 1 = 1$,

$(BE_3) 1 \to x = x$,

$(BE_4) x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in X$.

Proposition 5.2 Every GW_d-FI algebra $(X, \to, 0)$ is a BE-algebra, but the converse may not be true.

Proof. By Definition 5.1, Definition 5.2 and Proposition 5.1, the proof is completed.

Example 10. Let $X = \{0, a, b, 1\}$ be a set and operation $\to$ be defined as follows:

$\to$	0	a	b	1
0	1	a	b	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

It is easy to see that $(X, \to, 0)$ is a BE-algebra, but it is not GW_d-FI algebra, since $0 \to a = a \neq 1$.

Definition 5.3 A GW_d-FI algebra $(X, \to, 1)$ is said to be self-distributive if $x \to (y \to z) = (x \to y) \to (x \to z)$, for all $x, y, z \in X$.

Definition 5.4 ([27]) A Hilbert algebra is an algebra $(A, \to, 1)$ of type (2,0) such that the following axioms are fulfilled for every $x, y, z \in A$:

(H_1) $x \to (y \to x) = 1$;

(H_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

(H_3) If $x \to y = y \to x = 1$, then $x = y$.

From Definition 5.1, 5.3, and 5.4, we have the following results.

Proposition 5.3 Every self-distributive GW_d-FI algebra $(X, \to, 1)$ is a Hilbert algebra. In this case, $(X, \leqslant)$ is a poset by defining an order relation $\leqslant$ such that $x \leqslant y$ iff $x \to y = 1$ (called the natural order on X), with respect to this order, 1 is the greatest element of X.

Proof. (1) By using (5-2) of Proposition 5.1, we get that (H_1) holds. Using the distributive law again, we obtain that (H_2) holds. (GW_4) coincides with (H_3). The proof is completed.

(2) By (GW_1), $x \to x = 1$, we get $x \leqslant x$. By (GW_4), we obtain $x \leqslant y, y \leqslant x \Rightarrow x = y$. Now, let $x, y, z \in X$, by using self-distributive law again, from $x \leqslant y, y \leqslant z$, we get

$$x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = x \to (y \to z) = x \to 1 = 1.$$

Thus, $x \leqslant z$. Hence, $(X, \leqslant)$ is a partial order set.

Remark. In Proposition 5.3, the condition self-distributive is necessary. The following example is given to illustrate it.

Example 11. If we consider Example 8, then we have

$$(b \to (c \to a)) \to ((b \to c) \to (b \to a)) = (b \to 1) \to (1 \to a) = 1 \to a = a \neq 1.$$

Hence, in Example 8, self-distributive law is not satisfied. So $(X, \to, 1)$ is a GW_d-FI algebra, but it is not a Hilbert algebra.

5.2 Generalized W_d -FI algebras and W-eo algebra

Definition 5.5 ([30]) Let L be a non-empty set,$\to$: $L \times L \to L$ a binary operation and $\top$ a fixed element of L. The triple $(L; \to; \top)$ is a weak extended-order algebra, shortly w-eo algebra, if for all $a, b, c \in L$, the following conditions are satisfied:

$(O_1) a \to \top = \top$ (upper bound condition);

$(O_2) a \to a = \top$(reflexivity condition);

$(O_3) a \to b = b \to a = \top \Rightarrow a = b$ (antisymmetry condition);

$(O_4) a \to b = \top$ and $b \to c = \top \Rightarrow a \to c = \top$ (weak transitivity condition).

Proposition 5.4 Every self-distributive GW_d-FI algebra $(X, \to, 0)$ is a W-eo algebra.

Proof. By using (5-1) of Proposition 5.1, we get (O_1) hold. By (GW_1) and (GW_4), we get $(O_2), and (O_3)$ hold, respectively. We take $1 = \top$, by $a \to b = 1$ and $b \to c = 1$, we have $a \to c = 1 \to (a \to c) = (a \to b) \to (a \to c) = a \to (b \to c) = a \to 1 = 1$, so $a \to c = 1$. Hence, (O_4) holds, i.e., $(X, \to, 0)$ is a W-eo algebra.

6 Conclusion

The aim of this paper is to study the relations between W_d-FI-algebras and other logical algebras such as FI-algebras, RFI-algebras, CFI-algebras, BCK-algebras, Hilbert-algebras and L-algebras, etc. The concept of GW_d-FI-algebra is introduced, and some properties of it are investigated. For the future research, we will investigate new structures on GW_d-algebras.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Data availability statement

The research data are not shared.

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