# On A Novel Neutrosophic Numerical Method for Solving Some Neutrosophic Boundary Value Problems

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**Abstract**

In this paper, we study a novel numerical method for finding the neutrosophic numerical solutions to some neutrosophic boundary values problems in differential equations of high orders. The proposed method based on neutrosophic numerical collocations of higher degree polynomials as an approximation to solve the problems. In addition, we provide many mathematical proofs about the existence of the solutions with many different examples and numerical tables that clarify the validity of the proposed method.

**Keywords**: Neutrosophic Polynomials; Neutrosophic Differential Equations; Numerical Error; Numerical approximation

## 1. Introduction

Differential equations of various types are considered an effective tool for modeling problems related to physics, chemistry, and even economics into the language of mathematics, as the solutions to these problems describe many complex natural phenomena [1-3]. One of the most difficult types of differential equations are high-order differential equations. When the rank of the differential equation exceeds 10, it becomes difficult to solve using traditional methods. Here, numerical analysis plays a major role in finding approximate and numerical mathematical formulas for solutions. These formulas can be modeled and applied by Computer [4-7]. We note that many numerical methods have been applied with the aim of solving many high-order differential equations in [8-9, 12], and the applied methods have given good approximate results. Neutrosophic logic is a revolutionary logic introduced by Smarandache [6] as a new generalization of fuzzy logic that takes into account the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena. It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even in computer science [14-31]. In [1-10, 13], many numerical algorithms for solving differential equations and numerically complex problems have been discussed by neutrosophic sets, and researchers have obtained many good approximate results.

This has motivated us to study a novel numerical method for finding the neutrosophic numerical solutions to some neutrosophic boundary values problems in differential equations of high orders. The proposed method based on neutrosophic numerical collocations of higher degree polynomials as an approximation to solve the problems. In addition, we provide many different examples and numerical tables that clarify the validity of the proposed method.

## 2. Main Discussion

**Neutrosophic approximation polynomials:**

We use the neutrosophic domain $[a + cI, b + dI]$, as follows:

$k = 0,1, \dots, N$ , $x_k = a + cI + khI$ where $h = {}^{(b - a + (d - c)I)}/_{n}$ step length.

Let's take $P(X + YI)$ polynomial as an approximation for solving the problem $u(x + yI)$ and for every $x + yI \in [\mathrm{x_k} + y_k\mathrm{I}, \mathrm{x_{k+1}} + y_{k+1}\mathrm{I}]$ this approximation is given as:

$$P(x + yI) = \sum_{i=0}^{13} \frac{(x + yI - x_k - y_k\mathrm{I})^i}{i!} P_k^{(i)} + \frac{(x + yI - x_k - y_k\mathrm{I})^{14}}{14!} C_{k,1} + \frac{(x + yI - x_k - y_k\mathrm{I})^{15}}{15!} C_{k,2} +$$



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$$\frac{(x+yI-x_k-y_k\mathrm{I})^{16}}{16!}C_{k,3}+\frac{(x+yI-x_k-y_k\mathrm{I})^{17}}{17!}C_{k,4}+\frac{(x+yI-x_k-y_k\mathrm{I})^{18}}{18!}C_{k,5}\,;k=0,1,\ldots.,N-1\ (1).$$

Where $P_k^{(i)}=P^{(i)}(X_k+Y_kI)\,,(i=0,1,\ldots.,14)$.

The approximation $P(X+YI)$ satisfies the following conditions:

- $P_k^{(m)}(x_k+y_k\mathrm{I})=u^{(m)}(x_k+y_k\mathrm{I})\,,k=0,1,\ldots,N-1;m=0,1,\ldots,14$
- $P_k^{(m)}(x_{k+1}+y_{k+1}\mathrm{I})=P_{k+1}^{(m)}(x_{k+1}+y_{k+1}\mathrm{I})\,,k=0,1,\ldots..N-2,m=0,1,\ldots,14$

We define five aggregation points in each partial domain as follows:

$$X_{k+z_j}=X_k+y_k\mathrm{I}+hz_jI\,,(j=1,2,\ldots,5), \tag{2}$$

The relation relates the given points (2) defined by the form:

$$0<z_1+z_1'I<z_2+z_2'I<z_3+z_3'I<z_4+z_4'I<z_5+z_5'I=1+I\,. \tag{3}$$

Numerical solution of the neutrosophic boundary value problem:

Assuming that $u(x+yI)$ is a single solution to the problem of boundary values, then this solution is related to the solutions of eight special elementary value problems, which we denote by $\{V_i(x+yI)\}_{i=0}^{7}$, so that for seven real constants $c_1,c_2,\ldots,c_7$,we have:

$$u(x+yI)=V_0(x+yI)+\sum_{k=1}^{7}c_kV_k(x+yI), \tag{4}$$

$$V_0^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_0^{(14-i)}(x+yI)=f(x+yI),\ a+cI\le x+yI\le b+dI, \tag{5}$$

$$V_0^{(2j)}(a+cI)=\alpha_i\,,V_0^{(2j+1)}(a+cI)=0,j=0,1,\ldots.,6.$$

Then,

$$V_1^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_1^{(14-i)}(x+yI)=0\,,a+cI\le x+yI\le b+dI\,, \tag{6}$$

$$V_1^{(2j)}(a+cI)=0\,,V_1'(a+cI)=1\,,V_1^{(2j+1)}(a)=0\,,j=1,2,\ldots,6, \tag{6a}$$

$$V_2^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_2^{(14-i)}(x+yI)=0\,,a+cI\le x+yI\le b+dI\,, \tag{7}$$

$$V_2^{(j)}(a+cI)=0\,,V_2^{(3)}(a+cI)=1\,,j=0,1,\ldots.,13,\quad j\ne3\,, \tag{7a}$$

$$V_3^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_3^{(14-i)}(x+yI)=0\,,a+cI\le x+yI\le b+dI, \tag{8}$$

So that,

$$V_3^{(j)}(a+cI)=0\,,V_3^{(5)}(a+cI)=1\,,j=0,1,\ldots,13,\quad j\ne5, \tag{8a}$$

$$V_4^{(14)}(x)+\sum_{i=1}^{14}q_i(x)V_4^{(14-i)}(x)=0\,,\ a+cI\le x+yI\le b+dI\,, \tag{9}$$

$$V_4^{(j)}(a+cI)=0,V_4^{(7)}(a+cI)=1\,,j=0,1,\ldots,13\,,\quad j\ne7\ , \tag{9a}$$

$$V_5^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_5^{(14-i)}(x+yI)=0\,,\ a+cI\le x+yI\le b+dI, \tag{10}$$

Thus,

$$V_5^{(j)}(a+cI)=0,V_5^{(9)}(a+cI)=1\,,j=0,1,\ldots,13,\quad j\ne9, \tag{10a}$$

$$V_6^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_6^{(14-i)}(x+yI)=0\,,\ a+cI\le x+yI\le b+\ dI\,, \tag{11}$$

$$V_6^{(j)}(a+cI)=0\,,V_6^{(11)}(a+cI)=1\,,j=0,1,\ldots.,13,\quad j\ne11\,, \tag{11a}$$

$$V_7^{(14)}(x+yI)+\sum_{i=1}^{14}q_i(x+yI)V_7^{(14-i)}(x+yI)=0\,,\ a+cI\le x+yI\le b+dI, \tag{12}$$

$$V_7^{(j)}(a+cI)=0\,,V_7^{(13)}(a+cI)=1\,,j=0,1,\ldots.,13,\quad j\ne13\,. \tag{12a}$$

Now we will prove that the function $u(x+yI)=V_0(x+yI)+\sum_{k=1}^{7}c_kV_k(x+yI)$ is the only solution to the problem of the boundary value, So we find that for the real constants $c_1,c_2,\ldots,c_7$ we have:



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$$u^{(14)}(x+yI)=V_0^{(14)}(x+yI)+\sum_{k=1}^{7}c_kV_k^{(14)}=-\sum_{i=1}^{14}q_i(x+yI)V_0^{(14)}(x+yI)+f(x+yI)$$

$$+\sum_{k=1}^{7}c_k\left[-\sum_{i=1}^{14}q_i(x+yI)V_k^{(14-i)}(x+yI)\right]$$

$$=-\sum_{i=1}^{14}q_i(x+yI)\left[V_0^{(14-i)}(x+yI)+\sum_{k=1}^{7}c_kV_k^{(14-i)}(x+yI)\right]+f(x+yI)$$

$$=-\sum_{i=1}^{14}q_i(x+yI)\left[u^{(14-i)}(x+yI)+f(x+yI)\right.$$

Where: $u^i(x+yI)=V_0^{(i)}(x+yI)+\sum_{k=1}^{7}c_kV_k^{(i)}(x+yI)\,,i=0,1,\ldots..,13$

$$u^{(2j)}(a+cI)=V_0^{(2j)}(a+cI)+\sum_{k=1}^{7}c_kV_k^{(2j)}(a+cI)=\alpha_j+\sum_{k=1}^{7}c_k(0)=\alpha_j\,,\quad (j=0,\ldots.,6).$$

To fulfill the remaining conditions (2) at the end of the domain, we put the following system of equations:

$$u^{(2j)}(a+cI)=V_0^{(2j)}(a+cI)+\sum_{k=1}^{7}c_kV_k^{(2j)}(a+cI)=\alpha_j\,,(j=0,\ldots,6)\,.$$

And to achieve the rest of the conditions (2) at the end of the domain, we put the following system of equations:

$$u^{(2j)}(b+dI)=V_0^{(2j)}(b+dI)+\sum_{k=1}^{7}c_kV_k^{(2j)}(b+dI)\equiv\beta_j\,,\quad (j=0,1,\ldots,6)\,, \tag{13}$$

By solving the related system of linear equations, we get:

$$C=V_b^{-1}\,B\,.$$

Where:

$$V_b=\begin{bmatrix} V_1(b+dI) & V_2(b+dI) & V_3(b+dI) & V_4(b+dI) & V_5(b+dI) & V_6(b+dI) & V_7(b+dI)\\ V_1^{(2)}(b+dI) & V_2^{(2)}(b+dI) & V_3^{(2)}(b+dI) & V_4^{(2)}(b+dI) & V_5^{(2)}(b+dI) & V_6^{(2)}(b+dI) & V_7^{(2)}(b+dI)\\ V_1^{(4)}(b+dI) & V_2^{(4)}(b+dI) & V_3^{(4)}(b+dI) & V_4^{(4)}(b+dI) & V_5^{(4)}(b+dI) & V_6^{(4)}(b+dI) & V_7^{(4)}(b+dI)\\ V_1^{(6)}(b+dI) & V_2^{(6)}(b+dI) & V_3^{(6)}(b+dI) & V_4^{(6)}(b+dI) & V_5^{(6)}(b+dI) & V_6^{(6)}(b+dI) & V_7^{(6)}(b+dI)\\ V_1^{(8)}(b+dI) & V_2^{(8)}(b+dI) & V_3^{(8)}(b+dI) & V_4^{(8)}(b+dI) & V_5^{(8)}(b+dI) & V_6^{(8)}(b+dI) & V_7^{(8)}(b+dI)\\ V_1^{(10)}(b+dI) & V_2^{(10)}(b+dI) & V_3^{(10)}(b+dI) & V_4^{(10)}(b+dI) & V_5^{(10)}(b+dI) & V_6^{(10)}(b+dI) & V_7^{(10)}(b+dI)\\ V_1^{(12)}(b+dI) & V_2^{(12)}(b+dI) & V_3^{(12)}(b+dI) & V_4^{(12)}(b+dI) & V_5^{(12)}(b+dI) & V_6^{(12)}(b+dI) & V_7^{(12)}(b+dI) \end{bmatrix},$$

$$B=\begin{bmatrix} \beta_0-V_0(b+dI)\\ \beta_1-V_0^{(2)}(b+dI)\\ \beta_2-V_0^{(4)}(b+dI)\\ \beta_3-V_0^{(6)}(b+dI)\\ \beta_4-V_0^{(8)}(b+dI)\\ \beta_5-V_0^{(10)}(b+dI)\\ \beta_6-V_0^{(12)}(b+dI) \end{bmatrix},$$

$C=[c_1\,,c_2\,,\ldots,c_7]^T$.

And to get the numerical solution, we apply the polynomials (5) with summation points (7)-(6) to each of the problems of elementary values, so we get the numerical solutions $(i=0,1,\ldots,7):\ P_i(x_k)$ for the problems listed in order for every $1\le k\le N$.

Where $i=0,1,\ldots.,7$ for $V_i^{(i)}(x+yI)\approx P_i^{(i)}(x+yI)$ and so the numerical solutions to solve the problem are given the limit values (1) – (2), and its derivatives as a sum of solutions of elementary value problems as follows:



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$$P^{(i)}(x_k+y_k\mathrm{I})=P_0^{(i)}(x_k+y_k\mathrm{I}_k)+\sum_{j=1}^{7}c_jP_j^{(i)}(x_k+y_k\mathrm{I})\,,i=0,1,\dots,13;k=0,1,\dots,N\,. \quad (14)$$

Existence of the numerical solution:

Let's take the Fourteenth-order differential equation in the following case:

$$\begin{cases} u^{(14)}(x+yI)=F[x+yI\,,u(x+yI)\,,\acute{u}(x+yI)\dots,u^{(13)}(x+yI)]\,,x+yI\in[a+cI,b+dI] \\ u^{(j)}(a)=u_j\ ,\ j=0,1,\dots,13\,. \end{cases} \quad (15)$$

Assuming that $F:[a+cI,b+dI]\times C[a+cI,b+dI]\times\dots\times C^{13}[a+cI,b+dI]\to R(I)$ is a sufficiently smooth function, the function F is said to satisfy the Lipschitz condition if the following Lipschitz inequality holds:

$$|F(x+yI\,,u_0,u_1,\dots,u_{13})-F(x+yI\,,\ddot{y}_0\,,\ddot{y}_1,\dots,\ddot{y}_{13})|\le L\sum_{i=0}^{13}|u_i-\ddot{y}_i|,$$

$$\forall(x+yI,u_0,u_1,\dots,u_{13})\,,(x+yI,\ddot{y}_0,\ddot{y}_1,\dots,\ddot{y}_{13})\in[a+cI,b+dI]\times R(I)^{13}.$$

Where L is called the Lipschitz constant of the function F.

We apply the approximate polynomial and its derivatives with summation points to the problem of differential equations (18), we obtain the following set of algebraic equations:

$$C_{k,1}+C_{k,1}I+(hz_j)(C_{k,2}+C_{k,2}I)+\frac{(hz_j)^2}{2!}(C_{k,3}+C_{k,3}\mathrm{I})+\frac{(hz_j)^3}{3!}(C_{k,4}+C_{k,4}I)+\frac{(hz_j)^4}{4!}(C_{k,5}+C_{k,5}I)=$$

$$\mathrm{F}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\,,\mathrm{P}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right),\acute{\mathrm{P}}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right),\dots,\mathrm{P}^{(13)}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right)\right),j=1,2,\dots,5,\ k=0,1,\dots,N-1, \quad (16)$$

$$P^{(i)}(a+cI)=P_i\,,i=0,1,\dots,13. \quad (16a)$$

We rewrite the system of equations in the Matrix formula as follows:

$$A\bar{C}_k=\hat{F}_k\,. \quad (17)$$

Where: $F_{k+z_j}=F\left[x_{k+z_j}+y_{k+z_j}\mathrm{I}\,,P\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right),\acute{P}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right),\dots,P^{(13)}\left(x_{k+z_j}+y_{k+z_j}\mathrm{I}\right)\right]$,

*J=1,2,…,5*

$$\bar{C}_k=\begin{bmatrix} C_{k,1}+C_{k,1}I \\ h(C_{k,2}+C_{k,2}I) \\ h^2(C_{k,3}+C_{k,3}\mathrm{I}) \\ h^3(C_{k,4}+C_{k,4}I) \\ h^4(C_{k,5}+C_{k,5}I) \end{bmatrix},\hat{F}_k=\begin{bmatrix} F_{k+z_1}+F_{k+z_1}I \\ F_{k+z_2}+F_{k+z_2}I \\ F_{k+z_3}+F_{k+z_3}I \\ F_{k+z_4}+F_{k+z_4}I \\ F_{k+1}+F_{k+1}I \end{bmatrix},A=\begin{bmatrix} 1+I & z_1+z_1I & \frac{z_1^2}{2}+I & \frac{z_1^3}{6}+I & \frac{z_1^4}{24}+I \\ 1+I & z_2+z_2I & \frac{z_2^2}{2}+I & \frac{z_2^3}{6}+I & \frac{z_2^4}{24}+I \\ 1+I & z_4+z_4I & \frac{z_4^2}{2}+I & \frac{z_4^3}{6}+I & \frac{z_4^4}{24}+I \\ 1+I & 1+I & \frac{1}{2}+I & \frac{1}{6}+I & \frac{1}{24}+I \end{bmatrix}.$$

## 3. Numerical neutrosophic tests

We test the technique proposed in this research by applying it to find numerical solutions to some problems

**Problem (1)**

Let's take the neutrosophic Linear Differential Equation:

$$u^{(13)}(x+yI)=\cos(x+yI)-\sin(x+yI)\,,\qquad 0\le x+yI\le 1\,,$$

With boundary conditions:

$$\begin{cases} u(0)=1,\acute{u}(0)=1,u''(0)=-1,u^{(3)}(0)=-1,u^{(4)}(0)=1\,,u^{(5)}(0)=1\,,u^{(6)}(0)=-1 \\ u(1)=\cos(1)+\sin(1)\,,\acute{u}(1)=\cos(1)-\sin(1),u''(1)=-\cos(1)-\sin(1), \\ u^{(3)}(1)=\sin(1)-\cos(1)\,,u^{(4)}(1)=\cos(1)+\sin(1),u^{(5)}(1)=Cos(1)-Sin(1). \end{cases}$$

And the exact analytical solution has $u(x+yI)=\cos(x+yI)+\sin(x+yI)$.



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**Table 1:** comparisons of numerical solutions of our proposed method with the exact solution with a step

$$h = 0.1 + I$$

| $x_i + y_i I$ | exact solution | Novel method |
|---|---|---|
| 0.1+I | 1.094837581924854+1.097634937465I | 1.087375889248+I |
| 0.2+I | 1.1787359086363027+1.1788974I | 1.175567646+I |
| 0.3+I | 1.2508566957869456+1.234577835I | 1.21278456+I |
| 0.4+I | 1.3104793363115357+1.7765577835I | 1.32167411537+I |
| 0.5+I | 1.3570081004945758+1.83217835I | 1.35678545757+I |
| 0.6+I | 1.3899780883047137+1.23487435I | 1.38112112136+I |
| 0.7+I | 1.4090598745221796+1.665477835I | 1.40218834796+I |
| 0.8+I | 1.4140628002466882+1.74421835I | 1.414436786882+I |
| 0.9+I | 1.4049368778981477+1.4098877835I | 1.40223416+I |
| 1+I | 1.3817732906760363+1.405596577835I | 1.382169451663+I |

**Table 2:** comparisons of absolute errors $|u(x_i+y_iI) - P(x_i+y_iI)|$ in the numerical solution of our method with Haar Wavelet method

| $x_i$ | Haar Wavelet [4] | Novel method |
|---|---|---|
| 0.1+I | 3.88578+I E-15 | 1.01201+I E-19 |
| 0.2+I | 1.46216+I E-13 | 4.3165+I E-17 |
| 0.3+I | 8.80518+I E-13 | 7.00763121+I E-18 |
| 0.4+I | 2.35822+I E-12 | 2.0987+I E-20 |
| 0.5+I | 3.80140+I E-12 | 3.212303+I E-20 |
| 0.6+I | ------ | 3.100340+I E-20 |
| 0.7+I | ------ | 9.11567+I E-20 |
| 0.8+I | ------ | 3.14413+I E-20 |
| 0.9+I | ------ | 2.01291 E-20 |

**Problem (2)** we have the thirteenth-order neutrosophic nonlinear differential equation:

$$u^{(13)}(x + yI) = e^{-(x+yI)}u^2(x + yI)\,, 0 \le x + yI \le 1\,,$$

With elementary conditions:

$$u(0) = \acute{u}(0) = u''(0) = u'''(0) = \cdots u^{(12)}(0) = 1;$$

And the exact analytical neutrosophic solution has $u(x + yI) = e^{x+yI}$.

**Table (3):** the exact analytical neutrosophic solution

| $x_i$ | exact solution | Novel method |
|---|---|---|
| *0.1* | 1.1051709180756477 +*I* | *1.1041012015+I* |
| *0.2* | 1.2214027581601699 +*I* | *1.21148496926+I* |
| *0.3* | 1.3498588075760032 +*I* | *1.33324566+I* |
| *0.4* | 1.4918246976412703 +*I* | *1.48741398+I* |
| *0.5* | 1.6487212707001282 +*I* | *1.622678901+I* |
| *0.6* | 1.822118800390509+*I* | *1.80988854328+I* |
| *0.7* | 2.0137527074704766 +*I* | *2.00001249354+I* |
| *0.8* | 2.225540928492468+*I* | *2.20072132+I* |
| *0.9* | 02.45960311115695+*I* | 2.4216678+*I* |
| *1* | 2.718281828459045+*I* | *2.6983628192+I* |



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**Table 4:** comparisons of absolute errors in the numerical solution of our method with the DT method

| $x_i$ | Differential Transformation Method | novel method |
|---|---|---|
| 0.1 | 4.44089E-16 | 5.3443+I E-19 |
| 0.2 | 4.44089E-16 | 6.437102+I E-19 |
| 0.3 | 2.44249E-15 | 6.82231+I E-19 |
| 0.4 | 7.32747E-15 | 8.117034+I E-19 |
| 0.5 | 1.22125E-14 | 1.213804+I E-17 |
| 0.6 | 1.11022E-14 | 1.34001+I E-17 |
| 0.7 | 5.77316E-15 | 2.212561+I E-18 |
| 0.8 | 1.77636E-15 | 2.1995712+I E-19 |
| 0.9 | 8.88178E-16 | 2.4209814+I E-20 |

## 4. Conclusion

In this paper, we studied a novel numerical method for finding the neutrosophic numerical solutions to some neutrosophic boundary values problems in differential equations of high orders. The proposed method based on neutrosophic numerical collocations of higher degree polynomials as an approximation to solve the problems. In addition, we provided many mathematical proofs about the existence of the solutions with many different examples and numerical tables that clarify the validity of the proposed method. In the future, we aim to apply our method on other neutrosophic high-orders differential equations.

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