Elementary Examination of NeutroAlgebras and AntiAlgebras viz-a-viz the Classical Number Systems

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Abstract

The objective of this paper is to examine NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems.
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1 Introduction

The notions of NeutroAlgebra and AntiAlgebra were recently introduced by Florentin Smarandache.[1] Smarandache in[2] revisited the notions of NeutroAlgebra and AntiAlgebra and in[3] he studied Partial Algebra, Universal Algebra, Effect Algebra and Boole's Partial Algebra and showed that NeutroAlgebra is a generalization of Partial Algebra. In the present Short Communication, we are going to examine NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems. For more details about NeutroAlgebras, AntiAlgebras, NeutroAlgebraic Structures and AntiAlgebraic Structures, the readers should see.[1–3]

Let U be a universe of discourse and let X be a nonempty subset of U. Suppose that A is an item (concept, attribute, idea, proposition, theory, algebra, structure etc.) defined on the set X. By neutrosophication approach, X can be split into three regions namely: $< A >$ the region formed by the sets of all elements where $< A >$ is true with the degree of truth (T), $< antiA >$ the region formed by the sets of all elements where $< A >$ is false with the degree of falsity (F) and $< neutA >$ the region formed by the sets of all elements where $< A >$ is indeterminate (neither true nor false) with the degree of indeterminacy (I). It should be noted that depending on the application, $< A >$, $< antiA >$ and $< neutA >$ may or may not be disjoint but they are exhaustive that is; their union is X. If A represents Function, Operation, Axiom, Algebra etc, then we can have the corresponding triplets $< Function, NeutroFunction, AntiFunction >$, $< Operation, NeutroOperation, AntiOperation >$, $< Axiom, NeutroAxiom,$ $AntiAxiom >$ and $< Algebra, NeutroAlgebra, AntiAlgebra >$ etc.

Definition 1.1. [1]

(i) A NeutroAlgebra X is an algebra which has at least one NeutroOperation or one NeutroAxiom that is; axiom that is true for some elements, indeterminate for other elements, and, false for other elements.

(ii) An AntiAlgebra X is an algebra endowed with a law of composition such that the law is false for all the elements of X.

Definition 1.2. [1] Let X and Y be nonempty subsets of a universe of discourse U and let $f : X \to Y$ be a function. Let $x \in X$ be an element. We define the following with respect to $f(x)$ the image of x:

(i) Inner-defined or Well-defined: This corresponds to $f(x) \in Y$ (True)(T). In this case, f is called a Total Inner-Function which corresponds to the Classical Function.

(ii) Outer-defined: This corresponds to $f(x) \in U - Y$ (Falsehood) (F). In this case, f is called a Total Outer-Function or AntiFunction.

(iii) Indeterminacy: This corresponds to $f(x) =$ indeterminacy (Indeterminate) (I); that is, the value $f(x)$ does exist, but we do not know it exactly. In this case, f is called a Total Indeterminate Function.

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2 Subject Matter

In what follows, we will consider the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of natural, integer, rational, real and complex numbers respectively and noting that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. Let $+, -, \times, \div$ be the usual binary operations of addition, subtraction, multiplication and division of numbers respectively. Using elementary approach, we will examine whether or not the abstract systems $(\mathbb{N}, *), (\mathbb{Z}, *), (\mathbb{Q}, *), (\mathbb{R}, *), (\mathbb{C}, *)$ are NeutroAlgebras or and AntiAlgebras where $* = +, -, \times, \div$.

(1) Let $X = \mathbb{N}$.

(i) It is clear that $(X, +)$ and $(X, \times)$ are neither NeutroAlgebras nor AntiAlgebras.

(ii) For some $x, y \in X$, $x - y \in X$ (True) (Inner) or $x - y \notin X$ (False) (Outer). However, for all $x, y \in X$ with $x \leq y$, $x - y \notin X$ (False) (Outer) and for all $x, y \in X$ with $x > y$, we have $x - y \in X$ (True) (Inner). This shows that $-$ is a NeutroOperation over X and $\therefore (X, -)$ is a NeutroGroupoid. The operation $-$ is not commutative for all $x \in X$. This shows that $-$ is AntiCommutative over X. We claim that $-$ is NeuroAssociative over X.

Proof. For $x > y, z = 0$, we have $x - (y - z) = (x - y) - z$, or $x - y + 0 = x - y - 0 > 0$ (degree of Truth) (T). However, for $x > y, z \neq 0$, we have $x - (y - z) \neq (x - y) - z$ (degree of Falsehood) (F). For $x < y, c = 0$, we have $x - y + 0 = x - y - 0 < 0$ (degree of Indeterminacy) (I). This shows that $-$ is NeutroAssociative and $\therefore (X, -)$ is a NeutroSemigroup. □

(iii) For all $x \in X$, $x \div 1 \in X$ (True) (Inner). For some $x, y \in X$, $x \div y \notin X$ (False) (Outer). However, if x is a multiple of y including 1, then $x \div y \in X$ (True) (Inner). This shows that $\div$ is a NeutroOperation and therefore, $(X, \div)$ is a NeutroGroupoid. It can be shown that $\div$ is NeutroAssociative over X and therefore, $(X, \div)$ is a NeutroSemigroup.

The equation $ax = b$ is not solvable for some $a, b \in X$. However, if b is a multiple of a including 1, then the equation is solvable and the solution is called a NeutroSolution. Also, the equation $acx^2 + bd = (ad + bc)x$ is not solvable for some $a, b, c, d \in X$. However, if b is a multiple of a including 1 and c is a multiple of d including 1, the equation is solvable and the solutions are called NeutroSolutions.

Let $\circ$ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} 0 & \text{if } x = y \\ -\alpha & \text{if } x < y \\ -\beta & \text{if } x > y \end{cases}$$

where $\alpha, \beta \in \mathbb{N}$ such that $\alpha \leq \beta$. It is clear that $\circ$ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(2) Let $X = \mathbb{Z}$.

(i) $(X, +)$ and $(X, \times)$ are neither NeutroAlgebras nor AntiAlgebras.

(ii) For all $x, y, z \in X$ such that $x, y = 0, 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False (Outer) so that $-$ is NeutroCommutative over X. However,if $x, y, z = 0$, then $x - (y - z) = (x - y) - z = 0 \in X$ (True), otherwise for other elements, the result is False and consequently, $-$ is NeutroAssociative over X and hence $(X, -)$ is a NeutroSemigroup.

(iii) For all $x \in X$, $x \div \pm 1 \in X$ (True) (Inner). For all $x \in X$, $x \div 0 = indeterminate$ (Indeterminacy). For some $x, y \in X$, $x \div y \notin X$ (False) (Outer) however, if x is a multiple of y including ± 1, then $x \div y \in X$ (True) (Inner). This shows that $\div$ is a NeutroOperation over X and $\therefore (X, \div)$ is a NeutroGroupoid. It can also be shown that $(X, \div)$ is a NeutroSemigroup.

The equation $ax = b$ is not solvable for some $a, b \in X$. If $a = 0$, the solution is indeterminate (Indeterminacy). However, if b is a multiple of a including ± 1, then the equation is solvable and the solution is called a NeutroSolution. Also, the equation $acx^2 + (ad - bc)x - bd = 0$ is not solvable for some $a, b, c, d \in X$. However, if b is a multiple of a including ± 1 and c is a multiple of d including ± 1, the equation is solvable and the solutions are called NeutroSolutions.

For all $x, y \in X$, let $\circ$ be a binary operation defined by $x \circ y = \ln(xy)$. If $x, y = 0$, we have $x \circ y =$ indeterminate (Indeterminacy) (I). If $x > 0, y < 0$, we have $x \circ y =$ indeterminate (Indeterminacy) (I). If $x > 0, y > 0$, we have $x \circ y =$ False (F) except when $x = y = 1$. These show that $\circ$ is a NeutroOperation over X and $\therefore (X \circ)$ is a NeutroAlgebra.

Let $\circ$ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -1/2 & \text{if } x < y \\ 1/2 & \text{if } x > y \end{cases}$$

It is clear that $\circ$ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(3) Let $X = \mathbb{Q}$.

(i) $(X, +)$ and $(X, \times)$ are neither NeutroAlgbras nor AntiAlgebras.

(ii) For all $x, y, z \in X$ such that $x, y, z = 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeuroCommutative over X. Also,if $x, y, z = 0$, then $x - (y - z) = (x - y) - z = 0 \in X$ (True), otherwise for other elements, the result is False and consequently, $-$ is NeutroAssociative over X and $(X, -)$ is a NeutroSemigroup.

(iii) For all $0 \neq x, y \in X$, $x \div y \in X$ (True) (Inner) but for all $x \in X$, $x \div 0 =$ indeterminate (Indeterminacy). $\therefore (X, \div)$ is a NeutroAlgebra which we call a NeutroField.

For all $x, y \in X$, let $\circ$ be a binary operation defined by $x \circ y = e^{x \div y}$. If $x, y = 0$, we have $x \circ y =$ indeterminate (Indeterminacy) (I). If $x > 0, y = 0$, we have $x \circ y =$ indeterminate (Indeterminacy) (I). If $x > 0, y > 0$, we have $x \circ y =$ False (F). These show that $\circ$ is a NeutroOperation over X and $\therefore (X \circ)$ is a NeutroAlgebra.

Let $\circ$ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -e & \text{if } x \leq y \\ e & \text{if } x \geq y \end{cases}$$

where e is the base of Naperian Logarithm. It is clear that $\circ$ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(4) Let $X = \mathbb{R}$.

(i) $(X, +)$ and $(X, \times)$ are neither NeutroAlgebras nor PartialAlgebras.

(ii) For all $x, y \in X$ such that $x, y = 0, \pm 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeuroCommutative over X.

(iii) For all $0 \neq x, y \in X$, $x \div y \in X$ (True) (Inner) but for all $x \in X$, $x \div 0 =$ indeterminate (Indeterminacy). It can be shown that $\div$ is NeutroAssociative over X. Hence, $(X, \div)$ is a NeutroSemigroup and therefore, it is a NeutroAlgebra which we call a NeutroField.

Let $\circ$ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -\sqrt{-1} & \text{if } x \leq y \\ \sqrt{-1} & \text{if } x \geq y \end{cases}$$

It is clear that $\circ$ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(5) Let $X = \mathbb{C}$.

(i) $(X, +)$ and $(X, \times)$ are neither NeutroAlgebras nor AntilAlgebras.

(ii) For all $z, w \in X$ such that $z, w = 0, \pm i$, we have $z - w = w - z = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeutroCommutative over X.

(iii) For all $0 \neq z, w \in X$, $z \div w \in X$ (True) (Inner) but for all $z \in X$, $z \div 0 =$ indeterminate (Indeterminacy). Therefore, $(X, \div)$ is a NeutroAlgebra which we call a NeutroField.

Let $\circ$ be a binary operation defined for all $z, w \in X$ by

$$z \circ w = \begin{cases} i & \text{if } \mid z \mid = \mid w \mid \\ j & \text{if } \mid z \mid \leq \mid w \mid \\ k & \text{if } \mid z \mid \geq \mid w \mid \end{cases}$$

where $ijk = -1$. It is clear that $\circ$ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

Theorem 2.1. *For all prime number $n \geq 2$, $(\mathbb{Z}_n, +, \times)$ is a NeutroAlgebra called a NeutroField.*

Proof. Suppose that $n \geq 2$ is a prime number. Clearly, 1 is the multiplicative identity element in $\mathbb{Z}_n$. For all $0 \neq x \in \mathbb{Z}_n$, there exist a unique $y \in \mathbb{Z}_n$ such that $x \times y = 1$ (True) (T). However, for $0 = x \in \mathbb{Z}_n$, there does not exist any unique $y \in \mathbb{Z}_n$ such that $x \times y = 1$ (False) (F). This shows that $(\mathbb{Z}_n, \times)$ is a NeutroGroup. Since $(\mathbb{Z}_n, +)$ is an abelian group, it follows that $(\mathbb{Z}_n, +, \times)$ is a NeutroDivisionRing called a NeutroField. □

3 Conclusion

We have in this paper examined NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of natural, integer, rational, real and complex numbers respectively. In our future papers, we hope to study more algebraic properties of NeutroAlgebras and NeutroSubalgebras and NeutroMorphisms between them.

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