On The Topological Space of Some n- Refined Neutrosophic Real Intervals and Its Open Sets For $4 \leq n \leq 5$

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Abstract

This paper is dedicated to studying for the first time the building of a topological space based on the intervals defined over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we define a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterize the formula of open sets through these two topological spaces with some illustrated examples.

Keywords: 4-refined neutrosophic numbers; 5-refined neutrosophic numbers; Partial order relation; Neutrosophic interval topology; Open set

1. Introduction

Many neutrosophic structures were built over the idea of splitting the indeterminacy element to many different sub-indeterminacies. Many authors used this idea to study the generalized structures related to this splitting operation such that algebraic rings, matrices, spaces, and modules [4-8]. In the literature, we find many different refined neutrosophic structures. For example refined neutrosophic structures [11, 19] n-refined neutrosophic structures, and n-cyclic refined neutrosophic structures [20-22]. Neutrosophic topological spaces were studied by many authors [1-3], especially open sets, closed sets, and compact sets [9-10, 15, 17]. The concept of n-refined neutrosophic rings was presented in [13]. These rings are considered as an extension of refined neutrosophic rings. Recently AL-Husban et al. Discuss the new structure such neutrosophic set and its application [23-38].

In this work, we focus on the partial order relations over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we define a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterize the formula of open sets through these two topological spaces with some illustrated examples.

2. Main discussion

Definition 2.1

Let $\mu = \{x + yI_1 + zI_2 + tI_3 + qI_4 \ ; x, y, z, t, q \in \mathbb{R}\}$ and

$I_1.I_2 = I_2.I_1 = I_1.I_3 = I_3.I_1 = I_1.I_4 = I_4.I_1 = I_1,\ I_2.I_3 = I_3.I_2 = I_2,$

$I_4 = I_4.I_2 = I_2, I_3.I_4 = I_4.I_3 = I_3, {I_1}^2 = I_1, {I_2}^2 = I_2, {I_3}^2 = I_3, {I_4}^2 = I_4.$

μ is called the real 4-refined neutrosophic ring.

Remark 1

For $x = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in \mu$, we denote it by $x = x_0 + \sum_{i=1}^{4} x_iI_i$.

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Theorem 2.1

μ is partially ordered set.

Proof:

We define the relation $(\leq)$ as follows:

$u = u_0 + \sum_{i=1}^{4} u_i I_i \leq v = v_0 + \sum_{i=1}^{4} v_i I_i$, if and only if:

$$\begin{cases} u_0 \leq v_0 \\ \sum_{i=0}^{4} u_i \leq \sum_{i=0}^{4} v_i \\ \sum_{i\neq 1} u_i \leq \sum_{i\neq 1} v_i \\ \sum_{i\neq 1,2} u_i \leq \sum_{i\neq 1,2} v_i \\ \sum_{i\neq 1,2,3} u_i \leq \sum_{i\neq 1,2,3} v_i \end{cases}.$$

We will show that $(\leq)$ is a partial order relation:

Let $u = u_0 + \sum_{i=1}^{4} u_i I_i$, $v = v_0 + \sum_{i=1}^{4} v_i I_i$, $w = w_0 + \sum_{i=1}^{4} w_i I_i$, $u \leq u$ that is because:

$$\begin{cases} u_0 \leq u_0 \\ \sum_{i=0}^{4} u_i \leq \sum_{i=0}^{4} u_i \\ u_0 + u_2 + u_3 + u_4 \leq u_0 + u_2 + u_3 + u_4 \\ u_0 + u_3 + u_4 \leq u_0 + u_3 + u_4 \\ u_0 + u_4 \leq u_0 + u_4 \end{cases}$$

If $u \leq v$ and $v \leq u$, then:

$$\begin{cases} u_0 \leq v_0 \, , v_0 \leq u_0 \\ \sum_{i=0}^{4} u_i \leq \sum_{i=0}^{4} v_i \, , \sum_{i=0}^{4} v_i \leq \sum_{i=0}^{4} u_i \\ u_0 + u_2 + u_3 + u_4 \leq v_0 + v_2 + v_3 + v_4 \\ v_0 + v_2 + v_3 + v_4 \leq u_0 + u_2 + u_3 + u_4 \\ u_0 + u_3 + u_4 \leq v_0 + v_3 + v_4, \;\; v_0 + v_3 + v_4 \leq u_0 + u_3 + u_4 \\ u_0 + u_4 \leq v_0 + v_4 \; , v_0 + v_4 \leq u_0 + u_4 \end{cases}$$

Hence:

$$\begin{cases} u_0 = v_0 \\ u_0 + u_4 = v_0 + v_4 \\ u_0 + u_3 + u_4 = v_0 + v_3 + v_4 \\ u_0 + u_2 + u_3 + u_4 = v_0 + v_2 + v_3 + v_4 \\ u_0 + u_1 + u_2 + u_3 + u_4 = v_0 + v_1 + v_2 + v_3 + v_4 \end{cases} \Longrightarrow \begin{cases} u_0 = v_0, u_1 = v_1 \\ u_2 = v_2, u_3 = v_3 \\ u_4 = v_4 \end{cases} \Longrightarrow u = v.$$

If $u \leq v$, and $v \leq w$, then:

$$\begin{cases} u_0 \leq v_0, \sum_{i=0}^{4} u_i \leq \sum_{i=0}^{4} v_i \\ u_0 + u_2 + u_3 + u_4 \leq v_0 + v_2 + v_3 + v_4 \\ u_0 + u_3 + u_4 \leq v_0 + v_3 + v_4 \\ u_0 + u_4 \leq v_0 + v_4 \end{cases}$$

And $$\begin{cases} v_0 \leq w_0 \\ \sum_{i=0}^{4} v_i \leq \sum_{i=0}^{4} w_i \\ v_0 + v_2 + v_3 + v_4 \leq w_0 + w_2 + w_3 + w_4 \\ v_0 + v_3 + v_4 \leq w_0 + w_3 + w_4 \\ v_0 + v_4 \leq w_0 + w_4 \end{cases}$$

Thus: $$\begin{cases} u_0 \leq w_0 \\ \sum_{i=0}^{4} u_i \leq \sum_{i=0}^{4} w_i \\ u_0 + u_4 \leq w_0 + w_4 \\ u_0 + u_3 + u_4 \leq w_0 + w_3 + w_4 \\ u_0 + u_2 + u_3 + u_4 \leq w_0 + w_2 + w_3 + w_4 \end{cases}$$

So that $u \leq w$.

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Definition 2.2

Let u, v be two 4-refined neutrosophic real numbers with $u \leq v$, then we define:

$]u, v[= \{x \in \mu \ ; u < x < v\}$,

$[u, v[= \{x \in \mu \ ; u \leq x < v\}$,

$]u, v] = \{x \in \mu \ ; u < x \leq v\}$,

$[u, v] = \{x \in \mu \ ; u \leq x \leq v\}$.

We say that $]u, v[$ is an open 4-refined neutrosophic interval, and $[u, v]$ is a closed one.

Theorem 2.2

Let $]u, v[,]w, t[$ be two 4-refined neutrosophic open intervals, then:

$]u, v[\cap]w, t[$ is an open interval of $\emptyset$.

Proof:

If $u = u_0 + \sum_{i=1}^{4} u_i I_i$, $v = v_0 + \sum_{i=1}^{4} v_i I_i$, $w = w_0 + \sum_{i=1}^{4} w_i I_i$, $t = t_0 + \sum_{i=1}^{4} t_i I_i$, and $u < v$, $w < t$ with $]u, v[\cap]w, t[\neq \emptyset$,

then there exists $m = m_0 + \sum_{i=1}^{4} m_i I_i$ such that:

$m \in]u, v[, m \in]w, t[$.

Define:

$q = q_0 + \sum_{i=1}^{4} q_i I_i$, $p = p_0 + \sum_{i=1}^{4} p_i I_i$, with:

$q_0 = \max(u_0, w_0)$, $q_4 = \max(u_0 + u_4, w_0 + w_4) - \max(u_0, w_0)$, $q_3 = \max(u_0 + u_3 + u_4, w_0 + w_3 + w_4) - \max(u_0 + u_4, w_0 + w_4)$, $q_2 = \max(u_0 + u_2 + u_3 + u_4, w_0 + w_2 + w_3 + w_4) - \max(u_0 + u_3 + u_4, w_0 + w_3 + w_4)$, $q_1 = \max(\sum_{i=0}^{4} u_i, \sum_{i=0}^{4} w_i) - \max(u_0 + u_2 + u_3 + u_4, w_0 + w_2 + w_3 + w_4)$

$p_0 = \min(v_0, t_0)$, $p_4 = \min(v_0 + v_4, t_0 + t_4) - \min(v_0, t_0)$, $p_3 = \min(v_0 + v_3 + v_4, t_0 + t_3 + t_4) - \min(v_0 + v_4, t_0 + t_4)$, $p_2 = \min(v_0 + v_2 + v_3 + v_4, t_0 + t_2 + t_3 + t_4) - \min(v_0 + v_3 + v_4, t_0 + t_3 + t_4)$, $p_1 = \min(\sum_{i=0}^{4} v_i, \sum_{i=0}^{4} w_i) - \min(v_0 + v_2 + v_3 + v_4, t_0 + t_2 + t_3 + t_4)$, we have:

$m > q$ that is because:

$$\begin{cases} m_0 > u_0, m_0 > w_0 \Longrightarrow m_0 > q_0 \\ \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} u_i, \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} w_i \Longrightarrow \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} q_i \\ m_0 + m_4 > u_0 + u_4, m_0 + m_4 > w_0 + w_4 \Longrightarrow m_0 + m_4 > q_0 + q_4 \\ m_0 + m_3 + m_4 > u_0 + u_3 + u_4, \ m_0 + m_3 + m_4 > w_0 + w_3 + w_4, \\ m_0 + m_3 + m_4 > q_0 + q_3 + q_4, m_0 + m_2 + m_3 + m_4 > u_0 + u_2 + u_3 + u_4 \\ m_0 + m_2 + m_3 + m_4 > w_0 + w_2 + w_3 + w_4, m_0 + m_2 + m_3 + m_4 > q_0 + q_2 + q_3 + q_4 \end{cases}$$

$m < p$ that is because:

$$\begin{cases} m_0 < v_0, m_0 < t_0 \Longrightarrow m_0 < p_0 \\ \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} v_i, \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} t_i \Longrightarrow \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} p_i \\ m_0 + m_4 < v_0 + v_4, m_0 + m_4 < t_0 + t_4, m_0 + m_4 < p_0 + p_4 \\ m_0 + m_3 + m_4 < v_0 + v_3 + v_4, \ m_0 + m_3 + m_4 < t_0 + t_3 + t_4, \\ m_0 + m_3 + m_4 < p_0 + p_3 + p_4, m_0 + m_2 + m_3 + m_4 < v_0 + v_2 + v_3 + v_4 \\ m_0 + m_2 + m_3 + m_4 < t_0 + t_2 + t_3 + t_4, m_0 + m_2 + m_3 + m_4 < p_0 + p_2 + p_3 + p_4 \end{cases}$$

Thus: $]u, v[\cap]w, t[\subseteq]q, p[$

Now, let $m = m_0 + \sum_{i=1}^{4} m_i I_i \in]q, p[$, then:

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$$m > q \Rightarrow \begin{cases} m_0 > u_0 \, , m_0 > w_0 \\ \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} u_i \, , \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} w_i \\ m_0 + m_4 > u_0 + u_4 \ \ , m_0 + m_4 > w_0 + w_4 \\ m_0 + m_3 + m_4 > u_0 + u_3 + u_4, \ \ m_0 + m_3 + m_4 > w_0 + w_3 + w_4 \\ m_0 + m_2 + m_3 + m_4 > u_0 + u_2 + u_3 + u_4 \\ m_0 + m_2 + m_3 + m_4 > w_0 + w_2 + w_3 + w_4 \end{cases} \Rightarrow m > u \text{ and } m > w.$$

$$m < p \ \Rightarrow \begin{cases} m_0 < v_0 \, , m_0 < t_0, \\ \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} v_i \, , \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} t_i, \\ m_0 + m_4 < v_0 + v_4 \ \ , m_0 + m_4 < t_0 + t_4, \\ m_0 + m_3 + m_4 < v_0 + v_3 + v_4, \ \ m_0 + m_3 + m_4 < t_0 + t_3 + t_4, \\ m_0 + m_2 + m_3 + m_4 < v_0 + v_2 + v_3 + v_4, \\ m_0 + m_2 + m_3 + m_4 < t_0 + t_2 + t_3 + t_4, \end{cases} \Rightarrow m < t \text{ and } m < v.$$

Hence $]q,p[\subseteq]u,v[\cap \ ]w,t[$ and $]u,v[\cap]w,t[=]q,p[$.

Remark 2

$[u,v] \cap [w,t] = [q,p]$, i.e. the intersection of two 4-refined neutrosophic real closed intervals is a closed interval or $\emptyset$.

Definition 2.3

We define $\tau = \{\mu, \emptyset, s_i\}$; $s_i = \cup_{i\in I}]u_i, v_i[$, with $u_i, v_i \in \mu$.

Theorem 2.3

(μ, τ) is a topological space.

Proof:

$\mu \in \tau, \emptyset \in \tau$.

Let $s_i, s_j \in \tau$, then:

$s_i = \cup_{i\in I}]u_i, v_i[\, , s_j = \cup_{j\in I}]u_j, v_j[$, then:

$s_i \cap s_j = \bigcup_{k\in K}]p_k, q_k[\in \tau$.

Let $\{(s_i)_j\}$ be a family of τ, then:

$\cup_{j\in J} (s_i)_j = \cup_{j\in J} (\cup_{i\in I}]u_i, v_i[) \in \tau$, thus τ is a topology and (μ, τ) is a topological space.

Example 2.1

Consider: $u_1 = 2 + I_1 + I_2 + I_3 + I_4 \, , v_1 = 3 + 2I_1 + I_2 + 3I_3 + 2I_4, u_2 = 1 + 2I_1 + I_2 + I_3 + 2I_4, v_1 = 2 + 3I_1 + 3I_2 + I_3 + 3I_4$, we have: $u_1 < v_1, u_2 < v_2,]u_1, v_1[\in \tau,]u_2, v_2[\in \tau,]u_1, v_1[\cup]u_2, v_2[\in \tau,]u_1, v_1[\cap]u_2, v_2[=]q, p[$;

$q = 2 + 2I_1 + I_2 + I_3 + I_4 \, , p = 2 + 2I_1 + 3I_2 + I_3 + 3I_4$.

Remark 3

Open sets of μ are the union of open intervals.

The complements of open sets are closed sets.

Definition 2.4

Let $\mu = \{x + yI_1 + zI_2 + tI_3 + qI_4 + sI_5 \ ; x, y, z, t, q, s \ \in \mathbb{R}\}$ and

$I_1.I_2 = I_2.I_1 = I_1.I_3 = I_3.I_1 = I_1.I_4 = I_4.I_1 = I_5.I_1 = I_1, \ I_2.I_3 = I_3.I_2 = \ I_2,$

$I_4 = I_4.I_2 = I_5.I_2 = I_2, I_3.I_4 = I_4.I_3 = I_5.I_3 = I_3, {I_1}^2 = I_1, {I_2}^2 = I_2, {I_3}^2 = I_3, {I_4}^2 = I_4, {I_5}^2 = I_5.$

$I_5 I_4 = I_4 I_5 = I_4.$

μ is called the real 5-refined neutrosophic ring.

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Remark 4

For $x = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4 + x_5 I_5 \in \mu$, we denote it by $x = x_0 + \sum_{i=1}^{5} x_i I_i$.

Theorem 2.4

μ is partially ordered set.

Proof:

We define the relation $(\leq)$ as follows:

$u = u_0 + \sum_{i=1}^{5} u_i I_i \leq v = v_0 + \sum_{i=1}^{5} v_i I_i$, if and only if:

$$\begin{cases} u_0 \leq v_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i \\ \sum_{i\neq 1} u_i \leq \sum_{i\neq 1} v_i \\ \sum_{i\neq 1,2} u_i \leq \sum_{i\neq 1,2} v_i \\ \sum_{i\neq 1,2,3} u_i \leq \sum_{i\neq 1,2,3} v_i \\ \sum_{i\neq 1,2,3,4} u_i \leq \sum_{i\neq 1,2,3,4} v_i \end{cases}.$$

We will show that $(\leq)$ is a partial order relation:

Let $u = u_0 + \sum_{i=1}^{5} u_i I_i$, $v = v_0 + \sum_{i=1}^{5} v_i I_i$, $w = w_0 + \sum_{i=1}^{5} w_i I_i$, $u \leq u$ that is because:

$$\begin{cases} u_0 \leq u_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} u_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq u_0 + u_2 + u_3 + u_4 + u_5 \\ u_0 + u_3 + u_4 + u_5 \leq u_0 + u_3 + u_4 + u_5 \\ u_0 + u_4 + u_5 \leq u_0 + u_4 + u_5 \\ u_0 + u_5 \leq u_0 + u_5 \end{cases}$$

If $u \leq v$ and $v \leq u$, then:

$$\begin{cases} u_0 \leq v_0, v_0 \leq u_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i, \sum_{i=0}^{5} v_i \leq \sum_{i=0}^{5} u_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq v_0 + v_2 + v_3 + v_4 + v_5 \\ v_0 + v_2 + v_3 + v_4 + v_5 \leq u_0 + u_2 + u_3 + u_4 + u_5 \\ u_0 + u_3 + u_4 + u_5 \leq v_0 + v_3 + v_4 + v_5, \ v_0 + v_3 + v_4 + v_5 \leq u_0 + u_3 + u_4 + u_5 \\ u_0 + u_4 + u_5 \leq v_0 + v_4 + v_5, v_0 + v_4 + v_5 \leq u_0 + u_4 + u_5 \\ u_0 + u_5 \leq v_0 + v_5, v_0 + v_5 \leq u_0 + u_5 \end{cases}$$

Hence:

$$\begin{cases} u_0 = v_0 \\ u_0 + u_4 + u_5 = v_0 + v_4 + v_5 \\ u_0 + u_3 + u_4 + u_5 = v_0 + v_3 + v_4 + v_5 \\ u_0 + u_2 + u_3 + u_4 + u_5 = v_0 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_5 = v_0 + v_5 \end{cases} \Rightarrow \begin{cases} u_0 = v_0, u_1 = v_1 \\ u_2 = v_2, u_3 = v_3 \\ u_4 = v_4 \\ u_5 = v_5 \end{cases} \Rightarrow u = v.$$

If $u \leq v$, and $v \leq w$, then:

$$\begin{cases} u_0 \leq v_0, \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq v_0 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_3 + u_4 + u_5 \leq v_0 + v_3 + v_4 + v_5 \\ u_0 + u_4 + u_5 \leq v_0 + v_4 + v_5 \\ u_0 + u_5 \leq v_0 + v_5 \end{cases}$$

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And $\begin{cases} v_0 \leq w_0 \\ \sum_{i=0}^{5} v_i \leq \sum_{i=0}^{5} w_i \\ v_0 + v_2 + v_3 + v_4 + v_5 \leq w_0 + w_2 + w_3 + w_4 + w_5 \\ v_0 + v_3 + v_4 + v_5 \leq w_0 + w_3 + w_4 + w_5 \\ v_0 + v_4 + v_5 \leq w_0 + w_4 + w_5 \\ v_0 + v_5 \leq w_0 + w_5 \end{cases}$

Thus: $\begin{cases} u_0 \leq w_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} w_i \\ u_0 + u_4 + u_5 \leq w_0 + w_4 + w_5 \\ u_0 + u_3 + u_4 + u_5 \leq w_0 + w_3 + w_4 + w_5 \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq w_0 + w_2 + w_3 + w_4 + w_5 \\ u_0 + u_5 \leq w_0 + w_5 \end{cases}$

So that $u \leq w$.

Definition 2.5

Let u, v be two 5-refined neutrosophic real numbers with $u \leq v$, then we define:

$]u, v[= \{x \in \mu \ ; u < x < v\}$,

$[u, v[= \{x \in \mu \ ; u \leq x < v\}$,

$]u, v] = \{x \in \mu \ ; u < x \leq v\}$,

$[u, v] = \{x \in \mu \ ; u \leq x \leq v\}$.

We say that $]u, v[$ is an open 5-refined neutrosophic interval, and $[u, v]$ is a closed one.

Theorem 2.5

Let $]u, v[,]w, t[$ be two 5-refined neutrosophic open intervals, then:

$]u, v[\cap]w, t[$ is an open interval of $\emptyset$.

Proof:

If $u = u_0 + \sum_{i=1}^{5} u_i I_i$, $v = v_0 + \sum_{i=1}^{5} v_i I_i$, $w = w_0 + \sum_{i=1}^{5} w_i I_i$, $t = t_0 + \sum_{i=1}^{5} t_i I_i$, and $u < v$, $w < t$ with$]u, v[\cap]w, t[\neq \emptyset$,

then there exists $m = m_0 + \sum_{i=1}^{5} m_i I_i$ such that: $m \in]u, v[, m \in]w, t[$.

Define:

$q = q_0 + \sum_{i=1}^{5} q_i I_i$, $p = p_0 + \sum_{i=1}^{5} p_i I_i$, with:

$q_0 = \max(u_0, w_0)$, $q_4 = \max(u_0 + u_4 + u_5, w_0 + w_4 + w_5) - \max(u_0 + u_5, w_0 + w_5)$, $q_3 = \max(u_0 + u_3 + u_4 + u_5, w_0 + w_3 + w_4 + w_5) - \max(u_0 + u_4 + u_5, w_0 + w_4 + w_5)$, $q_2 = \max(u_0 + u_2 + u_3 + u_4 + u_5, w_0 + w_2 + w_3 + w_4 + w_5) - \max(u_0 + u_3 + u_4 + u_5, w_0 + w_3 + w_4 + w_5)$,$q_1 = \max\left(\sum_{i=0}^{5} u_i, \sum_{i=0}^{5} w_i\right) - \max(u_0 + u_2 + u_3 + u_4 + u_5, w_0 + w_2 + w_3 + w_4 + w_5)$, $q_5 = \max(u_0 + u_5, w_0 + w_5) - \max(u_0, w_0)$,

$p_0 = \min(v_0, t_0)$, $p_4 = \min(v_0 + v_4 + v_5, t_0 + t_4 + t_5) - \min(v_0, t_0)$, $p_3 = \min(v_0 + v_3 + v_4 + v_5, t_0 + t_3 + t_4 + t_5) - \min(v_0 + v_4 + v_5, t_0 + t_4 + t_5)$, $p_2 = \min(v_0 + v_2 + v_3 + v_4 + v_5, t_0 + t_2 + t_3 + t_4 + t_5) - \min(v_0 + v_3 + v_4 + v_5, t_0 + t_3 + t_4 + t_5)$,$p_1 = \min\left(\sum_{i=0}^{5} v_i, \sum_{i=0}^{5} w_i\right) - \min(v_0 + v_2 + v_3 + v_4 + v_5, t_0 + t_2 + t_3 + t_4 + t_5)$, $p_5 = \min(v_0 + v_5, t_0 + t_5) - \min(v_0, t_0)$,, we have:

$m > q$ that is because:

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$$\begin{cases} m_0 > u_0\,, m_0 > w_0 \Longrightarrow m_0 > q_0 \\ \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} u_i\,, \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} w_i \Longrightarrow \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} q_i \\ m_0 + m_4 + m_5 > u_0 + u_4 + u_5\,, m_0 + m_4 + m_5 > w_0 + w_4 + w_5 \Longrightarrow m_0 + m_4 + m_5 > q_0 + q_4 + q_5 \\ m_0 + m_3 + m_4 + m_5 > u_0 + u_3 + u_4 + u_5,\ m_0 + m_3 + m_4 + m_5 > w_0 + w_3 + w_4 + w_5, \\ m_0 + m_3 + m_4 + m_5 > q_0 + q_3 + q_4, m_0 + m_2 + m_3 + m_4 + m_5 > u_0 + u_2 + u_3 + u_4 + u_5 \\ m_0 + m_2 + m_3 + m_4 + m_5 > w_0 + w_2 + w_3 + w_4 + w_5, m_0 + m_2 + m_3 + m_4 + m_5 > q_0 + q_2 + q_3 + q_4 + q_5 \\ m_0 + m_5 > u_0 + u_5\,, m_0 + m_5 > w_0 + w_5 \Longrightarrow m_0 + m_5 > q_0 + q_5 \end{cases}$$

$m < p$ that is because:

$$\begin{cases} m_0 < v_0\,, m_0 < t_0 \Longrightarrow m_0 < p_0 \\ \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} v_i\,, \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} t_i \Longrightarrow \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} p_i \\ m_0 + m_4 + m_5 < v_0 + v_4 + v_5\,, m_0 + m_4 + m_5 < t_0 + t_4 + t_5, m_0 + m_4 + m_5 < p_0 + p_4 + p_5 \\ m_0 + m_3 + m_4 + m_5 < v_0 + v_3 + v_4 + v_5,\ m_0 + m_3 + m_4 + m_5 < t_0 + t_3 + t_4 + t_5, \\ m_0 + m_3 + m_4 + m_5 < p_0 + p_3 + p_4 + p_5, m_0 + m_2 + m_3 + m_4 + m_5 < v_0 + v_2 + v_3 + v_4 + v_5 \\ m_0 + m_2 + m_3 + m_4 + m_5 < t_0 + t_2 + t_3 + t_4 + t_5, m_0 + m_2 + m_3 + m_4 + m_5 < p_0 + p_2 + p_3 + p_4 + p_5 \\ m_0 + m_5 < v_0 + v_5\,, m_0 + m_5 < t_0 + t_5, m_0 + m_5 < p_0 + p_5 \end{cases}$$

Thus: $]u, v[\, \cap\,]w, t[\, \subseteq\,]q, p[$

Now, let $m = m_0 + \sum_{i=1}^{5} m_i I_i \in\,]q, p[$, then:

$$m > q \Longrightarrow \begin{cases} m_0 > u_0\,, m_0 > w_0 \\ \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} u_i\,, \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} w_i \\ m_0 + m_4 + m_5 > u_0 + u_4 + u_5\,, m_0 + m_4 + m_5 > w_0 + w_4 + w_5 \\ m_0 + m_3 + m_4 > u_0 + u_3 + u_4 + u_5,\ m_0 + m_3 + m_4 > w_0 + w_3 + w_4 + w_5 \\ m_0 + m_2 + m_3 + m_4 + m_5 > u_0 + u_2 + u_3 + u_4 \\ m_0 + m_2 + m_3 + m_4 + m_5 > w_0 + w_2 + w_3 + w_4 + w_5 \end{cases} \Longrightarrow m > u \text{ and } m > w.$$

$$m < p \Longrightarrow \begin{cases} m_0 < v_0\,, m_0 < t_0, \\ \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} v_i\,, \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} t_i, \\ m_0 + m_4 + m_5 < v_0 + v_4 + v_5\,, m_0 + m_4 + m_5 < t_0 + t_4 + t_5, \\ m_0 + m_3 + m_4 + m_5 < v_0 + v_3 + v_4 + v_5,\ m_0 + m_3 + m_4 + m_5 < t_0 + t_3 + t_4 + t_5, \\ m_0 + m_2 + m_3 + m_4 + m_5 < v_0 + v_2 + v_3 + v_4 + v_5, \\ m_0 + m_2 + m_3 + m_4 + m_5 < t_0 + t_2 + t_3 + t_4 + t_5, \end{cases} \Longrightarrow m < t \text{ and } m < v.$$

Hence $]q, p[\, \subseteq\,]u, v[\, \cap\,]w, t[$ and $]u, v[\, \cap\,]w, t[\, =\,]q, p[$.

Remark 5

$[u, v] \cap [w, t] = [q, p]$, i.e. the intersection of two 5-refined neutrosophic real closed intervals is a closed interval or $\emptyset$.

Definition 2.6

We define $\tau = \{\mu, \emptyset, s_i\}$; $s_i = \bigcup_{i\in I}]u_i, v_i[$, with $u_i, v_i \in \mu$.

Theorem 2.6

(μ, τ) is a topological space.

Proof:

$\mu \in \tau, \emptyset \in \tau$.

Let $s_i, s_j \in \tau$, then:

$s_i = \bigcup_{i\in I}]u_i, v_i[\,, s_j = \bigcup_{j\in I}]u_j, v_j[$, then:

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$s_i \cap s_j = \bigcup_{k\in K}]p_k, q_k[\in \tau.$

Let $\{(s_i)_j\}$ be a family of τ, then:

$\cup_{j\in J}(s_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[) \in \tau$, thus τ is a topology and (μ, τ) is a topological space.

Remark 6

Open sets of μ are the union of open intervals with respect to the partial order relation.

The complements of open sets are closed sets.

Example 2.2

Consider $u = 2 + I_1 + I_2 + I_3 + I_4 + I_5, v = 2 + 2I_1 + I_2 + 2I_3 + I_4 + 2I_5$,

Let $x = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 + x_5I_5 \in [u, v]$, then:

$$\begin{cases} x_0 = 2 \\ 3 \le x_0 + x_5 \le 4 \\ 4 \le x_0 + x_4 + x_5 \le 5 \\ 5 \le x_0 + x_3 + x_4 + x_5 \le 7 \\ 6 \le x_0 + x_2 + x_3 + x_4 + x_5 \le 8 \\ 7 \le x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \le 10 \end{cases}$$

Thus

$$\begin{cases} x_0 = 2 \\ 1 \le x_5 \le 2 \\ 2 \le x_4 + x_5 \le 3 \\ 3 \le x_3 + x_4 + x_5 \le 5 \\ 4 \le x_2 + x_3 + x_4 + x_5 \le 6 \\ 5 \le x_1 + x_2 + x_3 + x_4 + x_5 \le 8 \end{cases}$$

Theorem 2.7

Let (μ, τ) be the topological space of 4-refined neutrosophic numbers intervals, and (Δ, β) be the topological space of 5-refined neutrosophic numbers intervals, then:

$(\mu \times \Delta, (\tau, \beta)) = \{(\emptyset, \emptyset), (\mu, \Delta), (s_i, t_i); s_i \in \tau, t_i \in \beta\}$, is a topological space.

Proof:

$(\mu, \Delta) \in (\tau, \beta), (\emptyset, \emptyset) \in (\tau, \beta)$.

Let $(s_i, t_i), (s_j, t_j) \in (\tau, \beta)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[, s_j = \cup_{j\in I}]u_j, v_j[, t_i = \cup_{i\in I}]m_i, n_i[, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[, \bigcup_{k\in K}]h_k, f_k[) \in (\tau, \beta).$

Let $\{(s_i, t_i)_j\}$ be a family of (τ, β), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[, \cup_{i\in I}]m_i, n_i[) \in (\tau, \beta)$, thus (τ, β) is a topology and $(\mu \times \Delta, (\tau, \beta))$ is a topological space.

Theorem 2.8

Let (μ, τ) be the topological space of 4-refined neutrosophic numbers intervals, then:

$(\mu \times \mu, (\tau, \tau)) = \{(\emptyset, \emptyset), (\mu, \mu), (s_i, t_i); s_i \in \tau, t_i \in \tau\}$, is a topological space.

Proof:

$(\mu, \mu) \in (\tau, \tau), (\emptyset, \emptyset) \in (\tau, \tau)$.

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Let $(s_i, t_i), (s_j, t_j) \in (\tau, \tau)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[, s_j = \cup_{j\in I}]u_j, v_j[, t_i = \cup_{i\in I}]m_i, n_i[, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[, \bigcup_{k\in K}]h_k, f_k[) \in (\tau, \tau)$.

Let $\{(s_i, t_i)_j\}$ be a family of (τ, τ), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[, \cup_{i\in I}]m_i, n_i[) \in (\tau, \tau)$, thus (τ, τ) is a topology and $(\mu \times \mu, (\tau, \tau))$ is a topological space.

Theorem 2.9

Let (μ, τ) be the topological space of 5-refined neutrosophic numbers intervals, then:

$(\mu \times \mu, (\tau, \tau)) = \{(\emptyset, \emptyset), (\mu, \mu), (s_i, t_i); s_i \in \tau, t_i \in \tau\}$, is a topological space.

Proof:

$(\mu, \mu) \in (\tau, \tau), (\emptyset, \emptyset) \in (\tau, \tau)$.

Let $(s_i, t_i), (s_j, t_j) \in (\tau, \tau)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[, s_j = \cup_{j\in I}]u_j, v_j[, t_i = \cup_{i\in I}]m_i, n_i[, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[, \bigcup_{k\in K}]h_k, f_k[) \in (\tau, \tau)$.

Let $\{(s_i, t_i)_j\}$ be a family of (τ, τ), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[, \cup_{i\in I}]m_i, n_i[) \in (\tau, \tau)$, thus (τ, τ) is a topology and $(\mu \times \mu, (\tau, \tau))$ is a topological space.

3. Conclusion

In this paper we studied for the first time the building of a topological space based on the intervals defined over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we defined a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterized the formula of open sets through these two topological spaces with some illustrated examples.

References

[1] I. M. Hanafy, A. A. Salama and K.M. Mahfouz (2013).Neutrosophic crisp events and its probability. International Journal of Mathematics and Computer Applications Research,3(1),171-178.

[2] Kim, J., Lee, J. G., & Hur, K. (2021). Intuitionistic neutrosophic crisp sets and their application to topology. Infinite Study.

[3] A. M. AL-Odhari. (2015).On infra topological space. International Journal of Mathematical Archive, 6(11), 179-184.

[4] Abobala, M., Bal, M., & Hatip, A. (2021). A review on recent advantages in algebraic theory of neutrosophic matrices. International Journal of Neutrosophic Science, 68-86.

[5] Abobala, M., Ziena, B. M., Doewes, R., & Hussein, Z. (2022). The Representation of Refined Neutrosophic Matrices by Refined Neutrosophic Linear Functions. Int. J. Neutrosophic Sci, 19, 342-349.

[6] Ahmad, K. D., Thjeel, N. N., Zahra, M. A., & Jaleel, R. A. (2022). On the Classification of n-Refined Neutrosophic Rings and Its Applications in Matrix Computing Algorithms and Linear Systems. International Journal Of Neutrosophic Science, 18(4).

[7] Abobala, M. (2020). A study of AN H-substructures in n-refined neutrosophic vector spaces. International Journal of Neutrosophic Science, 9, 74-85.

[8] Abobala, M., On Refined Neutrosophic Matrices and Their Applications in Refined Neutrosophic Algebraic Equations, Journal of Mathematics, Hindawi, 2021.

[9] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., Refined Neutrosophic Rings II", International Journal of Neutrosophic Science, Vol. 2(2), pp. 89-94, 2020.

[10] Hatip, A., and Olgun, N., "On Refined Neutrosophic R-Module", International Journal of Neutrosophic Science, Vol. 7, pp.87-96, 2020.

DOI: https://doi.org/10.54216/IJNS.250326
Received: July 15, 2024 Revised: August 08, 2024 Accepted: October 25, 2024

[11] Smarandache, F., and Abobala, M., n-Refined Neutrosophic Rings, International Journal Of Neutrosophic Science, 2020.

[12] Abobala, M., "On the Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic Sets and Systems, Vol. 45, 2021.

[13] AL-Nafee, A. B., Al-Hamido, R. K., & Smarandache, F. (2019). Separation axioms in neutrosophic crisp topological spaces. Infinite Study.

[14] Saber, W. A., N. M., A. (2023). N-refined Neutrosophic Fuzzy of Some Topological Concepts. International Journal of Neutrosophic Science, (), 80-87.

[15] Riad K. Al-Hamido. (2018). Neutrosophic crisp bi-topological spaces. Neutrosophic Sets and Systems 21, 66-73.

[16] Ali, R., "A Short Note on the Solution of n-Refined Neutrosophic Linear Diophantine Equations", International Journal of Neutrosophic Science, Vol. 15, 2021.

[17] Ibrahim, M., and Abobala, M., "An Introduction to Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[18] Abobala, M, "n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, an Application to Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95. 2020.

[19] Mohammed, H. Y., Mohammed, F. M., & Al-mahbashi, G. (2024). On Q^*-closed Sets in Fuzzy Neutrosophic Topology: Principles, Proofs, and Examples. Neutrosophic Systems with Applications, 20, 67-74. https://doi.org/10.61356/j.nswa.2024.20352

[20] Vetrivel, G., Mullai, M., & Buvaneshwari, R. (2024). Forgotten Topological Index and its Properties on Neutrosophic Graphs. Neutrosophic Systems with Applications, 21, 36-45. https://doi.org/10.61356/j.nswa.2024.21356

[21] Raed Hatamleh , Ayman Hazaymeh.(2025),On Some Topological Spaces Based On Symbolic n-Plithogenic Intervals, International Journal of Neutrosophic Science, Vol. 25 Issue.1 PP. 23-37. Doi: https://doi.org/10.54216/IJNS.250102.

[22] Raed Hatamleh , Ayman Hazaymeh,(2025),On The Topological Spaces of Neutrosophic Real Intervals, International Journal of Neutrosophic Science, Vol. 25 Issue. 1 PP. 130-136, (Doi: https://doi.org/10.54216/IJNS.250111.

[23] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri, Abdallah Al-Husban,(2024), On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2≤n≤3, Neutrosophic Sets and Systems, Vol. 68,pp.8 25.(Doi: 10.5281/zenodo.11406449).

[24] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems,vol.67, pp. 169-178.(DOI:10.5281/zenodo.11151930).

[25] A. Rajalakshmi, Raed Hatamleh, Abdallah Al-Husban, K. Lenin Muthu Kumaran, M. S. Malchijah raj. (2025). various (ζ1, ζ2) neutrosophic ideals of an ordered ternary semigroups. 32 (3), PP: 400-417.

[26] Raed Hatamleh, Abdallah Al-Husban, N. Sundarakannan, M. S. Malchijah Raj. (2025). Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism. 32 (3), PP: 418-435.

[27] Abu, Ibraheem. , Al-Husban, Abdallah. , J., Lejo. , J., Jamil. , Palanikumar, M., Balaji, G..(2024). Selection process real-life application for new type complex neutrosophic sets using various aggregation operators. International Journal of Neutrosophic Science, 23(4) 136-153.

[28] Selvaraj, S.. , Gharib, Gharib. , Al-Husban, Abdallah. , Al, Maha. , Lenin, K.. , Palanikumar, Murugan. , Sundareswari, K.. (2024) new algebraic approach towards interval-valued neutrosophic cubic vague set based on subbisemiring over bisemiring. International Journal of Neutrosophic Science, 23(4), pp. 272-292.

[29] C. Sivakumar, Mowafaq Omar Al-Qadri, Abdallah shihadeh, Ahmed Atallah Alsaraireh, Abdallah Al-Husban, P. Maragatha Meenakshi, N. Rajesh, M. Palanikumar. (2024). q-rung square root interval-valued sets with respect to aggregated operators using multiple attribute decision making. Journal of, 23(3), pp. 154-174.

[30] Abu, Ibraheem.,T., T.,Al-Husban, Abdallah.,Alahmade, Ayman.,Azhaguvelavan, S.. , Palanikumar, Murugan. (2024) Computer purchasing using new type neutrosophic sets and its extension based on aggregation operators. International Journal of Neutrosophic Science, 24(1), pp. 171-185.

[31] Raja, K., Meenakshi, P. M., Al-Husban, A., Al-Qadri, M. O., Rajesh, N., & Palanikumar, M. (2024). Multi-criteria group decision making approach based on a new type of neutrosophic vague approach is used to select the shares of the companies for purchase. Full Length Article, 23(3), 296-96.

DOI: https://doi.org/10.54216/IJNS.250326
Received: July 15, 2024 Revised: August 08, 2024 Accepted: October 25, 2024

[32] J., Lejo. , Damrah, Sadeq. , M., Mutaz. , Al-Husban, Abdallah. , Palanikumar, M.. (2024). Type-I extension Diophantine neutrosophic interval valued soft set in real life applications for a decision making. International Journal of Neutrosophic Science, 24(4), pp. 151-164 .

[33] Abu, Ibraheem. , Al-Husban, Abdallah. , J., Lejo. , J., Jamil. , Palanikumar, M.. , Balaji, G.. (2024). Selection process real-life application for new type complex neutrosophic sets using various aggregation operators. International Journal of Neutrosophic Science, 23(4), pp. 136-153.

[34] Shihadeh, Abdallah. , Mahmoud, Wael. , Bataineh, Malik. , Al-Tarawneh, Hassan. , Alahmade, Ayman. , Al-Husban, Abdallah. (2024) On the Geometry of Weak Fuzzy Complex Numbers and Applications to the Classification of Some A-Curves. International Journal of Neutrosophic Science, 23(4), pp. 369-375.

[35] Abualhomos, Mayada. , Mahmoud, Wael. , Bataineh, Malik. , Omar, Mowafaq. , Alahmade, Ayman. , Al-Husban, Abdallah. (2024). An Effective Algorithm for Solving Weak Fuzzy Complex Diophantine Equations in Two Variables. International Journal of Neutrosophic Science, 23(4), pp. 386-394.

[36] Abualhomos, Mayada. , Atallah, Ahmed. , Bataineh, Malik. , Abu-Alkishik, Nabeela. , Al-Shbeil, Isra. , Al-Husban, Abdallah.(2024). On The Real Inner Products and Orthogonality for Plithogenic Vector Spaces of Orders 4 and 5. International Journal of Neutrosophic Science, 24 (1), pp. 94-103.

[37] Abu, Ibraheem. , Al-Husban, Abdallah. , M., Mutaz. , A., Ahmed. , Shatarah, Amani. , Mousa, Norah.(2024) On the Characterization of Some m-Plithogenic Vector Spaces and Their AH-Substructures Under the Condition $6 \leq \dim SPV \leq 10$. International Journal of Neutrosophic Science, 24(3), pp. 258-267.

[38] Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Al-Qadri, M. O., & Al-Husban, A. (2024). On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2 \leq 3$. Neutrosophic Sets and Systems, 68, 8-25.

DOI: https://doi.org/10.54216/IJNS.250326
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