A Note on Two-Fold Neutrosophic and Fuzzy Topological Space Based on Real Numbers

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Abstract

The objective of this paper is to introduce for the first time the concept of two-fold neutrosophic and fuzzy topological space defined over real numbers, where we combine the two-fold neutrosophic sets with real numbers to get a novel topological space based on them. Also, we present many of its elementary properties and special subsets such as two-fold neutrosophic open sets, two-fold neutrosophic closed sets, and two-fold neutrosophic closure. Many examples and theorems will be provided to clarify the validity of our approach.

Keywords: Two-fold algebra; Two-fold neutrosophic topology; Two-fold neutrosophic open set; Two-fold neutrosophic closed set

1. Introduction

Two-fold neutrosophic algebras are new algebraic structures presented by Smarandache [1] by combining neutrosophic values of truth, falsity, and indeterminacy with classical algebraic sets. These ideas were used by many authors to generalize other famous algebraic structures such as two-fold fuzzy number theoretical systems [2-3], two-fold modules and spaces [4], and two-fold fuzzy rings [5]. Also, they were used in the study of some special two-fold complex functions such as Gamma function [7], and in extending n-refined neutrosophic rings [6]. Neutrosophic topological spaces and their related structures were studied by many authors, their special subsets such as closed and open sets have been discussed in details for many different types of neutrosophic topological spaces [8-13].

This has motivated us to introduce for the first time the concept of two-fold neutrosophic and fuzzy topological space defined over real numbers, where we combine the two-fold neutrosophic sets with real numbers to get a novel topological space based on them. Also, we present many of its elementary properties and special subsets such as two-fold neutrosophic open sets, two-fold neutrosophic closed sets, and two-fold neutrosophic closure. Many examples and theorems will be provided to clarify the validity of our approach.

2. Main Discussion

Definition 2.1

Let R be the real field, we define:

$T_R = \{x_{(y,t_y,i_y,f_y)}\ ; x, y \in R, t_y, i_y, f_y \in [0,1]\}$.

The components (t_y, i_y, f_y) refer to neutrosophic values of truth, indeterminacy and falsity.

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Definition 2.2

Let N be a topological space on ℝ, the corresponding two-fold neutrosophic topological space is defined as follows:

$T_N = \{x_{(y,t_y,i_y,f_y)}\ ; x \in N, y \in R, t_y, i_y, f_y \in [0,1]\}$.

Definition 2.3

Let $x_{(y,t_y,i_y,f_y)} = A$, $z_{(l,t_l,i_l,f_l)} = B \in T_N$, we define:

$A \wedge B = (x \wedge z)_{(c,t_c,i_c,f_c)}\ ; c = \min(y,l), t_c = \min(t_l,t_y), i_c = \max(i_y,i_l), f_c = \max(f_l,f_y)$.

$A \vee B = (x \vee z)_{(c,t_c,i_c,f_c)}\ ; c = \max(y,l), t_c = \max(t_l,t_y), i_c = \min(i_y,i_l), f_c = \min(f_l,f_y)$.

The complement of A is defined as follows:

$A^c = (x^c)_{(y,f_y,1-i_y,t_y)}$

The two-fold empty set is defined as:

$\emptyset_N = \{(\emptyset)_{(y,0,1,1)}\ \forall\ y \in \mathbb{R}\}$.

Definition 2.4

Consider $A = (x)_{(y,t_y,i_y,f_y)}$, $B = (z)_{(l,t_l,i_l,f_l)}$, then:

$A \subseteq B$ if and only if: $\begin{cases} x \subseteq z \\ t_y \le t_l, i_y \ge i_l, f_y \ge f_l \end{cases}$

Theorem 2.1

Let $A, B, c\ \in T_N$ with:

$A = (x)_{(y,t_y,i_y,f_y)}$, $B = (z)_{(l,t_l,i_l,f_l)}$, $C = (k)_{(s,t_s,i_s,f_s)}$, then:

1] $A \wedge A = A$, $A \wedge \emptyset_N = \emptyset_N$

2] $A \vee A = A$, $A \vee \emptyset_N = A$

3] $A \wedge B = B \wedge A$, $A \vee B = B \vee A$

4] $A \wedge (B \wedge C) = (A \wedge B) \wedge C$, $A \vee (B \vee C) = (A \vee B) \vee C$

5] $(A^c)^c = A$

6] $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$

Definition 2.5

Open sets in T_N are:

$O_{T_N} = \left\{A = (x)_{(y,t_y,i_y,f_y)}; x \in \tau_1, y \in \tau_2\right\}$; τ_1 is the topology defined over N, τ_2 is the neutrosophic topology defined over ℝ.

Closed sets on T_N are:

$C_{T_N} = \{A^c\ ; A \in O_{T_N}\}$.

Theorem 2.2

Let $A, B \in O_{T_N}$, then:

1] $A \wedge B \in O_{T_N}$

2] $A \vee B \in O_{T_N}$

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Theorem 2.3

Let $A, B \in C_{T_N}$, then:

1] $A \wedge B \in C_{T_N}$

2] $A \vee B \in C_{T_N}$

Definition 2.6

Let $A \in T_N$, then we define:

$int(A) = \bigcup_{i \in I} B_i \ \ ; B_i \in O_{T_N}, B_i \subseteq A.$

Theorem 2.4

Let $A \in T_N$, then:

1] $int(A) \subseteq A$

2] $int(A \wedge B) = int(A) \wedge int(B)$

3] $int(int(A)) = int(A)$

4] $A \subseteq B \Longrightarrow int(A) \subseteq int(B)$

5] $int(A) \vee int(B) \subseteq int(A \vee B)$

6] $A \in O_{T_N} \Leftrightarrow A = int(A)$

Definition 2.7

Let $A \in T_N$, then we define:

$cl(A) = \bigcap_{i \in I} B_i \ ; B_i \in C_{T_N}, A \subseteq B_i.$

Theorem 2.5

Let $A, B \in T_N$, then:

1] $A \subseteq cl(A), cl(cl(A)) = cl(A).$

2] $A \subseteq B \Longrightarrow cl(A) \subseteq cl(B)$

3] $cl(A \vee B) = cl(A) \vee cl(B)$

4] $cl(A \wedge B) \subseteq cl(A) \wedge cl(B)$

5] $A \in C_{T_N} \Leftrightarrow A = cl(A)$

Theorem 2.6

Let $A \in T_N$, then:

1] $int(A^c) = (cl(A))^c.$

2] $cl(A^c) = (int(A))^c.$

Definition 2.8

Let $A \in T_N$,we define: $ext(A) = int(A^c).$

Theorem 2.7

Let $A, B \in T_N$, then:

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1] $ext(A \vee B) = ext(A) \wedge ext(B)$

2] $ext(A) \vee ext(B) \subseteq ext(A \wedge B)$

Definition 2.9

Let $A, B \in T_N$, we define:

$fr(A) = cl(A) \cap cl(A^c)$

Theorem 2.8

Let $A, B \in T_N$, then:

1] $(fr(A))^c = ext(A) \vee int(A)$

2] $cl(A) = int(A) \vee fr(A)$

3] $fr(A) \wedge int(A) = \emptyset$

4] $fr(int(A)) \subseteq fr(A)$

Proof of theorem (.21):

1] $A \wedge A = (x)_{(y,t_y,i_y,f_y)} \wedge (x)_{(y,t_y,i_y,f_y)} = (x \wedge x)_{(y,t_y,i_y,f_y)} = (x)_{(y,t_y,i_y,f_y)} = A.$

$A \wedge \emptyset_N = \emptyset_N$ clearly.

2] $A \vee A = (x \vee x)_{(y,t_y,i_y,f_y)} = (x)_{(y,t_y,i_y,f_y)} = A.$

$A \vee \emptyset_N = \emptyset_N$ clearly.

3] It holds directly from the definition.

4] $A \wedge (B \wedge C) = (x \wedge y \wedge k)_{(m,t_m,i_m,f_m)}$; $m = \min(y,l,s), t_m = \min(t_y,t_l,t_s)$, $i_m = \max(i_y,i_l,i_s), f_m = \max(f_y,f_l,f_s)$, thus $A \wedge (B \wedge C) = (A \wedge B) \wedge C.$

The second formula can be proved by the same.

5] $(A^c)^c = (x^c)^c_{(y,t_y,i_y,f_y)} = (x)_{(y,t_y,i_y,f_y)} = A.$

6] we have: $(x \vee z)^c = x^c \wedge z^c$, $(x \wedge z)^c = x^c \vee z^c$, and:

$(A \vee B)^c = (x^c \wedge z^c)_{(m,t_m,i_m,f_m)}$; $(A \wedge B)^c = (x^c \vee z^c)_{(n,t_n,i_n,f_n)}$

So that: $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$

Proof of theorem (.22):

1]-2] Since $x \wedge z, x \vee z \in \tau_1$ and $\min(y,l)$, $\max(y,l) \in \tau_2$,

$$\begin{cases} \min(t_y,t_l), \max(t_y,t_l) \in [0,1] \\ \min(i_y,i_l), \max(i_y,i_l) \in [0,1] \\ \min(f_y,f_l), \max(f_y,f_l) \in [0,1] \end{cases}$$

Hence $A \wedge B = (x \wedge z)_{(\min(y,l),\min(t_y,t_l),\max(i_y,i_l),\max(f_y,f_l))} \in O_{T_N}$,

$A \vee B = (x \vee z)_{(\min(y,l),\max(t_y,t_l),\min(i_y,i_l),\min(f_y,f_l))} \in O_{T_N}$.

Theorem (.23) can be proved by similar argument of **theorem (.22).**

Proof of theorem (.24):

1] Since $B_i \subseteq A$, then $\bigvee_{i \in I} B_i \subseteq A$ and $int(A) \subseteq A$.

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2] Since $int(x \wedge z) = int(x) \wedge int(z)$, we get the proof.

3] Since $int(x) = int(int(x))$, then $int\big(int(A)\big) = int(A)$.

4] $A \subseteq B \Longrightarrow x \subseteq z$, hence $int(x) \subseteq int(z)$ so that: $int(A) \subseteq int(B)$.

5] $int(x) \vee int(z) \subseteq int(x \vee z)$, so that the proof holds.

6] $x \in O_N \Leftrightarrow x = int(x)$, thus $A \in O_{T_N} \Leftrightarrow A = int(A)$.

Proof of theorem (.25):

We have the following facts:

$$\begin{cases} x \subseteq cl(x), cl\big(cl(x)\big) = cl(x) \\ x \subseteq z \Longrightarrow cl(x) \subseteq cl(z) \\ cl(x \vee z) = cl(x) \vee cl(z) \\ cl(x \wedge z) \subseteq cl(x) \wedge cl(z) \\ x \in C_N \Leftrightarrow x = cl(x) \end{cases}$$

From these facts, we get: the results from 1 to 5.

Proof of theorem (.26), (.27):

We have the following facts:

$$\begin{cases} int(x^c) = (cl(x))^c \\ cl(x^c) = (int(x))^c \\ ext(x \vee z) = ext(x) \wedge ext(z) \\ ext(x) \vee ext(z) \subseteq ext(x \wedge z) \end{cases}$$

Hence, we get the proof.

Proof of theorem (.28):

We have the following facts:

$$\begin{cases} (fr(x))^c = ext(x) \vee int(x) \\ cl(x) = int(x) \vee fr(x) \\ fr(x) \wedge int(x) = \emptyset \\ fr(int(x)) \subseteq fr(x) \end{cases}$$

These facts imply the desired proof.

Definition 2.10

Let R be the real field, we define:

$T_R = \{x_{(y,t_y f_y)}\,; x, y \in R, t_y, f_y \in [0,1]\}$.

The components (t_y, f_y) refer to fuzzy values of truth and falsity.

Definition 2.11

Let N be a topological space on $\mathbb{R}$, the corresponding two-fold fuzzy topological space is defined as follows:

$T_N = \{x_{(y,t_y,f_y)}\,; x \in N, y \in R, t_y, f_y \in [0,1]\}$.

Definition 2.12

Let $x_{(y,t_y,f_y)} = A$, $z_{(l,t_l,f_l)} = B \in T_N$, we define:

$A \wedge B = (x \wedge z)_{(c,t_c,f_c)}$; $c = \min(y, l), t_c = \min\big(t_l, t_y\big), f_c = \max\big(f_l, f_y\big)$.

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$A \vee B = (x \vee z)_{(c,t_c,i_c,f_c)}$; $c = \max(y,l), t_c = \max(t_l,t_y), f_c = \min(f_l,f_y)$.

The complement of A is defined as follows:

$A^c = (x^c)_{(y,f_y,,t_y)}$

The two-fold empty set is defined as:

$\emptyset_N = \{(\emptyset)_{(y,0,,1)} \ \forall\, y \in \mathbb{R}\}$.

Definition 2.13

Consider $A = (x)_{(y,t_y,f_y)}$, $B = (z)_{(l,t_l,f_l)}$, then:

$A \subseteq B$ if and only if: $\begin{cases} x \subseteq z \\ t_y \le t_l,, f_y \ge f_l \end{cases}$

Theorem 2.9

Let $A, B, c \ \in T_N$ with:

$A = (x)_{(y,t_y,f_y)}$, $B = (z)_{(l,t_l,f_l)}$, $C = (k)_{(s,t_s,f_s)}$, then:

1] $A \wedge A = A$, $A \wedge \emptyset_N = \emptyset_N$

2] $A \vee A = A$, $A \vee \emptyset_N = A$

3] $A \wedge B = B \wedge A$, $A \vee B = B \vee A$

4] $A \wedge (B \wedge C) = (A \wedge B) \wedge C$, $A \vee (B \vee C) = (A \vee B) \vee C$

5] $(A^c)^c = A$

6] $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$.

Definition 2.14

Open sets in T_N are:

$O_{T_N} = \left\{A = (x)_{(y,t_y,,f_y)}; x \in \tau_1, y \in \tau_2\right\}$; τ_1 is the topology defined over N, τ_2 is the neutrosophic topology defined over $\mathbb{R}$.

Closed sets on T_N are:

$C_{T_N} = \{A^c \, ; A \in O_{T_N}\}$.

Theorem 2.10

Let $A, B \in O_{T_N}$, then:

1] $A \wedge B \in O_{T_N}$

2] $A \vee B \in O_{T_N}$

Theorem 2.11

Let $A, B \in C_{T_N}$, then:

1] $A \wedge B \in C_{T_N}$

2] $A \vee B \in C_{T_N}$

Definition 2.15

Let $A \in T_N$, then we define:

$int(A) = \bigcup_{i \in I} B_i \quad ; B_i \in O_{T_N}, B_i \subseteq A$.

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Theorem 2.12

Let $A \in T_N$, then:

1] $int(A) \subseteq A$

2] $int(A \wedge B) = int(A) \wedge int(B)$

3] $int\big(int(A)\big) = int(A)$

4] $A \subseteq B \Longrightarrow int(A) \subseteq int(B)$

5] $int(A) \vee int(B) \subseteq int(A \vee B)$

6] $A \in O_{T_N} \Leftrightarrow A = int(A)$

Definition 2.16

Let $A \in T_N$, then we define:

$cl(A) = \bigcap_{i \in I} B_i \ ; \ B_i \in C_{T_N}, A \subseteq B_i$.

Theorem 2.13

Let $A, B \in T_N$, then:

1] $A \subseteq cl(A), cl\big(cl(A)\big) = cl(A)$.

2] $A \subseteq B \Longrightarrow cl(A) \subseteq cl(B)$

3] $cl(A \vee B) = cl(A) \vee cl(B)$

4] $cl(A \wedge B) \subseteq cl(A) \wedge cl(B)$

5] $A \in C_{T_N} \Leftrightarrow A = cl(A)$.

Theorem 2.14

Let $A \in T_N$, then:

1] $int(A^c) = (cl(A))^c$.

2] $cl(A^c) = (int(A))^c$.

Definition 2.17

Let $A \in T_N$,we define: $ext(A) = int(A^c)$.

Theorem 2.15

Let $A, B \in T_N$, then:

1] $ext(A \vee B) = ext(A) \wedge ext(B)$

2] $ext(A) \vee ext(B) \subseteq ext(A \wedge B)$

Definition 2.18

Let $A, B \in T_N$, we define:

$fr(A) = cl(A) \cap cl(A^c)$

Theorem 2.16

Let $A, B \in T_N$, then:

1] $(fr(A))^c = ext(A) \vee int(A)$

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2] $cl(A) = int(A) \vee fr(A)$

3] $fr(A) \wedge int(A) = \emptyset$

4] $fr(int(A)) \subseteq fr(A)$

Proof of theorem (.29):

1] $A \wedge A = (x)_{(y,t_y,f_y)} \wedge (x)_{(y,t_y,,f_y)} = (x \wedge x)_{(y,t_y,f_y)} = (x)_{(y,t_y,,f_y)} = A.$

$A \wedge \emptyset_N = \emptyset_N$ clearly.

2] $A \vee A = (x \vee x)_{(y,t_y,f_y)} = (x)_{(y,t_y,,f_y)} = A.$

$A \vee \emptyset_N = \emptyset_N$ clearly.

3] It holds directly from the definition.

4] $A \wedge (B \wedge C) = (x \wedge y \wedge k)_{(m,t_m,f_m)}$; $m = \min(y, l, s), t_m = \min(t_y, t_l, t_s)$, $f_m = \max(f_y, , f_s)$, thus $A \wedge (B \wedge C) = (A \wedge B) \wedge C.$

The second formula can be proved by the same.

5] $(A^c)^c = (x^c)^c_{(y,t_y,,f_y)} = (x)_{(y,t_y,,f_y)} = A.$

6] we have: $(x \vee z)^c = x^c \wedge z^c$, $(x \wedge z)^c = x^c \vee z^c$, and:

$(A \vee B)^c = (x^c \wedge z^c)_{(m,t_m,,f_m)}$; $(A \wedge B)^c = (x^c \vee z^c)_{(n,t_n,,f_n)}$

So that: $(A \vee B)^c = A^c \wedge B^c$, $(A \wedge B)^c = A^c \vee B^c$

Proof of theorem (2.10):

1]-2] Since $x \wedge z, x \vee z \in \tau_1$ and $\min(y, l)$, $\max(y, l) \in \tau_2$,

$$\begin{cases} \min(t_y, t_l), \max(t_y, t_l) \in [0,1] \\ \min(f_y, f_l), \max(f_y, f_l) \in [0,1] \end{cases}$$

Hence $A \wedge B = (x \wedge z)_{(\min(y,l),\min(t_y,t_l), \quad ,\max(f_y,f_l))} \in O_{T_N}$,

$A \vee B = (x \vee z)_{(\min(y,l),\max(t_y,t_l), \quad ,\min(f_y,f_l))} \in O_{T_N}$.

Theorem (2.11) can be proved by similar argument of **theorem (210.).**

Proof of theorem (.212):

1] Since $B_i \subseteq A$, then $\bigvee_{i \in I} B_i \subseteq A$ and $int(A) \subseteq A$.

2] Since $int(x \wedge z) = int(x) \wedge int(z)$, we get the proof.

3] Since $int(x) = int(int(x))$, then $int(int(A)) = int(A)$.

4] $A \subseteq B \Longrightarrow x \subseteq z$, hence $int(x) \subseteq int(z)$ so that: $int(A) \subseteq int(B)$.

5] $int(x) \vee int(z) \subseteq int(x \vee z)$, so that the proof holds.

6] $x \in O_N \Leftrightarrow x = int(x)$, thus $A \in O_{T_N} \Leftrightarrow A = int(A)$.

Proof of theorem (.213):

We have the following facts:

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$$\begin{cases} x \subseteq cl(x), cl\big(cl(x)\big) = cl(x) \\ x \subseteq z \Longrightarrow cl(x) \subseteq cl(z) \\ cl(x \vee z) = cl(x) \vee cl(z) \\ cl(x \wedge z) \subseteq cl(x) \wedge cl(z) \\ x \in C_N \Leftrightarrow x = cl(x) \end{cases}$$

From these facts, we get: the desired results

Proof of theorem (.214), (.215):

We have the following facts:

$$\begin{cases} int(x^c) = (cl(x))^c \\ cl(x^c) = (int(x))^c \\ ext(x \vee z) = ext(x) \wedge ext(z) \\ ext(x) \vee ext(z) \subseteq ext(x \wedge z) \end{cases}$$

Hence, we get the proof.

Proof of theorem (.216):

We have the following facts:

$$\begin{cases} (fr(x))^c = ext(x) \vee int(x) \\ cl(x) = int(x) \vee fr(x) \\ fr(x) \wedge int(x) = \emptyset \\ fr(int(x)) \subseteq fr(x) \end{cases}$$

These facts imply the desired proof.

3. Conclusion

In this paper, we have introduced for the first time the concept of two-fold neutrosophic and fuzzy topological space defined over real numbers, where we combined the two-fold neutrosophic sets with real numbers to get a novel topological space based on them. Also, we presented many of its elementary properties and special subsets such as two-fold neutrosophic open sets, two-fold neutrosophic closed sets, and two-fold neutrosophic closure. Many examples and theorems have been provided to clarify the validity of our approach.

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