Collection of Bi-Univalent Functions Using Bell Distribution Associated With Jacobi Polynomials

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Abstract

The aim of this study is to present novel collections of bi-univalent functions, which are characterized using the Bell Distribution. These collections are delineated through the application of Jacobi polynomials. We have established bounds for the Taylor-Maclaurin coefficients, particularly $|a_2|$ and $|a_3|$. Additionally, we have investigated the Fekete-Szegö functional issues pertinent to functions within these subclasses. By concentrating on particular parameters in our principal findings, we have identified numerous new insights.

Keywords: Jacobi polynomials; analytic functions; univalent functions; bi-univalent functions; Fekete-Szegö problem.

1 Introduction and preliminaries

Legendre introduced orthogonal polynomials in 1784.[1] These polynomials are frequently employed for solving ordinary differential equations under specific model constraints. Additionally, they have a crucial role in approximation theory.[2]

Two polynomials, Y_n and Y_m, are said to be orthogonal if they have orders n and m respectively

$$\int_{\epsilon}^{\iota} Y_n(x)Y_m(x)v(x)dx = 0, \qquad \text{for} \qquad n \neq m. \tag{1}$$

Assuming $v(x)$ is non-negative within the interval (ϵ, ι), all polynomials of finite order $Y_n(x)$ possess a clearly defined integral. Jacobi polynomials belong to the class of orthogonal polynomials.

This paper analytically examines a newly defined subclass $\mathfrak{G}_{\Sigma}(\varrho, \daleth, \varsigma, x)$ of bi-univalent functions utilizing Jacobi polynomials.

Let $\mathcal{A}$ represent the class of analytic functions Θ within the open unit disk $\mathbb{U} = z \in \mathbb{C} : |z| < 1$, normalized such that $\Theta(0) = 0$ and $\Theta'(0) = 1$. Consequently, each $\Theta \in \mathcal{A}$ can be expressed as:

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$$\Theta(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad (z \in \mathbb{U}). \tag{2}$$

Moreover, the set of all univalent functions $\Theta \in \mathcal{A}$ is denoted by $\mathcal{S}$ (for more information, refer to[3]).

The field of geometric function theory can greatly benefit from the powerful tools provided by differential subordination of analytic functions. Miller and Mocanu[4] originally introduced the concept of differential subordination. Further references can be found in reference.[5] Miller and Mocanu's book[6] provides a comprehensive overview of the advancements in this area, including publication dates.

It is well established that for an analytic and univalent function $\Theta(z)$ mapping a domain $\mathbb{D}1$ onto a domain $\mathbb{D}2$, the inverse function $g(z) = \Theta^{-1}(z)$ is defined as

$$g\left(\Theta(z)\right) = z, \quad (z \in \mathbb{D}_1).$$

This function is analytic and univalent. Additionally, according to,[3] every function $\Theta \in \mathcal{S}$ has an inverse map Θ^{-1} which satisfies

$$\Theta^{-1}(\Theta(z)) = z \quad (z \in \mathbb{U}),$$

and

$$\Theta\left(\Theta^{-1}(\varpi)\right) = \varpi \quad \left(|\varpi| < r_0(\Theta); r_0(\Theta) \geq \frac{1}{4}\right).$$

In fact, the inverse function is given by

$$\Theta^{-1}(\varpi) = \varpi - a_2^2\varpi + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2a_3 + a_4)\varpi^4 + \cdots. \tag{3}$$

A function $\Theta \in \mathcal{A}$ is bi-univalent in $\mathbb{U}$ if both $\Theta(z)$ and $\Theta^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ represent the class of bi-univalent functions in $\mathbb{U}$ as defined in (2). For further details on the class Σ, refer to.[7–19]

2 Bell Distribution and Jacobi polynomial

In 2018, Castellares et al. introduced the Bell distribution,[20] which is suitable for count data with over-dispersion. The Bell distribution is an improvement over the Bell numbers.[21,22] The probability density function of a discrete random variable X, which follows the Bell distribution, is expressed as:

$$\mathcal{P}(X = m) = \frac{\daleth^m e^{e^{(-\daleth^2)+1}} \mathcal{B}_m}{m!}; \qquad m = 1, 2, 3, \cdots, \tag{4}$$

where $\mathcal{B}_m = \frac{1}{e}\sum\limits_{k=0}^{\infty} \frac{k^m}{m!}$ is the Bell numbers, $m \geq 2$, and $\daleth > 0$.

Example of the Bell numbers are $\mathcal{B}_2 = 2, \mathcal{B}_3 = 5, \mathcal{B}_4 = 15$ and $\mathcal{B}_5 = 52$.

Now, we will introduce a new power series whose coefficients represent the probabilities of the Bell distribution

$$\mathcal{B}(\daleth, z) = z + \sum_{n=2}^{\infty} \frac{\daleth^{n-1}\mathcal{B}_n}{(n-1)!\, e^{\daleth^2 - 1}} z^n, \qquad (z \in \mathbb{U}), \tag{5}$$

where $\daleth > 0$.

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Now, we considered the linear operator $\mathbb{P}_{\daleth} : \mathcal{A} \to \mathcal{A}$ defined by the convolution (or Hadamard product)

$$\mathbb{P}_{\daleth}\Theta(z) = \mathcal{B}(\daleth, z) * f(z) \tag{6}$$
$$= z + \sum_{n=2}^{\infty} \frac{\daleth^{n-1} e^{1-\daleth^2} \mathcal{B}_n}{(n-1)!} a_n z^n, \qquad (z \in \mathbb{U}), \tag{7}$$
$$= z + \frac{2\daleth}{e^{\daleth^2-1}} a_2 z^2 + \frac{5\daleth^2}{2e^{\daleth^2-1}} a_3 z^3 + \frac{15\daleth^3}{3! e^{\daleth^2-1}} a_4 z^4 + \cdots . \tag{8}$$

For $n, n+\vartheta, n+\varsigma$ are nonnegative integers, a generating function of Jacobi polynomials is defined by

$$J_n(x,z) = 2^{\daleth+\varsigma} R^{-1} (1-x+R)^{-\daleth} (1+x+R)^{-\varsigma},$$

where $R = R(x,z) = (1-2zx+x^2)^{0.5}$, $\daleth > -1$, $\varsigma > -1$, $x \in [-1,1]$ and $z \in \mathbb{U}$, (see[23]).

For a fixed x, the function $J_n(x,z)$ is analytic in $\mathbb{U}$, allowing it to be represented by a Taylor series expansion as follows:

$$J_n(x,z) = \sum_{n=0}^{\infty} P_n^{(\daleth,\varsigma)}(x) z^n, \tag{9}$$

where $P_n^{(\daleth,\varsigma)}(x)$ is Jacobi polynomial of degree n.

The Jacobi polynomial $P_n^{(\daleth,\varsigma)}(x)$ satisfies a second-order linear homogeneous differential equation:

$$(1-x^2)y'' + (\varsigma - \daleth - (\daleth+\varsigma+2)x)y' + n(n+\daleth+\varsigma+1)y = 0.$$

Jacobi polynomials can alternatively be characterized by the following recursive relationships:

$$P_n^{(\daleth,\varsigma)}(x) = (a_{n-1}z - b_{n-1}) P_{n-1}^{(\daleth,\varsigma)}(x) - c_{n-1} P_{n-2}^{(\daleth,\varsigma)}(x), \ n \geqslant 2,$$

where

$a_n = \frac{(2n+\daleth+\varsigma+1)(2n+\daleth+\varsigma+2)}{2(n+1)(n+\daleth+\varsigma+1)}$, $b_n = \frac{(2n+\daleth+\varsigma+1)(\varsigma^2-\daleth^2)}{2(n+1)(n+\daleth+\varsigma+1)(2n+\daleth+\varsigma)}$ and $c_n = \frac{(2n+\daleth+\varsigma+2)(n+\daleth)(n+\varsigma)}{(n+1)(n+\daleth+\varsigma+1)(2n+\daleth+\varsigma)}$, with the initial values

$$P_0^{(\daleth,\varsigma)}(x) = 1, \ P_1^{(\daleth,\varsigma)}(x) = (\daleth+1) + \frac{1}{2}(\daleth+\varsigma+2)(x-1) \text{ and} \tag{10}$$
$$P_2^{(\daleth,\varsigma)}(x) = \frac{(\daleth+1)(\daleth+2)}{2} + \frac{1}{2}(\daleth+2)(\daleth+\varsigma+3)(x-1) + \frac{1}{8}(\daleth+\varsigma+3)(\daleth+\varsigma+4)(x-1)^2.$$

To begin, we introduce certain special instances of the polynomials $P_n^{(\daleth,\varsigma)}$:

1. For $\daleth = \varsigma = 0$, we get the Legendre Polynomials.

[title=Legendre Polynomials $L_n(x)$ within the Unit Circle, xlabel=Re(z), ylabel=Im(z), axis equal, grid=major, xmin=-1.2, xmax=1.2, ymin=-1.2, ymax=1.2, samples=100, xtick=-1, -0.5, 0, 0.5, 1, ytick=-1, -0.5, 0, 0.5, 1] [domain=0:360, samples=200, smooth, thick, gray] (cos(x), sin(x));

[domain=-1:1, thick, blue] (x, 1); $L_0(x)$

[domain=-1:1, thick, red] (x, x); $L_1(x)$

[domain=-1:1, thick, green] (x, (1/2)*(3*x$^2 - 1$)); $\mathrm{L}_2(x)$

[domain=-1:1, thick, orange] (x, (1/2)*(5*x$^3 - 3*x$)); $\mathrm{L}_3(x)$

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2. For $\daleth = \varsigma = -0.5$, this results in the Chebyshev Polynomials of the first kind.

[title=Chebyshev Polynomials of the First Kind $T_n(x)$ within the Unit Circle, xlabel=Re(z), ylabel=Im(z), axis equal, grid=major, xmin=-1.2, xmax=1.2, ymin=-1.2, ymax=1.2, samples=100, xtick=-1, -0.5, 0, 0.5, 1, ytick=-1, -0.5, 0, 0.5, 1] [domain=0:360, samples=200, smooth, thick, gray] (cos(x), sin(x));

[domain=-1:1, thick, blue] (x, 1); $T_0(x)$

[domain=-1:1, thick, red] (x, x); $T_1(x)$

[domain=-1:1, thick, green] (x, 2*x$^2 - 1); \mathrm{T}_2(x)$

[domain=-1:1, thick, orange] (x, 4*x$^3 - 3 * x); \mathrm{T}_3(x)$

3. For $\daleth = \varsigma = 0.5$, this results in the Chebyshev Polynomials of the second kind.

[title=Chebyshev Polynomials of the Second Kind $U_n(x)$ within the Unit Circle, xlabel=Re(z), ylabel=Im(z), axis equal, grid=major, xmin=-1.2, xmax=1.2, ymin=-1.2, ymax=1.2, samples=100, xtick=-1, -0.5, 0, 0.5, 1, ytick=-1, -0.5, 0, 0.5, 1] [domain=0:360, samples=200, smooth, thick, gray] (cos(x), sin(x));

[domain=-1:1, thick, blue] (x, 1); $U_0(x)$

[domain=-1:1, thick, red] (x, 2*x); $U_1(x)$

[domain=-1:1, thick, green] (x, 4*x$^2 - 1); \mathrm{U}_2(x)$

[domain=-1:1, thick, orange] (x, 8*x$^3 - 4 * x); \mathrm{U}_3(x)$

4. For $\daleth = \varsigma$, we get the Gegenbauer Polynomials and each is replaced by $(\daleth - 0.5)$.

In recent years, many scholars have explored the connection between bi-univalent functions and orthogonal polynomials. Some notable references in this field are.[24–38] However, there seems to be a lack of research on bi-univalent functions related to Jacobi polynomials in existing literature. The main goal of this paper is to initiate an investigation into the properties of bi-univalent functions associated with Jacobi polynomials. To accomplish this, we will consider the following definitions.

3 Definition and Examples

In this section, we introduce and analyze a novel subclass of bi-univalent functions defined within the unit disk. This subclass is established by leveraging the concept of subordination. To construct this new class, we employ the Bell Distribution in conjunction with subordination via Jacobi polynomials.

Definition 3.1. Let $\daleth > -1$, $\varsigma > -1$, $\varrho \geq 0$, $x \in (\frac{1}{2}, 1]$ and n, $n+\vartheta$, $n+\varsigma$ are nonnegative integers. A function $\Theta \in \Sigma$ given by (2) is said to be in the class $\mathfrak{G}_\Sigma(\varrho, \daleth, \varsigma, x)$ if the following subordinations are satisfied:

$$(1-\varrho)\frac{\mathbb{P}_\neg \Theta(z)}{z} + \varrho(\mathbb{P}_\neg \Theta(z))' \prec J_n(x, z) \tag{11}$$

and

$$(1-\varrho)\frac{\mathbb{P}_\neg g(\varpi)}{\varpi} + \varrho(\mathbb{P}_\neg g(\varpi))' \prec J_n(x, \varpi), \tag{12}$$

where the function J_n is given by (9) and the function $g(\varpi) = \Theta^{-1}(\varpi)$ is defined by (3).

Example 3.2. Let $\daleth > -1$, $\varsigma > -1$, $x \in (\frac{1}{2}, 1]$ and n, $n+\vartheta$, $n+\varsigma$ are nonnegative integers. A function $\Theta \in \Sigma$ given by (2) is said to be in the class $\mathfrak{G}_\Sigma(0, \daleth, \varsigma, x)$ if the following subordinations are satisfied:

$$\frac{\mathbb{P}_\neg \Theta(z)}{z} \prec J_n(x, z)$$

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and

$$\frac{\mathbb{P}_{\daleth} g(\varpi)}{\varpi} \prec J_n(x,\varpi),$$

where the function J_n is given by (9) and the function $g(\varpi) = \Theta^{-1}(\varpi)$ is defined by (3).

Example 3.3. Let $\daleth > -1$, $\varsigma > -1$, $x \in (\frac{1}{2}, 1]$ and n, $n+\vartheta$, $n+\varsigma$ are nonnegative integers. A function $\Theta \in \Sigma$ given by (2) is said to be in the class $\mathfrak{G}_{\Sigma}(1,\daleth,\varsigma,x)$ if the following subordinations are satisfied:

$$(\mathbb{P}_{\daleth}\Theta(z))' \prec J_n(x,z)$$

and

$$(\mathbb{P}_{\daleth} g(\varpi))' \prec J_n(x,\varpi),$$

where the function J_n is given by (9) and the function $g(\varpi) = \Theta^{-1}(\varpi)$ is defined by (3).

4 Coefficient bounds of the subclass $\mathfrak{G}_{\Sigma}(\varrho,\daleth,\varsigma,x)$

This section focuses on determining initial coefficient bounds for the subclass $\mathfrak{G}_{\Sigma}(\varrho,\daleth,\varsigma,x)$.

Theorem 4.1. *If $\Theta \in \Sigma$ as defined by equation (2), it belongs to the class $\mathfrak{G}_{\Sigma}(\varrho,\daleth,\varsigma,x)$. Then*

$$|a_2| \leq \frac{e^{(\daleth^2-1)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|\sqrt{\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}}{\sqrt{\Upsilon(x,\daleth,\varsigma)}},$$

and

$$|a_3| \leq \frac{e^{2(\daleth^2-1)}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2}{4(1+\varrho)^2\daleth^2} + \frac{2e^{\daleth^2-1}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]}{5(1+2\varrho)\daleth^2},$$

where

$$\Upsilon(x,\vartheta,\varsigma) = \daleth^2\left[\begin{array}{c} 5(1+2\varrho)\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2 e^{(\daleth^2-1)} \\ -8(1+\varrho)^2\left(\begin{array}{c}\frac{(\daleth+1)(\daleth+2)}{2}+\frac{1}{2}(\daleth+2)(\daleth+\varsigma+3)(x-1) \\ +\frac{1}{8}(\daleth+\varsigma+3)(\daleth+\varsigma+4)(x-1)^2\end{array}\right)\end{array}\right].$$

Proof. Suppose $\Theta \in \mathfrak{G}_{\Sigma}(\varrho,\daleth,\varsigma,x)$. According to equations (11) and (12), for all $z,\varpi \in \mathbb{U}$ and analytic functions r and s with $r(0) = s(0) = 0$ and $|r(z)| < 1$, $|s(\varpi)| < 1$, we can express:

$$(1-\varrho)\frac{\mathbb{P}_{\daleth}\Theta(z)}{z} + \varrho(\mathbb{P}_{\daleth}\Theta(z))' = J_n(x,r(z)) \tag{13}$$

and

$$(1-\varrho)\frac{\mathbb{P}_{\daleth} g(\varpi)}{\varpi} + \varrho(\mathbb{P}_{\daleth} g(\varpi))' = J_n(x,s(\varpi)). \tag{14}$$

Thus we have

$$(1-\varrho)\frac{\mathbb{P}_{\daleth}\Theta(z)}{z} + \varrho(\mathbb{P}_{\daleth}\Theta(z))' = 1 + P_1^{(\vartheta,\varsigma)}(x)b_1 z + \left[P_1^{(\vartheta,\varsigma)}(x)b_2 + P_2^{(\vartheta,\varsigma)}(x)b_1^2\right]z^2 + \cdots \tag{15}$$

and

$$(1-\varrho)\frac{\mathbb{P}_{\daleth} g(\varpi)}{\varpi} + \varrho(\mathbb{P}_{\daleth} g(\varpi))' = 1 + P_1^{(\vartheta,\varsigma)}(x)d_1\varpi + \left[P_1^{(\vartheta,\varsigma)}(x)d_2 + P_2^{(\vartheta,\varsigma)}(x)d_1^2\right])\varpi^2 + \cdots. \tag{16}$$

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It is well known that if

$$|r(z)| = \left|b_1 z + b_2 z^2 + b_3 z^3 + \cdots\right| < 1, \ \ (z \in \mathbb{U})$$

and

$$|s(\varpi)| = \left|d_1 \varpi + d_2^2 \varpi + d_3^3 \varpi + \cdots\right| < 1, \ \ (\varpi \in \mathbb{U}),$$

then

$$|b_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{17}$$

Comparing the coefficients in (15) and (16), we get

$$\frac{2(1+\varrho)\daleth}{e^{\daleth^2-1}} a_2 = P_1^{(\daleth,\varsigma)}(x) b_1, \tag{18}$$

$$\frac{5(1+2\varrho)\daleth^2}{2e^{\daleth^2-1}} a_3 = P_1^{(\daleth,\varsigma)}(x) b_2 + P_2^{(\daleth,\varsigma)}(x) b_1^2, \tag{19}$$

$$-\frac{2(1+\varrho)\daleth}{e^{\daleth^2-1}} a_2 = P_1^{(\daleth,\varsigma)}(x) d_1, \tag{20}$$

and

$$\frac{5(1+2\varrho)\daleth^2}{2e^{\daleth^2-1}} \left(2a_2^2 - a_3\right) = P_1^{(\daleth,\varsigma)}(x) d_2 + P_2^{(\daleth,\varsigma)}(x) d_1^2. \tag{21}$$

From (18) and (20) it follows that

$$b_1 = -d_1 \tag{22}$$

and

$$\frac{8(1+\varrho)^2\daleth^2}{e^{2(\daleth^2-1)}} a_2^2 = \left[P_1^{(\daleth,\varsigma)}(x)\right]^2 \left(b_1^2 + d_1^2\right). \tag{23}$$

If we add (19) and (21), we get

$$\frac{5(1+2\varrho)\daleth^2}{e^{\daleth^2-1}} a_2^2 = P_1^{(\daleth,\varsigma)}(x) \left(b_2 + d_2\right) + P_2^{(\daleth,\varsigma)}(x) \left(b_1^2 + d_1^2\right). \tag{24}$$

Substituting the value of $\left(b_1^2 + d_1^2\right)$ from (23) in the right hand side of (24), we get

$$\frac{\daleth^2}{e^{\daleth^2-1}} \left[5(1+2\varrho) - \frac{P_2^{(\daleth,\varsigma)}(x)}{\left[P_1^{(\daleth,\varsigma)}(x)\right]^2} \frac{8(1+\varrho)^2}{e^{(\daleth^2-1)}} \right] a_2^2 = P_1^{(\daleth,\varsigma)}(x) \left(b_2 + d_2\right). \tag{25}$$

Using (10), (17) and (25), we find that

$$|a_2| \leq \frac{e^{(\daleth^2-1)} \left|(\daleth+1) + \frac{1}{2}(\daleth+\varsigma+2)(x-1)\right| \sqrt{\left|(\daleth+1) + \frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}}{\sqrt{\Upsilon(x,\daleth,\varsigma)}},$$

where

$$\Upsilon(x,\daleth,\varsigma) = \daleth^2 \left[\begin{array}{c} 5(1+2\varrho)\left[(\daleth+1) + \frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2 e^{(\daleth^2-1)} \\ -8(1+\varrho)^2 \left(\begin{array}{c} \frac{(\daleth+1)(\daleth+2)}{2} + \frac{1}{2}(\daleth+2)(\daleth+\varsigma+3)(x-1) \\ +\frac{1}{8}(\daleth+\varsigma+3)(\daleth+\varsigma+4)(x-1)^2 \end{array} \right) \end{array} \right]$$

Moreover, if we subtract (21) from (19), we have

$$\frac{5(1+2\varrho)\daleth^2}{e^{\daleth^2-1}} \left(a_3 - a_2^2\right) = P_1^{(\daleth,\varsigma)}(x) \left(b_2 - d_2\right) + P_2^{(\daleth,\varsigma)}(x) \left(b_1^2 - d_1^2\right). \tag{26}$$

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Then, in view of (23) and (26) becomes

$$a_3 = \frac{e^{2(\daleth^2-1)}\left[P_1^{(\daleth,\varsigma)}(x)\right]^2}{8(1+\varrho)^2\daleth^2}\left(b_1^2+d_1^2\right) + \frac{e^{\daleth^2-1}}{5(1+2\varrho)\daleth^2}P_1^{(\daleth,\varsigma)}(x)\left(b_2-d_2\right).$$

Thus applying (10) and (17), we get

$$|a_3| \le \frac{e^{2(\daleth^2-1)}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2}{4(1+\varrho)^2\daleth^2} + \frac{2e^{\daleth^2-1}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]}{5(1+2\varrho)\daleth^2}.$$

Which completes the proof. □

5 Fekete–Szegö problem for the subclass $\mathfrak{G}_\Sigma(\varrho,\daleth,\varsigma,x)$

A prominent issue concerning coefficients of univalent analytic functions is the Fekete-Szegő inequality. Initially proposed by,[39] it asserts that for $\Theta \in \Sigma$, where Σ denotes

$$|a_3 - \tau a_2^2| \le 1 + 2e^{-2\tau/(1-\mu)}.$$

When τ is real, this bound is sharp.

In this section, for functions in the class $\mathfrak{G}_\Sigma(\varrho,\daleth,\varsigma,x)$ we provide Fekete–Szegö inequalities.

Theorem 5.1. *Let $\Theta \in \Sigma$ given by (2) belongs to the class $\mathfrak{G}_\Sigma(\varrho,\daleth,\varsigma,x)$. Then*

$$\left|a_3-\tau a_2^2\right| \le \begin{cases} \dfrac{2e^{\left(\daleth^2-1\right)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}{5(1+2\varrho)\daleth^2}, & |\tau-1|\le k(x)\\ \dfrac{2(1-\tau)e^{2\left(\daleth^2-1\right)}\left[P_1^{(\vartheta,\varsigma)}(x)\right]^3}{\daleth^2\left[5(1+2\varrho)\left[P_1^{(\daleth,\varsigma)}(x)\right]^2e^{(\daleth^2-1)}-8(1+\varrho)^2P_2^{(\daleth,\varsigma)}(x)\right]}, & |\tau-1|\ge k(x)\end{cases}$$

where

$$k(x) = \left|1-\frac{8(1+\varrho)^2P_2^{(\daleth,\varsigma)}(x)}{5(1+2\varrho)e^{(\daleth^2-1)}\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2}\right|.$$

Proof. From (25) and (26)

$$\begin{aligned} &a_3-\tau a_2^2\\ &= \frac{(1-\tau)\,e^{2\left(\daleth^2-1\right)}\left[P_1^{(\daleth,\varsigma)}(x)\right]^3(b_2+d_2)}{\daleth^2\left[5(1+2\varrho)\left[P_1^{(\daleth,\varsigma)}(x)\right]^2e^{(\daleth^2-1)}-8(1+\varrho)^2P_2^{(\daleth,\varsigma)}(x)\right]} + \frac{e^{\daleth^2-1}}{5(1+2\varrho)\daleth^2}P_1^{(\daleth,\varsigma)}(x)\left(b_2-d_2\right)\\ &= P_1^{(\daleth,\varsigma)}(x)e^{\left(\daleth^2-1\right)}\left[\left(h(\tau)+\frac{1}{5(1+2\varrho)\daleth^2}\right)b_2+\left(h(\tau)-\frac{1}{5(1+2\varrho)\daleth^2}\right)d_2\right], \end{aligned}$$

where

$$h(\tau) = \frac{(1-\tau)\,e^{\left(\daleth^2-1\right)}\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2}{\daleth^2\left[5(1+2\varrho)\left[P_1^{(\daleth,\varsigma)}(x)\right]^2e^{(\daleth^2-1)}-8(1+\varrho)^2P_2^{(\daleth,\varsigma)}(x)\right]},$$

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Then, in view of (10), we have

$$\left|a_3-\tau a_2^2\right| \leq \begin{cases} \frac{2e^{\left(\daleth^2-1\right)}\left|P_1^{(\daleth,\varsigma)}(x)\right|}{5(1+2\varrho)\daleth^2} & 0 \leq |h(\tau)| \leq \frac{1}{5(1+2\varrho)\daleth^2}, \\ 2e^{\left(\daleth^2-1\right)}\left|P_1^{(\daleth,\varsigma)}(x)\right||h(\tau)| & |h(\tau)| \geq \frac{1}{5(1+2\varrho)\daleth^2}. \end{cases}$$

Which completes the proof. □

6 Corollaries and Consequences

We use our main results in this section to derive each of the new corollaries and implications that follow.

Corollary 6.1. *Let $\Theta \in \Sigma$ given by (2) belongs to the class $\mathfrak{G}_\Sigma(0,\daleth,\varsigma,x)$. Then*

$$|a_2| \leq \frac{e^{\left(\daleth^2-1\right)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|\sqrt{\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}}{\sqrt{\Upsilon(x,\daleth,\varsigma)}},$$

$$|a_3| \leq \frac{e^{2\left(\daleth^2-1\right)}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2}{4\daleth^2}+\frac{2e^{\daleth^2-1}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]}{5\daleth^2},$$

and

$$\left|a_3-\tau a_2^2\right| \leq \begin{cases} \frac{2e^{\left(\daleth^2-1\right)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}{5\daleth^2}, & |\tau-1| \leq k(x) \\ \frac{2(1-\tau)e^{2\left(\daleth^2-1\right)}\left[P_1^{(\daleth,\varsigma)}(x)\right]^3}{\daleth^2\left[5\left[P_1^{(\daleth,\varsigma)}(x)\right]^2 e^{\left(\daleth^2-1\right)}-8P_2^{(\daleth,\varsigma)}(x)\right]}, & |\tau-1| \geq k(x) \end{cases}$$

where

$$\Upsilon(x,\daleth,\varsigma)=\daleth^2\left[\begin{array}{c} 5\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2 e^{\left(\daleth^2-1\right)} \\ -8\left(\begin{array}{c}\frac{(\daleth+1)(\daleth+2)}{2}+\frac{1}{2}(\daleth+2)(\daleth+\varsigma+3)(x-1) \\ +\frac{1}{8}(\daleth+\varsigma+3)(\daleth+\varsigma+4)(x-1)^2\end{array}\right)\end{array}\right].$$

and

$$k(x)=\left|1-\frac{8P_2^{(\daleth,\varsigma)}(x)}{5e^{\left(\daleth^2-1\right)}\left[P_1^{(\daleth,\varsigma)}(x)\right]^2}\right|$$

Corollary 6.2. *Let $\Theta \in \Sigma$ given by (2) belongs to the class $\mathfrak{G}_\Sigma(1,\daleth,\varsigma,x)$. Then*

$$|a_2| \leq \frac{e^{\left(\daleth^2-1\right)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|\sqrt{\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}}{\sqrt{\Upsilon(x,\daleth,\varsigma)}},$$

$$|a_3| \leq \frac{e^{2\left(\daleth^2-1\right)}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]^2}{16\daleth^2}+\frac{2e^{\daleth^2-1}\left[(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right]}{15\daleth^2},$$

and

$$\left|a_3-\tau a_2^2\right| \leq \begin{cases} \frac{2e^{\left(\daleth^2-1\right)}\left|(\daleth+1)+\frac{1}{2}(\daleth+\varsigma+2)(x-1)\right|}{15\daleth^2}, & |\tau-1| \leq k(x) \\ \frac{2(1-\tau)e^{2\left(\daleth^2-1\right)}\left[P_1^{(\daleth,\varsigma)}(x)\right]^3}{\daleth^2\left[15\left[P_1^{(\daleth,\varsigma)}(x)\right]^2 e^{\left(\daleth^2-1\right)}-32P_2^{(\daleth,\varsigma)}(x)\right]}, & |\tau-1| \geq k(x) \end{cases}$$

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where

$$\Upsilon(x,\urcorner,\varsigma)=\urcorner^{2}\left[\begin{array}{c} 15\left[(\urcorner+1)+\frac{1}{2}(\urcorner+\varsigma+2)(x-1)\right]^{2}e^{\left(\urcorner^{2}-1\right)} \\ -32\left(\begin{array}{c}\frac{(\urcorner+1)(\urcorner+2)}{2}+\frac{1}{2}\left(\urcorner+2\right)\left(\urcorner+\varsigma+3\right)(x-1) \\ +\frac{1}{8}(\urcorner+\varsigma+3)(\urcorner+\varsigma+4)(x-1)^{2}\end{array}\right)\end{array}\right].$$

and

$$k(x)=\left|1-\frac{32P_{2}^{(\urcorner,\varsigma)}(x)}{15e^{(\urcorner^{2}-1)}\left[P_{1}^{(\urcorner,\varsigma)}(x)\right]^{2}}\right|$$

7 Concluding Remark

In this study, we examined the concerns related to coefficients associated with the newly introduced subclasses $\mathfrak{G}_{\Sigma}(\varrho,\urcorner,\varsigma,x)$, $\mathfrak{G}_{\Sigma}(0,\urcorner,\varsigma,x)$, and $\mathfrak{G}_{\Sigma}(1,\urcorner,\varsigma,x)$ within the class of bi-univalent functions defined in the open unit disk $\mathbb{U}$. The definitions of these subclasses are provided in the text. Our investigation focuses on estimating the Fekete-Szegő functional problems and the Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within different subclasses of bi-univalent functions. Furthermore, by specializing the parameters in our main results, we have made several novel discoveries.

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