Sharp Estimates for the Zalcman Conjecture and Second Order Hankel Determinant

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Abstract
In this work, we found sharp estimates for the Zalcman conjecture and second order Hankel determinant for the inverse function when it belongs to the class of starlike functions with respect to symmetric points, denoted by $S^*_s$. These results are new.

Keyword: Univalent function, regular function; Mobius transformation; close-to-convex; bi-univalent; Hankel Determinant; Zalcman conjecture.

1. Introduction
Hankel matrices are use when we have a sequence of resulting data, and we want to achieve a Markov model. Analyzing the high and low values in the Hankel matrix can help estimate the parameters related to the model. In conclusion, Hankel matrices represent a powerful tool in linear algebra and numerical analysis, and are use in a variety of mathematical and engineering applications.

Let the space of all single-valued analytic maps in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, stand for $H$, the subset $A$ of $H$ be the collection of all functions with Maclaurin’s series, namely

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

With $a_1 = 1$, which have been normalized by $f(0)' - 1 = f(0) = 0$, where the coefficients $a_n z$ are complex constants. Geometrically, the constraints $f(0)' = 0$ and $f(0) = 1$ are related in shifting and amounts in spinning with expansion or contraction of the image domain respectively.

The subcollection $S$ of $A$ are univalent it projects $\mathbb{D}$ onto the complete $z$-plane, minus a crack on the $-ve$ real axis from $-0.25$ to $\infty$, because it could be seen by expressing in Koebe function is the significant one namely

$$K(z) = 0.25 \left\{ \frac{1}{(1 - z^2)^2} - 1 \right\}$$

The Area theorem has a significant impact on the approach of schlicht functions, proved first by [1]. If $t(z) = \frac{1}{z} + \sum_{n=1}^t d_n z^n$ is univalent function in $D$ taking a pole of first order at the point $z = 0$ then

$$\sum_{n=1}^t |d_n|^2 \leq 1$$

[2] evidence to prove that, $|a_t| \leq 2$ for $f \in S$ and $|a_t| \leq t$ for $t \geq 2$

Let $S$ denotes the collection of holomorphic functions in $\mathbb{D}$ expressed as

$$g(z) = 1 + \sum_{t=1}^\infty c_t z^t,$$

With properties $Reg(z) > 0, g(0) = 1$ and $|c_t| \leq 2$ for each $t \geq 1$ is sharp for the Mobius transformation, namely

$$M_0(z) = (1 + z)(1 - z)^{-1} = 1 + 2z + 2z^2 + 2z^3 + \cdots$$

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Definition 1. A regular function $f$ named bounded turning which derivative has real part, if
\[ \text{Re}(f(z)) > 0, \quad z \in \mathbb{D}, \] (6)
The class of bounded turning functions is represents by $\mathfrak{B} [3, 4].$ Conducted a systematic study about the traits of functions in this class and estimated that $|a_n| \leq 2, \ n \geq 2,$ the estimate is sharp for the function $-z + \log \left( \frac{1 + z}{1 - z} \right)$ see [5,6]. Also studied properties of functions in this class.

Definition 2. A holomorphic function $f \in \mathfrak{B}(a), \ a \in [0,1)$ (according to [7]), if
\[ \text{Re}(f(z)) > a, \quad z \in \mathbb{D} \] (7)
If $a = 0,$ then $\mathfrak{B}(0) = \mathfrak{R}$

Definition 3. A regular function $f$ is named starlike ($S^*$) with regard to $(0,0).$ Analytically i.e.
\[ \text{Re} \left( \frac{zf(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D} \] (8)

Definition 4. A holomorphic function $f$ is termed convex $(k),$ when $f$ in $\mathbb{D}$ onto a convex domain. Analytically
\[ \text{Re} \left( \frac{f(z) + zf(z)''}{f(z)'} \right) > 0, \quad z \in \mathbb{D} \] (9)
If $f(z) \in K$ then $|a_n| \leq 1,$ for every $t \geq 2,$ is sharp for $f(z) = \frac{z}{1 - z}.$ Further, each function $f \in K$ is starlike and the containment is proper, because the function $K(z) \in S^*,$ but not in $K.$ [3],from Definitions 3. And 4. there is a very close analytic relationship with functions in the families $K$ and $S^*$ such that $f \in K \iff zf' \in S^*$

Definition 5.[8]. A regular function $f \in S^*(\beta),$ with $\beta \in [0,1),$ if
\[ \text{Re} \left( \frac{zf(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{D} \] (10)
If $\beta = 0,$ then $S^*(0) = S^*,$ where $S^*(\alpha) = S^*$ when $\alpha \in [0,1)$ and $S^*(\alpha) \subseteq S^*(\beta)$ if $\alpha - \beta \geq 0$

Definition 6. A holomorphic function $f \in K(\beta), \ \beta \in [0,1),$ if
\[ \text{Re} \left( \frac{f(z) + zf(z)''}{f(z)'} \right) > \beta, \quad z \in \mathbb{D} \] (11)
[3], observed that $f \in K(\beta) \iff zf' \in S^*(\beta)$ such that $K \subset S^* \left( \frac{1}{2} \right) \subset S^*$

Definition 7.[9]. A holomorphic transformation $f,$ called close-to-convex, whose collection is denoted by $CC,$ if $\exists$ a function $g(z) \in S^*$ satisfying
\[ \text{Re} \left( \frac{zf(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D} \] (12)
Where $K \subset S^* \subset CC \subset S$ if $g(z) = z$ then $CC = \mathfrak{R}$

Definition 8.[10]. Proved the members of $f \in S^*_0$ in $CC$ is univalent if
\[ \text{Re} \left( \frac{2zf(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \mathbb{D} \] (13)

Definition 9. A holomorphic transformation $f$ is called as convex function with regard to symmetric points, if
\[ \text{Re} \left( \frac{2zf(z)}{f(z) - f(-z)} \right)' > 0, \quad z \in \mathbb{D} \] (14)
The collection of all these mappings is represented by $K_0,$ introduced by Das and Singh [11].

Remark. Any Hypergeometric function $\mathcal{F}_1(t_1, t_2, t_3; z),$ defined in the unit disc in a series of powers in $z$
\[ \mathcal{F}_1(t_1, t_2, t_3; z) = \sum_{n=0}^{\infty} \frac{(t_1)_n(t_2)_n(t_3)_n z^n}{n!} \] It is not defined if $t_2$ is a negative integer. Here $(t)_n$ is called (rising) Pochhammer symbol, described as:
\[ (t)_n = \begin{cases} 1 & n > 0 \\ t(t + 1) \ldots (t + n - 1) & n > 0 \end{cases} \]

An example on Hypergeometric function:
For $p, m, n \in \mathbb{N}$ then
\[ \mathcal{F}_1 \left[ 1, \frac{p}{m+n}; 1 + \frac{p}{m+n}; z^{m+n} \right] = 1 + \frac{p z^{m+n}}{m+n} + \frac{p z^{2(m+n)}}{2(m+n)+m} + \ldots + \frac{p z^{3(m+n)}}{3(m+n)+m} \]

And a function $F$ subordinate to $G,$ both analytic in $\mathbb{D},$ symbolized as $F < G,$ when there occurs a holomorphic function $w(z)$ in $\mathbb{D}$ fulfilling $w(0) = 1 = 0$ and $|w(z)| < 1, z \in \mathbb{D}$ named Schwarz’s map in this way

Definition 10. A regular function $f$ is bi-univalent, if the couple’s $f$ and its analytic extension namely $f^{-1}$ are univalent in $\mathbb{D},$ whose class is represented by $\Omega.$

According to Koebe $s \left( \frac{1}{4} \right)^{th}$ - theorem, every holomorphic and univalent function $\omega \in \mathbb{D}$ possesses an inverse

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denoted by \( \omega^{-1} \) satisfying
\[
z = \{ \omega^{-1}(\omega(z)) ; z \in \mathbb{D} \text{ and } \omega(\omega^{-1}(\omega)) = \omega, (|\omega| < p_0(f); p_0(f) \geq \frac{1}{4}) \}.
\]
Where
\[
\omega = \left\{ \omega + \sum_{n \geq 1} q_n \omega^n \right\} + \sum_{n \geq 1} a_n \left\{ \omega + \sum_{n \geq 2} q_n \omega^n \right\}.
\]

**Definition 11** Let \( k \in \mathbb{N} \). A domain \( \psi \) is called \( K \)-copies of symmetric, if \( \psi \) revolves with respect to the origin making an angle \( \frac{2\pi}{k} \) takes \( \psi \) onto itself in \( \mathbb{D} \) if
\[
f(\frac{2\pi}{k} z) = e^{\frac{2\pi}{k}} f(z), z \in \mathbb{D}
\]

[12], [13] and [14]. Introduce \( f \in A_p \) is holomorphic function is a member of \( \mathcal{R}_p \) if
\[
Re \left\{ \frac{f(z)'}{P_{2p-1}} \right\} > 0, z \in \mathbb{D}
\]

For \( p = 1 \) in (17), we get \( \mathcal{R}_1(0) = \mathcal{R} \).

**Definition 12.** A function \( f \) given in (1.7.1) to be in \( \mathcal{R}_p(\alpha) \) with \( \alpha \in [0,1) \) if
\[
Re \left\{ \frac{f(z)'}{P_{2p-1}} \right\} - \alpha, z \in \mathbb{D}
\]

[15, 16], defined the Hankel det of order \( q \) for the regular function \( f \) specified in (1) as
\[
H_{q,t}(f) = \begin{vmatrix} a_t & a_{t+1} & \cdots & a_{t+q-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t+q-1} & a_{t+q} & \cdots & a_{t+2q-2} \end{vmatrix}
\]

Here \( a_1 = 1 \) and \( q, t \) are integers, positive in nature. The determinant \( H_{q,t}(f) \) has been investigated by many Mathematicians. [17, 18]. Applied by them in the evaluation of singularities of meromorphic mappings. [15], settled the Hankel det for schliet mappings by the inequality \( |H_{q,t}(f)| < k_t \frac{(1+2B)2^t}{2^t t^2} \), with \( \beta > 0.00025 \) and \( k \) dependent on \( q \) [19], has established a well-built outcome to a really mean univalent functions as
\[
|H_{2,n}(f)| < B n^\frac{1}{2}, \text{where } B \text{ is an absolute constant.}
\]

Given a subfamily of \( F \) of \( A \), for specific values of \( q \) and \( t \), estimating an upper bound \( H_{q,t}(f) \) is a problem of interest to study, when \( f \in F \). A familiar result is that \((a_q - a_2^2) = H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \) also called Fekete-Szego functional for the functions in \( S \). See [20, 21]. Supposed that
\[
f \in S \Rightarrow |a_j^2 - a_{2j-1}| \leq (j - 1)^2, \text{ for } j = 2, 3 \ldots
\]

[22], tested the Zalcman supposition for the functions in the class \( CC \). For \( f \in S \) [23]. Put forward a universal Zalcman hypothesis that
\[
|a_k a_l - a_{k+l-1}| \leq (k - 1)(l - 1), \text{ for } k, l = 2, 3, \ldots
\]

still it is an open problem. Further, he derived (20) for the mappings in the families \( S^* \) and \( S_R \).

**Some Lemma and Result**

**Lemma 13.** [24]. If \( g \in G \), then \( 2 \geq |c_t| \), for every \( t \geq 1 \), sharp for the mobius transformation, given in (5).

**Lemma 14.** Let \( g \in G \), given in (4) with \( c_1 \geq 0 \) then
\[
c_2 = \frac{1}{2} \left[ c_1^2 + x(4 - c_1^2) \right]
\]

And \( c_3 = \frac{1}{4} \left[ c_1^3 + (2c_1 x - c_1 x^2) + 2(1 - |x|^2) y(4 - c_1^2) \right] \), for \( x \) and \( y \) whose absolute value is at most unity, i.e.
\[
|x| \leq 1 \text{ and } |y| \leq 1.
\]

**Lemma 15.** [25]. \( g \in f \Rightarrow \) The bound \( 2 \geq |c_n| - \mu c_k c_{n-k} \) is sharp, for \( n, k \in \mathbb{N} \), where \( n > k \) and \( 0 \leq \mu \leq 1 \).

**Theorem 16.** If \( f \in S^*_g \) then \( |H_{2,1}(f^{-1})| \leq 1 \), the estimate is sharp for
\[
f_0 = z + z^3 + z^5 + z^7 + \ldots
\]

**Proof.** for \( f \in S^*_g \) as per definition \( 8 \), a function \( g(z) \), is given (4) occurs in such wise
\[
2z f(z)' \frac{f(z) - f(-z)}{f(z) - f(-z)} = g(z)
\]

Employing equivalent representation for \( f(-z) \), \( f(z) \) and \( g(z) \) in (21), then
\[
2z \sum_{n=1}^{\infty} na_n z^{n-1} = \left( \sum_{n=1}^{\infty} a_n z^n - \sum_{n=1}^{\infty} (-1)^n a_n z^n \right) \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right)
\]

(22)

An easy calculation gives
\[
2z(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 \ldots)
\]

(23)

Further simplification gives
\[
(4a_2 - 2c_1)z^2 + (4a_3 - 2c_2)z^3 + (2a_4 - 2c_3)z^4 + \ldots = 0
\]

(24)

After simplifying, the coefficients of \(z^i = 2,3,4\), are obtained as
\[
a_2 = \frac{c_1}{2}; a_3 = \frac{c_2}{2}; a_4 = \frac{(2c_3 + c_2)}{8} \ldots
\]

(25)

Simplifying the expressions (25) in sight of (15), we obtain
\[
q_2 = -\frac{c_1}{2}; q_3 = -\frac{c_2 + c_1^2}{2} \quad \text{and} \quad a_4 = -\frac{5c_1^2 + 9c_1c_2 - 2c_3}{8} \ldots
\]

(26)

Now, based on \(H_{2,1}(f)\), we can have
\[
H_{2,1}(f^{-1}) = \left| \frac{q_2}{q_3} \right| = q_3 - q_2^2
\]

(27)

Placing \(q_j, (j = 2,3)\) values from (26) in (27), we have
\[
H_{2,1}(f^{-1}) = \left( -\frac{c_2 + c_1^2}{2} \right) \left(\frac{c_1}{2}\right)^2
\]

(28)

After simplifying, we get
\[
H_{2,1}(f^{-1}) = \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right)
\]

(29)

Taking modulus and employing Lemma 15, we get
\[
\left| H_{2,1}(f^{-1}) \right| \leq \frac{1}{2} \left| c_2 - \frac{c_1^2}{2} \right| = 1
\]

(30)

From \(f_0\) we obtain \(q_2 = 0\) and \(q_3 = -1\) which follows the result.

**Theorem 17.** if \(f \in S^*_s \Rightarrow \left| H_{2,2}(f^{-1}) \right| \leq 1\), the equality holds for \(f_0\) specified under Theorem 16

**Proof.** for \(f \in S^*_s\), in view of \(H_{2,2}(f)\), we have

Placing \(q_j, (j = 2,3,4)\) values from (26) in (31), we have
\[
H_{2,2}(f^{-1}) = \left( -\frac{c_1}{2} \right) \left( -\frac{5c_1^2 + 9c_1c_2 - 2c_3}{8} \right) - \left( -\frac{c_2 + c_1^2}{2} \right)^2
\]

(32)

Which is equivalent to
\[
q_2 q_4 - q_3^2 = \frac{1}{16} [d_1 c_1 c_3 + d_2 c_2 c_3 + d_3 c_2^2 + d_4 c_4^2]
\]

(33)

Here \(d_1 = 2; d_2 = -1; d_3 = -4; d_4 = 1\)

Employing \(c_2\) and \(c_3\) values from Lemma 14 on right side of 33, appears as
\[
4 \left[ \frac{1}{4} d_1 c_1 c_3^2 + 2c_2 x - c_2 x^2 + 2(1 - x^2)|y| (4 - c_1^2) \right] + \frac{1}{2} d_2 c_2^2 (4 - c_1^2) + x (4 - c_1^2)
\]

(34)

Taking magnitude on either side, then employing the triangle inequity in (34), we have
\[
4|d_1 c_1 c_3 + d_2 c_2 c_3 + d_3 c_2^2 + d_4 c_4^2|
\]

(35)

From (33), we can now write
\[
|d_1 + 2d_2 + d_3 + 4d_4| = 0; |d_1 + 2d_2 + d_3| = 3; \{(|d_1| - |d_4|) c_1^2 - 2|d_1||c_1||y| + 4|d_3|\} (|4 - c_1^2||y|)
\]

(36)

Placing the values from (36), \(d_1\) from (32) in (34), it takes the form

\[
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\]

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\[ 4|d_1c_1e_1 + d_2c_1^2e_2 + d_3c_2^2 + d_4c_4^2| \\
\leq [(0)c_1^2 + 4c_1(4 - c_1^2)] |y| + 6c_1^2(4 - c_1^2)|x| + (-2c_1^2)|x| \\
+ (-2c_1^4c_1|y| + 16)(4 - c_1^2)|x|^2 \] (37)

Employing triangle inequality, displacing \(|x|\) with \(\rho\), besides \(1 \geq |y|\), designate \(c_1 = c \in [0,2]\), the right side of (37) takes the form

2. Conclusions

In this paper, we studied and presented properties of Zalcman conjecture where, we obtained some theorems and properties associated with a class defined by a second order Hankel determinant.

Reference


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